Available online at www.sciencedirect.com

ScienceDirect

C. R. Mecanique 335 (2007) 781–786

COMPTES RENDUS MECANIQUE

http://france.elsevier.com/direct/CRAS2B/

Discrete approach for modelling quasi-brittle damage: conditions on the set of directions

Renaud Bargellini [1], Damien Halm *, André Dragon

Laboratoire de mécanique et physique des matériaux (UMR 6617), ENSMA, BP 40109, 86961 Futuroscope cedex, France

Received 1 February 2007; accepted after revision 3 October 2007

Available online 9 November 2007

Presented by André Zaoui

Abstract

The aim of this Note is to investigate the discrete approach of anisotropic damage by microcracking by specifying the choice of the set of fixed directions that are to be effectively employed. Three conditions are presented, concerning the description of any damage configuration, and in particular isotropic and transversely isotropic ones. These conditions are given at two different orders in orientation (order 2 and order 4), as both tensorial terms intervene in the thermodynamic potential. The results show that the second order conditions are sufficient to represent a configuration of open microcracks, whereas fourth order conditions are necessary to represent a configuration of closed ones. ***To cite this article: R. Bargellini et al., C. R. Mecanique 335 (2007).***

Résumé

Approche discrète de l'endommagement quasi-fragile : conditions sur l'ensemble de directions. Cette Note poursuit le développement de l'approche discrète de l'endommagement anisotrope en précisant des conditions sur l'ensemble de directions fixes à utiliser. Trois conditions sont présentées, concernant la représentativité de toute configuration d'endommagement et en particulier des configurations isotrope et isotrope transverse. Ces conditions sont écrites aux ordres 2 et 4 en orientation du fait de la présence de ces deux types de termes dans le potentiel thermodynamique. Les résultats montrent que les conditions à l'ordre deux suffisent pour décrire une configuration de microfissures ouvertes, alors que les conditions d'ordre quatre sont nécessaires pour représenter une configuration de microfissures fermées. ***Pour citer cet article : R. Bargellini et al., C. R. Mecanique 335 (2007).***

Keywords: Damage; Anisotropic damage; Unilateral effect; Discrete approach

Mots-clés : Endommagement ; Endommagement anisotrope ; Effet unilatéral ; Approche discrète

* Corresponding author.
E-mail addresses: renaud-externe.bargellini@edf.fr (R. Bargellini), halm@lmpm.ensma.fr (D. Halm), andre.dragon@lmpm.ensma.fr (A. Dragon).

[1] Present address: Laboratoire de mécanique des structures industrielles durables (UMR CNRS/EDF 2832), 1, avenue du Général De Gaulle, 92141 Clamart cedex, France.

1631-0721/$ – see front matter
doi:10.1016/j.crme.2007.10.003

Version française abrégée

Constatant que la modélisation du comportement des milieux microfissurés, notamment de l'anisotropie induite et de l'effet unilatéral, par une variable tensorielle d'ordre deux engendre régulièrement des difficultés mathématiques (comme la non unicité du potentiel thermodynamique [2]), Bargellini et al. [1] proposent une approche originale de l'endommagement anisotrope, dite discrète dans la mesure où elle considère des directions fixes d'endommagement auxquelles elle associe des variables internes scalaires représentant la densité de microfissuration. L'objet de cette Note est de préciser les conditions sur l'ensemble de tenseurs directionnels utilisé ; les cas particuliers d'endommagement isotrope et isotrope transverse sont notamment traités. Ces conditions sont transcrites aux ordres deux et quatre en orientation, puisque, sous les principales hypothèses de non interaction des défauts et d'absence de contraintes résiduelles, l'énergie libre, obtenue par la théorie de représentation des fonctions tensorielles à partir de l'approche discrète, ne fait intervenir que des termes d'ordre deux lorsque les microfissures sont ouvertes, et que l'effet unilatéral est pris en compte par le truchement d'un terme d'ordre quatre en orientation. Un premier ensemble de directions, comprenant neuf entités et fondé sur la base canonique de l'ensemble des tenseurs de la forme $\mathbf{n} \otimes \mathbf{n}, \mathbf{n} \in \Re^3$, permet de vérifier les trois conditions à l'ordre deux, mais pas à l'ordre quatre. Afin de vérifier ces conditions à l'ordre quatre, un ensemble de soixante directions est proposé, fondé sur une base de l'ensemble des tenseurs de la forme $\mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n}, \mathbf{n} \in \Re^3$. Les résultats obtenus, pour une configuration isotrope transverse de défauts ouverts et fermés, par ces deux ensembles et par un modèle utilisant une variable d'ordre deux [5] sont comparés. L'étude des dégradations du module d'Young et du coefficient de Poisson montre que celles-ci sont isotropes transverses lorsque les microfissures sont ouvertes dans les trois cas ; mais lorsque les défauts sont fermés, seul l'ensemble de soixante systèmes aboutit à une dégradation isotrope transverse des propriétés élastiques. Ceci s'explique du fait de la présence de termes d'ordre quatre en orientation dans les termes de fermeture uniquement. Les conditions d'ordre quatre sont donc primordiales pour décrire une configuration de défauts fermés.

1. Introduction

This Note provides developments for the discrete approach of anisotropic damage by microcracks growth proposed by Bargellini et al. [1], summarized in Section 2, which notably avoids some inconveniences encountered when using spectral decompositions of tensorial variables (non-uniqueness of the free energy, see [2]) or strain decomposition into positive and negative parts (spurious dissipation, see [3], and non-conservativeness, see [4]). The reasoning that permits to define the considered fixed microcracks directions is completed: some conditions on their number and orientation are posed and two sets are proposed. The results obtained for a given fixed damage configuration are presented, commented and compared with the ones obtained with the model proposed by Halm and Dragon [5] using a classical second order variable.

2. A discrete approach for modelling anisotropic damage with unilateral effect

The discrete approach consists in considering p different fixed directional tensors $\mathbf{N_i} = \mathbf{n_i} \otimes \mathbf{n_i}$, where $\mathbf{n_i} \in \Re^3$ represents the normal to the crack, and in associating them with p scalar internal variables ρ_i representing evolving microcrack densities. Consequently p independent couples $(\rho_i, \mathbf{N_i})$, $i \in [1, \ldots, p]$, enter the thermodynamic potential (free energy) and constitutive equations. Contrary to the microplane approach (see [6]), which defines separately constitutive relations on various planes and uses spatial integration to build the macroscopic behavior, the macroscopic free energy is directly obtained according to the tensor functions representation theory ([7]) under the main assumptions of non-interacting microcracks, no residual stresses and no frictional sliding.

$$W(\boldsymbol{\epsilon}, \rho_i; \mathbf{N_i}) = \frac{\lambda}{2} \operatorname{tr}^2(\boldsymbol{\epsilon}) + \mu \operatorname{tr}(\boldsymbol{\epsilon}.\boldsymbol{\epsilon}) + \sum_{i=1}^{p} \rho_i \left[\alpha \left[\operatorname{tr}(\boldsymbol{\epsilon}.\boldsymbol{\epsilon}) - \frac{1}{2} \operatorname{tr}^2(\boldsymbol{\epsilon}) + \operatorname{tr}(\boldsymbol{\epsilon}) \operatorname{tr}(\boldsymbol{\epsilon}.\mathbf{N_i}) \right] + 2\beta \operatorname{tr}(\boldsymbol{\epsilon}.\boldsymbol{\epsilon}.\mathbf{N_i}) \right.$$
$$\left. - \left(\frac{3}{2}\alpha + 2\beta \right) H\big[-\operatorname{tr}(\boldsymbol{\epsilon}.\mathbf{N_i})\big] \operatorname{tr}^2(\boldsymbol{\epsilon}.\mathbf{N_i}) \right] \tag{1}$$

where $H(x)$ is the classical Heaviside function, λ and μ the elastic Lamé constants and α and β material parameters representing the degradation of elastic properties due to damage. This thermodynamic potential accounts for the unilateral effect and fulfills the continuity requirements ([8]). The microcrack closure term, namely $-(\frac{3}{2}\alpha + 2\beta)H[-\mathrm{tr}(\boldsymbol{\epsilon}.\mathbf{N_i})]\,\mathrm{tr}^2(\boldsymbol{\epsilon}.\mathbf{N_i})$, is of fourth order in orientation ($\mathrm{tr}^2(\boldsymbol{\epsilon}.\mathbf{N_i})$ can be written $(\mathbf{n_i}\otimes\mathbf{n_i}) : (\boldsymbol{\epsilon}\otimes\boldsymbol{\epsilon}) : (\mathbf{n_i}\otimes\mathbf{n_i})$), according to micromechanical works ([9]), and completely restores the Young's modulus in the direction normal to a closed microcrack. State laws are classically obtained by differentiating (1). Evolution laws of the internal density variables complete this model ([1]).

As the directional tensors $\mathbf{N_i}$ are fixed in the physical space $\Re^3$ of basis $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$, the definition of a 'sufficient' number (from the operational viewpoint) and of the specific orientations embodied by $\mathbf{N_i}$ is important. Section 3 gives some conditions that permit to define their set.

3. Definition of the number and orientations embodied by $\mathbf{N_i}$

The set of $\mathbf{N_i}$ is assumed to fulfil three objectives:

(i) it is able to capture any damage configuration represented by a tensorial variable;
(ii) the material degradation due to an isotropic damage configuration (i.e. with the same microcrack density ρ_0 in all directions) is isotropic;
(iii) in the same way, the material degradation due to a transversely isotropic damage configuration is transversely isotropic.

As the free energy (1) contains terms of second and fourth orders in orientation, conditions (i), (ii) and (iii) are expressed at these orders. In each case, one $\mathbf{N_i}$ set is defined and enters thermodynamic potential (1).

3.1. A nine system set fulfilling second order objectives

At the second order in orientation, the previous conditions read:

(i) any tensor of the type $\mathbf{n}\otimes\mathbf{n}$ is an additive combination of $\mathbf{N_i}$;
(ii) $(\exists a \in \Re)$ so that $\sum_i \rho_0\mathbf{N_i} = a\rho_0\mathbf{I_{d2}}$, where $\mathbf{I_{d2}}$ is the second order identity tensor;
(iii) for any subset $\{A\}_k$ of $\mathbf{N_i}$ directions belonging to the envelope of a cone of axis $\mathbf{e_1}$, $(\exists (a,b) \in \Re^2)$ so that $\sum_{\mathbf{N_i}\in\{A\}_k} \rho_0\mathbf{N_i} = \rho_0[a\mathbf{e_1}\otimes\mathbf{e_1} + b(\mathbf{I_{d2}} - \mathbf{e_1}\otimes\mathbf{e_1})]$.

As the set of tensors of the type $\{\mathbf{n}\otimes\mathbf{n}, \mathbf{n}\in\Re^3\}$ is of rank 6, there must be at least 6 $\mathbf{N_i}$ to fulfil condition (i). Starting from its canonical basis, the following nine tensors $\mathbf{N_i}$ can be defined (see [1]):

$$\begin{aligned}
&\mathbf{N_1} = \mathbf{e_1}\otimes\mathbf{e_1}, \qquad \mathbf{N_4} = \frac{1}{2}(\mathbf{e_1}+\mathbf{e_2})\otimes(\mathbf{e_1}+\mathbf{e_2}), \qquad \mathbf{N_7} = \frac{1}{2}(\mathbf{e_1}-\mathbf{e_2})\otimes(\mathbf{e_1}-\mathbf{e_2})\\
&\mathbf{N_2} = \mathbf{e_2}\otimes\mathbf{e_2}, \qquad \mathbf{N_5} = \frac{1}{2}(\mathbf{e_1}+\mathbf{e_3})\otimes(\mathbf{e_1}+\mathbf{e_3}), \qquad \mathbf{N_8} = \frac{1}{2}(\mathbf{e_1}-\mathbf{e_3})\otimes(\mathbf{e_1}-\mathbf{e_3})\\
&\mathbf{N_3} = \mathbf{e_3}\otimes\mathbf{e_3}, \qquad \mathbf{N_6} = \frac{1}{2}(\mathbf{e_3}+\mathbf{e_2})\otimes(\mathbf{e_3}+\mathbf{e_2}), \qquad \mathbf{N_9} = \frac{1}{2}(\mathbf{e_3}-\mathbf{e_2})\otimes(\mathbf{e_3}-\mathbf{e_2})
\end{aligned} \tag{2}$$

This set of nine tensorial parameters fulfils the three second order conditions. The following paragraph extends these conditions to the fourth order, as a fourth order term, namely $\mathrm{tr}^2(\boldsymbol{\epsilon}.\mathbf{N_i})$, intervenes in (1).

3.2. A sixty system set fulfilling fourth order objectives

At the fourth order in orientation, the previous conditions read:

(i) any tensor of the type $\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}$ is an additive combination of $\mathbf{N_i}\otimes\mathbf{N_i}$;
(ii) $(\exists (a',b') \in \Re^2)$ so that $\sum_i \rho_0\mathbf{N_i}\otimes\mathbf{N_i} = \rho_0(a'\mathbf{I_{d2}}\otimes\mathbf{I_{d2}} + b'\mathbf{I_{d2}}\,\underline{\overline{\otimes}}\,\mathbf{I_{d2}})$; the right-hand side of this condition corresponds to the stiffness tensor of an isotropic material (where $(\mathbf{A}\,\underline{\overline{\otimes}}\,\mathbf{B})_{ijkl} = \frac{1}{2}(A_{ik}B_{jl} + A_{il}B_{jk})$ is the symmetric tensor product);

(iii) for any subset $\{A\}_k$ of $\mathbf{N_i}$ directions belonging to the envelope of a cone of axis $\mathbf{e_1}$, $\sum_{\mathbf{N_i}\in\{A\}_k} \rho_0 \mathbf{N_i} \otimes \mathbf{N_i}$ is proportional to $\mathbf{C_{e1\text{-}trans}}$, where $\mathbf{C_{e1\text{-}trans}}$ is the stiffness tensor characteristic of transverse isotropy of axis $\mathbf{e_1}$.

The set (2) does not fulfil any of these three fourth order conditions. Another set is consequently built. In [10] a particular basis of $\{\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}, \mathbf{n}\in\Re^3\}$ is defined, respecting isotropic conditions at both second and fourth order, in which each direction $\mathbf{n_i}$ is parallel to the edge of a regular dodecahedron:

$$
\begin{aligned}
&\mathbf{n_1} = \mathbf{e_1}, && \mathbf{n_2} = \mathbf{e_2}, && \mathbf{n_3} = \mathbf{e_3}\\
&\mathbf{n_4} = \frac{t}{2}\mathbf{e_1} + \frac{1}{2}\mathbf{e_2} + \frac{1}{2t}\mathbf{e_3}, && \mathbf{n_5} = \frac{1}{2t}\mathbf{e_1} - \frac{t}{2}\mathbf{e_2} + \frac{1}{2}\mathbf{e_3}, && \mathbf{n_6} = \frac{1}{2}\mathbf{e_1} - \frac{1}{2t}\mathbf{e_2} - \frac{t}{2}\mathbf{e_3}\\
&\mathbf{n_7} = \frac{t}{2}\mathbf{e_1} - \frac{1}{2}\mathbf{e_2} + \frac{1}{2t}\mathbf{e_3}, && \mathbf{n_9} = \frac{1}{2t}\mathbf{e_1} + \frac{t}{2}\mathbf{e_2} + \frac{1}{2}\mathbf{e_3}, && \mathbf{n_8} = \frac{1}{2}\mathbf{e_1} + \frac{1}{2t}\mathbf{e_2} - \frac{t}{2}\mathbf{e_3}\\
&\mathbf{n_{10}} = \frac{t}{2}\mathbf{e_1} - \frac{1}{2}\mathbf{e_2} - \frac{1}{2t}\mathbf{e_3}, && \mathbf{n_{11}} = \frac{1}{2t}\mathbf{e_1} + \frac{t}{2}\mathbf{e_2} - \frac{1}{2}\mathbf{e_3}, && \mathbf{n_{12}} = \frac{1}{2}\mathbf{e_1} + \frac{1}{2t}\mathbf{e_2} - \frac{t}{2}\mathbf{e_3}\\
&\mathbf{n_{13}} = \frac{t}{2}\mathbf{e_1} + \frac{1}{2}\mathbf{e_2} - \frac{1}{2t}\mathbf{e_3}, && \mathbf{n_{15}} = \frac{1}{2t}\mathbf{e_1} - \frac{t}{2}\mathbf{e_2} - \frac{1}{2}\mathbf{e_3}, && \mathbf{n_{14}} = \frac{1}{2}\mathbf{e_1} - \frac{1}{2t}\mathbf{e_2} + \frac{t}{2}\mathbf{e_3}
\end{aligned}
\tag{3}
$$

where $t = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

The corresponding set of 15 $\mathbf{N_i}$ respects the isotropic conditions at both second and fourth order, with $a = 5, a' = 1$ and $b' = 2$. This set can be split in 5 different geometric groups $\{A\}_k$:

(i) a plane $P = (\mathbf{e_2}, \mathbf{e_3})$ containing directions $\mathbf{n_2} = \mathbf{e_2}$ and $\mathbf{n_3} = \mathbf{e_3}$; thus $\{A\}_1 = \{\mathbf{N_2}, \mathbf{N_3}\}$;
(ii) the envelope of a first cone C_1 of height $\frac{t}{2}$ and of axis $\mathbf{e_1}$, containing directions $\mathbf{n_4}, \mathbf{n_7}, \mathbf{n_{10}}$ and $\mathbf{n_{13}}$; thus $\{A\}_2 = \{\mathbf{N_4}, \mathbf{N_7}, \mathbf{N_{10}}, \mathbf{N_{13}}\}$;
(iii) the envelope of a second cone C_2 of height $\frac{1}{2}$ and of axis $\mathbf{e_1}$, containing directions $\mathbf{n_6}, \mathbf{n_8}, \mathbf{n_{12}}$ and $\mathbf{n_{14}}$; thus $\{A\}_3 = \{\mathbf{N_6}, \mathbf{N_8}, \mathbf{N_{12}}, \mathbf{N_{14}}\}$;
(iv) the envelope of a third cone C_3 of height $\frac{1}{2t}$ and of axis $\mathbf{e_1}$, containing directions $\mathbf{n_5}, \mathbf{n_9}, \mathbf{n_{11}}$ and $\mathbf{n_{15}}$; thus $\{A\}_4 = \{\mathbf{N_5}, \mathbf{N_9}, \mathbf{N_{11}}, \mathbf{N_{15}}\}$;
(v) direction $\mathbf{n_1} = \mathbf{e_1}$; thus $\{A\}_5 = \{\mathbf{N_1}\}$.

It can be verified that this set does not respect transverse isotropy conditions (iii), neither at second nor at fourth orders. It is consequently modified to fulfil these conditions, while keeping fulfilling conditions (i) and (ii). As all conditions are additive, one possibility is to add three sets of fifteen directions to the first one; each added set is obtained from the first one by a rotation around axis $\mathbf{e_1}$ of respective angles 45°, 90° and −45°. A set containing sixty directions is thus built; the different cone envelopes $\{A\}_k$ remain, with a multiplication of the direction they contain: $\{A\}_1$ now contains eight $\mathbf{n_i}$, $\{A\}_2$, $\{A\}_3$ and $\{A\}_4$ sixteen and $\{A\}_5$ four. The obtained set fulfils all conditions at both second and fourth orders.

Two sets have been defined above, fulfilling partially or completely the conditions posed at second and fourth orders in orientation. The next step is to compare the results obtained using one set or the other.

4. Comparative results

In this section, a material weakened by a transversely isotropic damage configuration of axis $\mathbf{e_1}$ is considered. The elastic properties degradations and restorations simulated with the discrete approach (defined in Section 2), using the two sets defined in Section 3, are compared with those obtained with a model using a single second order damage variable and a fourth-order closure term built from its eigenentities (e.g. [5]). The damage configuration considered is represented in the following way: for the discrete approach with the nine direction set, the same microcrack density is associated to each direction $\mathbf{N_4}$, $\mathbf{N_5}$, $\mathbf{N_7}$ and $\mathbf{N_8}$; for the discrete approach with the sixty direction set, the same microcrack density is associated to each direction in $\{A\}_3$; for the Halm–Dragon model, variable $\mathbf{D}$ is $\mathbf{D} = d(\mathbf{e_2}\otimes\mathbf{e_2} + \mathbf{e_3}\otimes\mathbf{e_3})$.

The corresponding elastic moduli are derived from the stiffness tensor corresponding to each model ([11]). In order to have a reliable comparison, the values of material parameters and internal variables are assumed to lead to the same

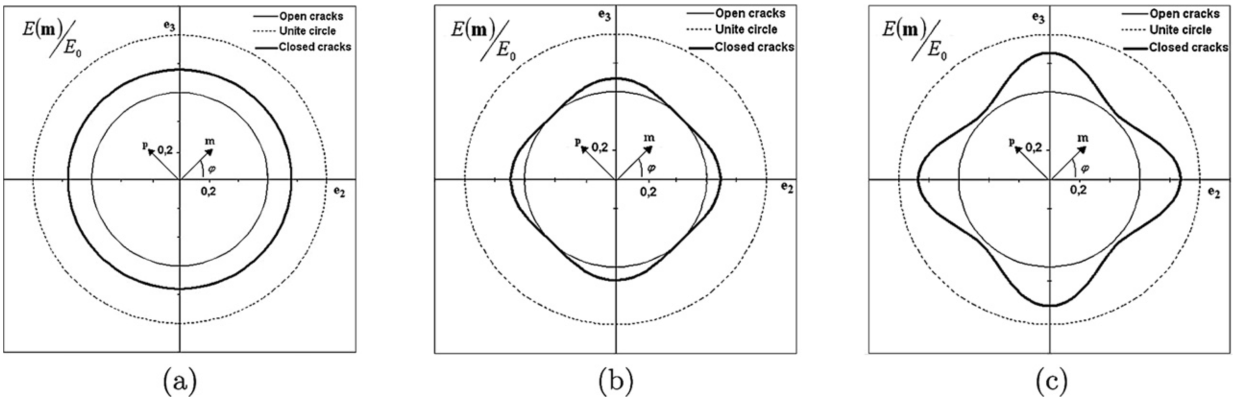

Fig. 1. Normalized Young's modulus for a transversely isotropic damage configuration: (a) discrete approach with 60 directions; (b) discrete approach with 9 directions; (c) Halm–Dragon model.

degradation of the Young's modulus when defects are open; in the three cases, $\frac{E^{\text{open}}}{E_0} = 60\%$, where E_0 is the Young's modulus of the sound material and E^{open} the Young's modulus of the weakened material when all defects are open. Moreover, the elastic initial moduli are identical in the three cases. Fig. 1 presents the simulated Young's modulus $E(\mathbf{m})$ related to a direction of unit vector $\mathbf{m} \in (\mathbf{e}_2, \mathbf{e}_3)$, for the three models. Poisson's ratio has also been investigated.

When microcracks are open, the Young's modulus and Poisson's ratio degradations simulated by the three models are transversely isotropic. The second order conditions ensure a coherent description when microcracks are open since only second order terms appear in the corresponding thermodynamic potentials (and consequently in the stiffness matrix). Note that as the fifteen direction system (3) presented in Section 3 does not fulfill the second order transverse isotropy condition, its use would lead to a non-transversely isotropic response, even when defects are open. When they are closed, only the set of sixty systems leads to a transversely isotropic response, as the closure terms are of fourth order and as only this system fulfils the fourth order transverse isotropy condition. The anisotropy obtained with the two other models is compared introducing an 'anisotropy ratio', equal to the relative difference between the maximum and the minimum of the modulus considered (for example for the Young's modulus: $\frac{E^{\text{closed}}_{\max} - E^{\text{closed}}_{\min}}{E^{\text{closed}}_{\max}}$); the smaller it is, the better transverse isotropy is kept. The Young's modulus and the Poisson's ratio are equal to 0.09 and 0.45, respectively, using the discrete approach with the set of nine systems, and to 0.27 and 0.35, respectively, with the Halm–Dragon model; if the nine system set keeps a correct restitution of the Young's modulus, the two models fail concerning Poisson's ratio restoration. Conclusions would be the same for an isotropic damage configuration. The fourth order conditions are thus of great importance to represent a configuration of closed microcracks.

5. Conclusion

Three conditions have been postulated to built the set of directions chosen in the discrete approach of anisotropic damage evolving unilateral effect. In particular, isotropic or transversely isotropic damage cases have been considered. These conditions are expressed at both second and fourth orders in orientation, as these two kinds of terms are necessary to describe correctly the material degradation when establishing a thermodynamic potential; for each level, one set (of respectively nine and sixty directional tensors) respecting the corresponding conditions is proposed. The comparative results show that the second order conditions permit one to represent an open microcracks configuration, whereas fourth order conditions are necessary when microcracks close.

References

[1] R. Bargellini, D. Halm, A. Dragon, Modelling of anisotropic damage by microcracks: Towards a discrete approach, Arch. Mech. 58 (2) (2006) 93–123.
[2] F. Cormery, H. Welemane, A critical review of some damage models with unilateral effect, Mech. Res. Com. 29 (2002) 391–395.
[3] I. Carol, K. Willam, Spurious energy dissipation/generation in stiffness recovery models for elastic degradation and damage, Int. J. Solids Struct. 33 (20–22) (1996) 2939–2957.

[4] N. Challamel, D. Halm, A. Dragon, On the non-conservativeness of a class of anisotropic damage models with unilateral effects, C. R. Mécanique 334 (2006) 414–418.
[5] D. Halm, A. Dragon, A model of anisotropic damage by mesocrack growth; unilateral effect, Int. J. Damage Mech. 5 (1996) 384–402.
[6] Z.P. Bazant, P.C. Prat, Microplane model for progressive fracture of concrete and rock, J. Eng. Mech. 114 (1988) 1672–1702.
[7] J.P. Boehler, Lois de comportement anisotrope des milieux continus, J. Mécanique 17 (1978) 153–190.
[8] A. Curnier, Q. He, P. Zysset, Conewise linear elastic materials, J. Elasticity 37 (1995) 1–38.
[9] M. Kachanov, Elastic solids with many cracks and related problems, in: J. Hutchinson, T. Wu (Eds.), Adv. Appl. Mech., vol. 30, Academic Press, New York, 1994, pp. 259–445.
[10] D. Elata, M.B. Rubin, Isotropy of strain energy functions which depend only on a finite number of directional strain measures, J. Appl. Mech. ASME 61 (1994) 284–289.
[11] M. Hayes, Connexions between the moduli for anisotropic elastic materials, J. Elasticity 2 (2) (1972) 135–141.