Available online at www.sciencedirect.com
ScienceDirect
C. R. Mecanique 335 (2007) 763–767
COMPTES RENDUS
MECANIQUE
http://france.elsevier.com/direct/CRAS2B/

# Asymptotics of Neumann harmonics when a cavity is close to the exterior boundary of the domain

Giuseppe Cardone [a], Sergey A. Nazarov [b], Jan Sokolowski [c,*], Jari Taskinen [d]

[a] *Università del Sannio, Dipartimento di Ingegneria, Piazza Roma, 21, 84100 Benevento, Italy*
[b] *Institute of Mechanical Engineering Problems, V.O., Bolshoi pr. 61, 199178 St. Petersburg, Russia*
[c] *Université Henri-Poincaré, Nancy 1, département de mathematiques, B.P. 239, 54506 Vandoeuvre les Nancy cedex, France*
[d] *University of Helsinki, Department of Mathematics and Statistics, P.O. Box 68, 00014 Helsinki, Finland*

Received 25 September 2007; accepted 2 October 2007

Available online 8 November 2007

Presented by Évariste Sanchez-Palencia

**Abstract**

We construct the asymptotics (as $\varepsilon \to 0$) of solutions to the Neumann problem for the Laplace equation and of the corresponding Dirichlet integral. The problem concerns a three-dimensional domain having two connected components of the boundary at the distance $\varepsilon > 0$. ***To cite this article: G. Cardone et al., C. R. Mecanique 335 (2007).***


**Résumé**

**Développements asymptotiques des solutions harmoniques d'un problème de Neumann lorsqu'une cavité est proche d'un bord extérieur du domaine.** Nous construisons les développements asymptotiques (lorsque $\varepsilon$ tend vers 0) des solutions d'un problème de Neumann pour l'équation de Laplace ainsi que le développement asymptotique de l'intégrale de Dirichlet correspondante. Le problème est défini dans un domaine tri-dimensionnel avec un bord ayant deux composantes connexes distantes de $\varepsilon > 0$. ***Pour citer cet article : G. Cardone et al., C. R. Mecanique 335 (2007).***


*Keywords:* Singularly perturbed Neumann problem; Touching surfaces; Dirichlet integral; Thin ligament

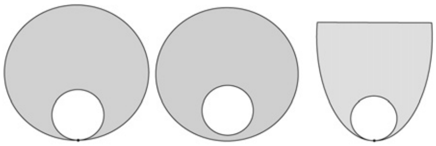
*Mots-clés :* Problème singularement perturbé ; Surfaces en contact ; Integrale Dirichlet ; Ligament fin

## 1. Formulation of the problem

Let $\Gamma$ and $\Gamma_0$ be smooth closed surfaces in the Euclidean space $\mathbb{R}^3$; assume that they have the only common point $O$ and that $\Gamma$ envelopes $\Gamma_0$. We introduce the Cartesian coordinate system $x = (y, z) = (y_1, y_2, z)$ centered at $O$ and such that the plane $\{x\colon\ z = 0\}$ is tangent to both surfaces at the point $O$. Given the small positive parameter $\varepsilon$, we set

* Corresponding author.
*E-mail addresses:* gcardone@unisannio.it (G. Cardone), srgnazarov@yahoo.co.uk (S.A. Nazarov), Jan.Sokolowski@iecn.u-nancy.fr (J. Sokolowski), taskinen@cc.helsinki.fi (J. Taskinen).

1631-0721/$ – see front matter 
doi:10.1016/j.crme.2007.10.001



(a) (b) (c)

Fig. 1. Sketch of the problem.

$\Gamma_\varepsilon = \{x\colon (y, z-\varepsilon) \in \Gamma_0\}$ and denote by $\Omega_\varepsilon$ the domain between the surfaces $\Gamma$ and $\Gamma_\varepsilon$ (see Fig. 1(a) and (b)). We consider the Neumann problem for the Laplace equation

$$-\Delta_x u_\varepsilon(x) = 0, \quad x \in \Omega_\varepsilon$$
$$\partial_\nu u_\varepsilon(x) = 0, \quad x \in \Gamma, \qquad \partial_\nu u_\varepsilon(x) = g(y, z-\varepsilon), \quad x \in \Gamma_\varepsilon \tag{1}$$

where $\partial_\nu$ stands for derivative along the outward normal and $g$ is a function in the Hölder space $C^{1,\alpha}(\Gamma_0)$, $\alpha \geqslant 1/2$, with zero mean value. For $\varepsilon > 0$, the boundary $\partial\Omega_\varepsilon$ is smooth and problem (1) has a solution $u_\varepsilon$ in the Sobolev space $H^1(\Omega_\varepsilon)$ which is unique under the orthogonality condition

$$\int_{\Omega_\varepsilon} u_\varepsilon(x)\,\mathrm{d}x = 0 \tag{2}$$

The limit boundary $\partial\Omega_0 = \Gamma_0 \cup \Gamma$ gets the singularity point $O$ of the specific type (cf. [1–3]). In the next section we show that, under the condition

$$g(0) \neq 0 \tag{3}$$

the solvability in $H^1(\Omega_0)$ of the limit problem, with respect to the singularly perturbed problem (1),

$$-\Delta_x v(x) = 0, \quad x \in \Omega_0, \qquad \partial_\nu v(x) = 0, \quad x \in \Gamma \setminus O, \qquad \partial_\nu v(x) = g(x), \quad x \in \Gamma_0 \setminus O \tag{4}$$

depends crucially on the exponent $m \in \{1, 2, \ldots\}$ in the formula

$$H(y) := H_+(y) + H_-(y) = \mathbf{H}(y) + \mathrm{O}\big(r^{2m+1}\big), \quad r := |y| \to 0 \tag{5}$$

Here $\mathbf{H}$ is a positive homogeneous polynomial of degree $2m$ and $H_\pm$ are smooth functions in the ball $\mathbb{B}_R = \{y\colon r < R\}$ which determine the set $\Lambda_\varepsilon = \{x \in \Omega_\varepsilon\colon y \in \mathbb{B}_R,\ |z| < d\}$ by the inequalities

$$-H_-(y) < z < \varepsilon + H_+(y) \tag{6}$$

Note that (6) remains valid at $\varepsilon = 0$ and $H_\pm(0) = 0$, $\nabla_y H_\pm(0) = 0$ because the surfaces $\Gamma$ and $\Gamma_0$ are smooth and tangential to each other.

The main goal of this Note is to describe the asymptotics as $\varepsilon \to 0^+$ of the solution $u_\varepsilon$ and of the energy functional

$$E(u_\varepsilon; \Omega_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} \big|\nabla_x u_\varepsilon(x)\big|^2\,\mathrm{d}x - \int_{\Gamma_\varepsilon} g(y, z-\varepsilon) u_\varepsilon(x)\,\mathrm{d}s_x = -\frac{1}{2}\int_{\Omega_\varepsilon} \big|\nabla_x u_\varepsilon(x)\big|^2\,\mathrm{d}x \tag{7}$$

In particular, we show that functional (7) has a finite limit as $\varepsilon \to 0^+$ if and only if the problem (3) is solvable in $H_0^1(\Omega_0)$.

## 2. The weighted trace inequality

The following proposition proves that, under condition (3), the right-hand side of the integral identity

$$(\nabla_x v, \nabla_x V)_{\Omega_0} = (g, V)_{\Gamma_0}, \quad V \in H^1(\Omega_0) \tag{8}$$

defines a continuous functional on $H^1(\Omega_0)$ provided $m = 1$ in (5):

**Proposition 2.1.** *If a function $v \in H^1(\Omega_0)$ satisfies the orthogonality condition* (2) *at $\varepsilon = 0$, the inequality*

$$\left\| \rho^{m-1} v; L^2(\Gamma_0) \right\| \leqslant c \left\| \nabla_x v; L^2(\Omega_0) \right\| \tag{9}$$

*is valid*; *here $\rho(x) = |x|$ and the constant $c$ is independent of $v$.*

It is possible to verify that a function $v \in H^1(\Omega_0)$ cannot satisfy identity (8) if the inequalities $m > 1$ and $g(0) \neq 0$ occur. Problem (8) admits a solution in $H^1(\Omega)$ if, e.g., $g(x) = \mathrm{O}(\rho(x)^{m-2+\delta})$ with any $\delta > 0$. In other words, the smooth function $g$, together with all its derivatives up to order $m-2$, must vanish at $O$. For the 'osculating' balls on Fig. 1(a), we have $m = 1$ and a solution $v \in H^1(\Omega_0)$ exists. At the same time, if the ball of radius $R_0$ touches the rotationally symmetric paraboloid with curvature $(2R_0)^{-1}$ at its tip (see Fig. 1(c)), the problem (8) in case (3) is not solvable in $H^1(\Omega_0)$ because $m = 2$.

The Neumann problem with the data $f \in L^2(\Omega_0)$ and $g \in L^2(\Gamma_0)$, $h \in L^2(\Gamma)$, satisfying the compatibility condition

$$\int_{\Omega_0} f(x)\,\mathrm{d}x + \int_{\Gamma_0} g(x)\,\mathrm{d}s_x + \int_{\Gamma} h(x)\,\mathrm{d}s_x = 0$$

has a solution in $H^1(\Omega_0)$ if and only if the function $y \mapsto r^{1-m}(g(y, H_+(y)) - h(y, -H_-(y)))$ belongs to $L^2(\mathbb{B}_R)$.

The following weighted inequality justifies the asymptotics constructed below for the solution $u_\varepsilon$ of problem (1):

**Proposition 2.2.** *If a function $u_\varepsilon$ satisfies the orthogonality condition* (2)*, the inequality*

$$\left\| \mathcal{R}_\varepsilon u_\varepsilon; L^2(\Omega_\varepsilon) \right\| + \left\| (\varepsilon + \rho^{2m})^{1/2} \mathcal{R}_\varepsilon u_\varepsilon; L^2(\partial\Omega_\varepsilon) \right\| \leqslant c \left\| \nabla_x u_\varepsilon; L^2(\Omega_\varepsilon) \right\| \tag{10}$$

*holds, where $\mathcal{R}_\varepsilon(x) = (\varepsilon^{1/(2m)} + \rho(x))^{-1}(1 + |\ln(\varepsilon^{1/(2m)} + \rho(x))|)^{-1}$ and the constant $c$ is independent of $u_\varepsilon$ and $\varepsilon \in (0, \varepsilon_0]$.*

## 3. The asymptotics in the case $m > 2$

The leading asymptotic term provides a boundary layer phenomenon and the solution $u_\varepsilon$ is mainly located on the ligament $\Lambda_\varepsilon$. Indeed, the ligament is thin in the vicinity of the point $O$ so that the standard asymptotic ansätz in thin domains (cf. [4,5]) and the coordinate scalings

$$y \mapsto \eta = \varepsilon^{-\mu} y, \quad \mu = (2m)^{-1}, \quad z \mapsto \zeta = \varepsilon^{-1} z \tag{11}$$

yield the second limit problem

$$-\nabla_\eta \cdot \big(1 + \mathbf{H}(\eta)\big) \nabla_\eta w(\eta) = g(0), \quad \eta \in \mathbb{R}^2 \tag{12}$$

Eq. (12) has a unique smooth solution with the following behavior at infinity:

$$w(\eta) = g(0)|\eta|^{2-2m} \Psi(\theta) + \mathrm{O}\big(|\eta|^{-2m}\big), \quad |\eta| \to +\infty \tag{13}$$

where $\theta = |\eta|^{-1}\eta \in \mathbb{S}$ and $\Psi$ is a smooth function on the unit circle $\mathbb{S}$. Note that, due to (13), the following integral converges:

$$\mathbf{I}(w) = \int_{\mathbb{R}^2} \big(1 + \mathbf{H}(\eta)\big) \big|\nabla_\eta w(\eta)\big|^2\,\mathrm{d}\eta \tag{14}$$

**Theorem 3.1.** *If $m \geqslant 3$, the following relation holds*:

$$\left\| \nabla_x (u_\varepsilon - \varepsilon^{-1+1/m} \chi w); L^2(\Omega_\varepsilon) \right\| \leqslant c \varepsilon^{\mu(3-m)} \big(1 + |\ln \varepsilon|\big)$$

*where the constant $c$ does not depend on $\varepsilon \in (0, \varepsilon_0]$, $\chi$ is a cut-off function which is equal to $\chi_0(y)$ on the ligament and vanishes on $\Omega_\varepsilon \setminus \Lambda_\varepsilon$, and $\chi_0 \in C^\infty(\mathbb{R}^2)$ is such that $\chi_0(y) = 0$ for $|y| > R$ and $\chi_0(y) = 1$ for $|y| < R/2$.*

We emphasize that, under condition (3), $\varepsilon^{-1+1/m} \|\nabla_x (\chi w); L^2(\Omega_\varepsilon)\| = \mathrm{O}(\varepsilon^{\mu(1-m)})$ and, hence, Theorem 3.1 proves $\varepsilon^{-1+1/m} \chi(x) w(\varepsilon^{-1/(2m)} y)$ to be the main asymptotic term of the solution $u_\varepsilon$.

**Corollary 3.2.** *If $m \geqslant 3$ and $g(0) \neq 0$, then*

$$E(u_\varepsilon; \Omega_\varepsilon) = -\frac{1}{2}\varepsilon^{-1+1/(2m)}\big(\mathbf{I}(w) + \mathrm{O}(\varepsilon^\mu)\big) \tag{15}$$

Since $m > 2$ and $-1 + 1/(2m) < 0$, relation (15) demonstrates that, under condition (3), the energy functional (7) tends to infinity as $\varepsilon \to 0^+$.

## 4. The asymptotics in the case $m = 1$

The solution $w$ to Eq. (12) takes the form

$$w(\eta) = g(0)\big(c_0 \ln|\eta| + \Psi(\theta)\big) + c_2 + \mathrm{O}\big(|\eta|^{-2}\big), \quad |\eta| \to +\infty \tag{16}$$

where $c_0$ is a certain constant and $c_2$ is arbitrary. Thus, the weighted Dirichlet integral (14) diverges. At the same time, Proposition 2.1 delivers the solution $v \in H^1(\Omega_0)$ of the first limit problem (4). According to [3], the following representation holds:

$$v(x) = \chi(x)g(0)\big(c_0 \ln|\eta| + \Psi(\theta)\big) + c_1 + \tilde{v}(x) \tag{17}$$

where $c_0$ and $\Psi$ are the same as in (16), the constant $c_1$ is fixed by the orthogonality condition (2) at $\varepsilon = 0$ and the remainder $\tilde{v}(x)$ decays as $\mathrm{O}(\rho(x))$ for $x \to 0$. Similarly to [1,7], we take the following approximate solution to the singularly perturbed problem (1):

$$U_\varepsilon(x) = \big(1 - \chi(x)\big)V(x) + \chi(x)\varepsilon^{-1+1/m} w(\varepsilon^{-\mu} y) + \big(1 - \chi(\varepsilon^{-1}x)\big)\tilde{v}(y, \mathfrak{z}) \tag{18}$$

Here $V$ is an extension of $v$ in the Hölder class $C^{2,\alpha}_{\mathrm{loc}}(\overline{\Omega} \setminus O)$ on the domain $\Omega \supset \Omega_0$ bounded by the surface $\Gamma$, $w$ is the solution (16) with $c_2 = -g(0)\mu c_0 \ln\varepsilon$ and the variable

$$\mathfrak{z} = \big(\varepsilon + H(y)\big)^{-1}\big(zH(y) - \varepsilon H_-(y)\big)$$

belongs to the interval $\Upsilon_0(y)$ when $z \in \Upsilon_\varepsilon(y) := (-H_-(y), \varepsilon + H_+(y))$.

**Theorem 4.1.** *If $m = 1$, the following relation holds*

$$\big\|\nabla_x(u_\varepsilon - U_\varepsilon); L^2(\Omega_\varepsilon)\big\| \leqslant c\varepsilon^\mu\big(1 + |\ln\varepsilon|\big)^2 \tag{19}$$

*where $U_\varepsilon$ denotes the asymptotic solution* (18) *and the constant $c$ does not depend on $\varepsilon \in (0, \varepsilon_0]$.*

The following assertion shows that the energy functional (7) gets the natural limit $E(v; \Omega_0)$ in the situation of Fig. 1(b) and (a):

**Corollary 4.2.** *If $m = 1$, then $|E(u_\varepsilon; \Omega_\varepsilon) - E(v; \Omega_0)| \leqslant c\varepsilon^\mu(1 + |\ln\varepsilon|)^2$.*

## 5. The asymptotics in the case $m = 2$

If (3) is valid, none of the limit problems any more has solutions $v$ and $w$ with finite Dirichlet integrals $\|\nabla_x v; L^2(\Omega_0)\|^2$ and (14), respectively. However, Eq. (12) still admits the solution $w$ in the form (13), while, in view of results in [3], problem (4) has the solution

$$v(x) = \chi(x)g(0)r^{-2}\Psi(\theta) + \hat{v}(x), \quad \hat{v} \in H^1(\Omega_0) \tag{20}$$

Then the approximate solution $U_\varepsilon$ of problem (1) keeps the form (18). Nevertheless, the energy functional is not bounded.

**Theorem 5.1.** *If $m = 2$, the conclusion of Theorem* 4.1 *remains valid with the bound $c\varepsilon^\mu(1 + |\ln\varepsilon|)$ in* (19).

**Corollary 5.2.** *If $m = 2$, then*

$$\left| E(u_\varepsilon; \Omega_\varepsilon) - \frac{1}{4m} |\ln \varepsilon| g^2(0) \int\limits_{\mathbb{S}} \Psi_0(\theta)\, \mathrm{d}\theta \right| \leqslant c$$

*where $\Psi_0$ is the function on the circle $\mathbb{S}$ from representation* (20) *and the constant $c$ is independent of $\varepsilon \in (0, \varepsilon_0]$.*

## 6. Discussion

The weighted trace inequality (9) is still valid in the $n$-dimensional domain $\Omega_0$ defined by formulas (6) and (5). Under the condition (3), the Neumann problem (4) and the similar Dirichlet problem have solutions in $H^1(\Omega_0)$ if and only if the following relations are valid, respectively:

$$2m < n + 1 \quad \text{and} \quad 2m < n - 1 \tag{21}$$

The Dirichlet integral $\|\nabla_x u_\varepsilon; L^2(\Omega_\varepsilon)\|^2$, where $u_\varepsilon$ is the solution of the Neumann or Dirichlet problem of type (1), has the finite limit $\|\nabla_x v; L^2(\Omega_0)\|^2$ provided the conditions (21), respectively, are satisfied. The conclusions remain valid if $\mathbf{H}$ in (5) is a positive homogeneous function of degree $2m > 1$ in the variables $y = (y_1, \ldots, y_{n-1})$.

The two-dimensional Dirichlet and Neumann problems in singularly perturbed domains of type $\Omega_\varepsilon$ with thin ligaments were considered in [6,1,2] and [7,8], respectively. Asymptotics of stresses in two-dimensional elasticity problems with thin ligaments were constructed in [9,10], although the question on asymptotic behavior of the elastic fields in the three-dimensional case is still open.

Let the surfaces $\Gamma_0$ and $\Gamma$ be given by the formulas

$$y_1^2 + y_2^2 + (z - R)^2 = R^2 \quad \text{and} \quad 2Rz = y_2^2 \tag{22}$$

in other words, $\Gamma_0$ is a sphere and $\Gamma$ the parabolic cylinder in the vicinity of the point $O$. In case (22), relation (5) is valid with $m = 1$ but the polynomial $\mathbf{H}(y) = (2R)^{-1} y_1^2$ is not positive since it degenerates at the $y_1$-axis. The authors know neither a condition for existence of a solution $v \in H^1(\Omega_0)$ to problem (4), nor the asymptotics of the solution $u_\varepsilon$ to problem (1) nor that of the Dirichlet integral.

## Acknowledgements

The second author acknowledges the financial support of RFBR (grant 06-01-00257).

## References

[1] V.G. Maz'ya, S.A. Nazarov, B.A. Plamenevskii, Asymptotics of the solution of the Dirichlet problem in domains with a thin crosspiece, Funkt. Anal. i Prilozhen. 16 (2) (1982) 39–46; English transl.: Funct. Anal. Appl. 16 (1982) 108–114.
[2] V.G. Maz'ya, S.A. Nazarov, B.A. Plamenevskij, The Dirichlet problem in domains with thin cross connections, Sibirsk. Mat. Zh. 25 (2) (1984) 161–179 (in Russian); English transl.: Sib. Math. J. 25 (4) (1984) 297–313.
[3] S.A. Nazarov, Asymptotics of the solution of the Neumann problem at a point of tangency of smooth components of the boundary of the domain, Izv. Ross. Akad. Nauk Ser. Mat. 58 (1) (1994) 92–120; English transl.: Math. Izvestiya 44 (1) (1995) 91–118.
[4] V.G. Maz'ya, S.A. Nazarov, B.A. Plamenevskij, Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains, Tbilisi Univ., Tbilisi, 1981 (in Russian); English transl.: in: Operator Theory: Advances and Applications, vol. 112, Birkhäuser-Verlag, Basel, 2000.
[5] S.A. Nazarov, Asymptotic Theory of Thin Plates and Rods. Dimension Reduction and Integral Estimates, Nauchnaya Kniga, Novosibirsk, 2001.
[6] D.I. Sherman, One particular Dirichlet problem for a doubly connected region whose boundaries are extremely close together in a narrow zone, Mech. Solids 15 (3) (1980) 76–87 (in Russian).
[7] S.A. Nazarov, O.R. Polyakova, Asymptotic expansions of eigenvalues of the Neumann problem in a domain with a thin bridge, Sibirsk. Mat. Zh. 33 (4) (1992) 80–96; English transl.: Sib. Math. J. 33 (4) (1992) 618–633.
[8] J.G. Huout, A. Munnier, On the motion and collisions of rigid bodies in an ideal fluid, Prépublications de l'IECN, 33 (2006).
[9] X. Markenscoff, J. Dundurs, Amplification of stresses in thin ligaments, Int. J. Solids Structures 29 (1992) 1883–1888.
[10] S.A. Nazarov, O.R. Polyakova, Deformation of elastic bodies with thin ligaments, Prikl. Mat. Mekh. 50 (5) (1992) 52–65; English transl.: J. Appl. Math. Mech. 56 (5) (1992) 651–664.