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Missing boundary data recovering for the Helmholtz problem

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Abstract

This Note is dedicated to the numerical treatment of the ill-posed Cauchy–Helmholtz problem. Resorting to the domain decomposition tools, these missing boundary data are rephrased through an 'interfacial' equation. This equation is solved via a preconditioned Richardson algorithm with dynamic relaxation. The efficiency of the proposed method is illustrated by some numerical experiments. ***To cite this article: R. Ben Fatma et al., C. R. Mecanique 335 (2007).***

Résumé

Reconstruction des données pour le problème de Helmholtz. Cette Note concerne le traitement numérique du problème de Cauchy–Helmholtz. On « emprunte » les outils de type décomposition de domaines pour exprimer le problème de complétion de données en terme d'équation « d'interface ». Cette équation est résolue via un algorithme de Richardson préconditionné avec relaxation dynamique. L'efficacité de la méthode est illustrée par quelques expériences numériques. ***Pour citer cet article : R. Ben Fatma et al., C. R. Mecanique 335 (2007).***

Keywords: Data completion; Inverse problem; Cauchy–Helmholtz problem; Interface operator; Steklov–Poincaré operator

Mots-clés : Complétion de données ; Problème inverse ; Problème de Cauchy–Helmholtz ; Operateur d'interface ; Opérateur de Steklov–Poincaré

Version française abrégée

On s'intéresse dans cette Note au problème de complétion de données pour l'équation de Helmholtz. Ce problème fait partie de la famille des problèmes inverses connus pour être mal posés au sens de Hadamard [1]. Le modèle qui nous concerne est celui de Cauchy : il consiste à calculer un champ de pression solution du système (1) dans lequel f et φ sont des données connues sur Γ_c. Tout comme pour le Laplacien (voir [2,3]), on démontre que le problème à résoudre (1) est équivalent à trouver l'unique trace λ définie sur Γ_i, partie de la frontière de Ω à compléter, vérifiant (3). Cette

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dernière relation conduit à la récriture du problème de Cauchy sous forme d'un problème faisant intervenir l'opérateur de Steklov–Poincaré (4).

La discrétisation du problème (4) est, dans notre cas, réalisée par une méthode d'éléments finis P_1 standard. Le système algébrique issu de la discrétisation est résolu par une méthode de Richardson préconditionnée et avec relaxation dynamique. Le paramètre de relaxation est défini à chaque itération grâce à la relation (10).

La validation de la convergence de cet algorithme est testée sur l'exemple analytique $u = \mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{\mathrm{i}y}$. Elle est démontrée par la Fig. 1. On observe que le logarithme décimal de l'erreur décroît en fonction du nombre d'itérations. Ici le coefficient q est fixé égal à 10.

La Fig. 2 illustre la répartition de la pression calculée ainsi que l'erreur de convergence sur le domaine de calcul. On remarque que le maximum de l'erreur de complétion se trouve sur Γ_i. Enfin et dans le but de tester la robustesse de notre algorithme nous avons bruité les données. Le Tableau 1 illustre la vitesse de convergence de l'algorithme ainsi que l'erreur de complétion pour différents niveaux de bruit blanc.

En conclusion, nous avons montré que l'algorithme de Richardson préconditionné permet de réaliser la complétion de données, à condition de choisir une relaxation à pas variable. L'écriture du problème de Cauchy sous forme d'un problème de frontière faisant intervenir l'opérateur de Steklov–Poincaré offre la possibilité d'élargir le choix des algorithmes de résolution à des techniques du type GMRES (voir [2]).

1. Introduction

This contribution is concerned with the recovering of both Dirichlet and Neumann data on some part of the domain boundary, starting from the knowledge of these data on another part of the boundary for the Helmholtz equation. This data completion question may be relevant by itself in some practical applications or may be a preliminary step to others.

For example, the methodology for the crack detection consists in considering a domain containing a crack located in an already known contour Γ. The idea is to break up the domain into two parts. Then two completion data problems are considered, for the two sub-domains. These two Cauchy problems are solved, and the jump $[u] = (u_1 - u_2)$ on Γ is computed. The crack is localized as is the support of $[u]$.

Among well-known examples, in the elliptic framework, let us mention the non-destructive thermal, electrical or mechanical inspection which allows the location of flaws from knowledge of heat flux–temperature, electrical current–potential or normal stress–displacement on the surface of manufactured product see [4–6] and references therein. However, to our knowledge there are very few results for missing boundary data recovering in the harmonic electromagnetic context.

The continuous problem of the data completion problem for the Helmholtz operator is formulated as follows. Let Ω be a bounded domain in $\mathbf{R}^2$ or $\mathbf{R}^3$. The boundary $\Gamma = \partial\Omega$, assumed regular, is split into Γ_c and Γ_i having both non-vanishing measure, whose outer normal direction is denoted by $\mathbf{n}$. Given a flux φ and the data f on the overdetermined boundary Γ_c and when no source is involved, recovering the data on the remainder (incomplete) part Γ_i of the boundary is accomplished by solving the Cauchy system that may be put under the following mathematical setting: *find u such that*

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega \\ \partial_n u = \varphi & \text{on } \Gamma_c \\ u = f & \text{on } \Gamma_c \end{cases} \tag{1}$$

In this Note, a reconstruction of the lacking data is proposed which uses the Domain Decomposition tools (DD). The analogy with DD can be understood as if the two subdomains are, in our case, twin ones (i.e. the same domain Ω) sharing the 'interface' Γ_i (i.e. the boundary where no data are known).

This Note is outlined as follows: in the next section the Cauchy–Helmholtz problem is rephrased in terms of an interfacial problem using the Steklov–Poincaré operator. Section 3 is devoted to the numerical resolution of the interfacial equation. Section 4 illustrates the numerical procedures. We end this with some comments.

2. A Steklov–Poincaré formulation for the Cauchy–Helmholtz problem

The methodology which allows to write the Cauchy–Helmholtz equation in terms of an interfacial equation has been already presented in the case of the Laplacian operator in [2,3].

We have a double condition on Γ_c, let λ be an auxiliary field defined on Γ_i, we introduce two well-defined boundary value problems having a Dirichlet data on Γ_i equal to λ. For the remaining part of the boundary Γ_c, we can combine different choices for boundary conditions (Dirichlet, Neumann or Robin). We select the Robin type boundary condition on Γ_c.

We consider therefore the two following well-posed Helmholtz problems: *Find $v(\lambda)$ and $w(\lambda)$ solutions of*

$$\begin{cases} \Delta v(\lambda) + k^2 v(\lambda) = 0 & \text{in } \Omega \\ \partial_n v(\lambda) - \mathrm{i}q v(\lambda) = \varphi - \mathrm{i}qf & \text{on } \Gamma_c \\ v(\lambda) = \lambda & \text{on } \Gamma_i \end{cases} \qquad \begin{cases} \Delta w(\lambda) + k^2 w(\lambda) = 0 & \text{in } \Omega \\ \partial_n w(\lambda) + \mathrm{i}q w(\lambda) = \varphi + \mathrm{i}qf & \text{on } \Gamma_c \\ w(\lambda) = \lambda & \text{on } \Gamma_i \end{cases} \tag{2}$$

Here q is a real constant. Its choice will be discussed later on.

Solving the Cauchy system (1) is achieved when the data extension λ makes v and w coincide, and the solution is then $u = v = w$. This leads to write an equation on Γ_i to be satisfied by λ:

$$\frac{\partial v(\lambda)}{\partial n} = \frac{\partial w(\lambda)}{\partial n} \tag{3}$$

One poses $v = v^0(\lambda) + v^*$ and $w = w^0(\lambda) + w^*$, where we have defined $(v^0(\lambda), w^0(\lambda))$ and (v^*, w^*) to be solutions of:

$$\begin{cases} \Delta v^0(\lambda) + k^2 v^0(\lambda) = 0 & \text{in } \Omega \\ \partial_n v^0(\lambda) - \mathrm{i}q v^0(\lambda) = 0 & \text{on } \Gamma_c \\ v^0(\lambda) = \lambda & \text{on } \Gamma_i \end{cases} \qquad \begin{cases} \Delta w^0(\lambda) + k^2 w^0(\lambda) = 0 & \text{in } \Omega \\ \partial_n w^0(\lambda) + \mathrm{i}q w^0(\lambda) = 0 & \text{on } \Gamma_c \\ w^0(\lambda) = \lambda & \text{on } \Gamma_i \end{cases}$$

and

$$\begin{cases} \Delta v^* + k^2 v^* = 0 & \text{in } \Omega \\ \partial_n v^* - \mathrm{i}q v^* = \varphi - \mathrm{i}qf & \text{on } \Gamma_c \\ v^* = 0 & \text{on } \Gamma_i \end{cases} \qquad \begin{cases} \Delta w^* + k^2 w^* = 0 & \text{in } \Omega \\ \partial_n w^* + \mathrm{i}q w^* = \varphi + \mathrm{i}qf & \text{on } \Gamma_c \\ w^* = 0 & \text{on } \Gamma_i \end{cases}$$

v^0 and w^0 are the Helmholtz extensions of λ from Γ_i into Ω, noted respectively $H_-(\lambda)$ and $H_+(\lambda)$, whereas v^* and w^* are two Helmholtz extensions of $(\varphi - \mathrm{i}qf)$ and $(\varphi + \mathrm{i}qf)$ from Γ_i into Ω, noted respectively $R_-(\varphi - \mathrm{i}qf)$ and $R_+(\varphi + \mathrm{i}qf)$.

The latter condition (3) amounts to the requirement that λ satisfies the Steklov–Poincaré type equation

$$S\lambda = \chi \quad \text{on } \Gamma_i \tag{4}$$

where

$$\chi := -\partial_n R_-(\varphi - \mathrm{i}qf) + \partial_n R_+(\varphi + \mathrm{i}qf) \tag{5}$$

and S is the Helmholtz–Cauchy–Steklov–Poincaré operator formally defined by

$$S(\lambda) := S_-(\lambda) - S_+(\lambda) = \partial_n H_-(\lambda) - \partial_n H_+(\lambda) \tag{6}$$

This operator, borrowed to the domain decomposition community, is widely used (see [7] and the references therein).

3. Data completion process and numerical results

Up to now we did not specify the discretization technique used to solve (4). Any stable numerical technique could be applied. One can adapt the description presented in [2] from the Laplacian operator to the Helmholtz one. Here, the Finite Element code MELINA ([8]) is used.

To conduct the numerical discussion, Ω is considered as a thick annular domain with radii $r_1 = 1$ and $r_2 = 1.5$. The internal boundary on which the data is lacking is denoted Γ_i and the external boundary on which the data is overspecified is denoted Γ_c. For this example, the data is provided by the harmonic function

$$u = \mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{\mathrm{i}y}$$

$f = u$ and $\varphi = \partial_n u$ are given on the boundary Γ_c.

The meshes we use are triangular, the finite elements are linear. The calculations are run on a uniform mesh with 100 nodes on Γ_i, 150 nodes on Γ_c and 1328 nodes on Ω. For the numerical reconstruction process we tried successively the Richardson algorithm, a preconditioned one, and finally a preconditioned Richardson algorithm with dynamic relaxation.

The Richardson iterations are described by the following iteration equation:

$$\lambda^{n+1} = \lambda^n + \omega(S\lambda^n - \chi) \tag{7}$$

where ω is a coefficient that we choose equal to 1, whereas the preconditioned Richardson algorithm is described by:

$$\lambda^{n+1} = \lambda^n + S_+^{-1}(S\lambda^n - \chi) = S_+^{-1}(S_-\lambda^n - \chi) \tag{8}$$

These two algorithms failed to converge. Let us recall, that this method, introduced by Koslov in [9], for the Laplacian operator, is relevant to the data completion problem. However, it was already noticed in [10] that the alternating algorithm of Koslov is no more relevant for the Helmholtz equation. To overcome this convergence failure, we propose in the following part to use the dynamic relaxation method as proposed by [11] for the Laplacian operator.

The introduction of the dynamic relaxation in the preconditioned Richardson algorithm consists in writing the iterative algorithm in the following way:

$$\lambda^{n+1} = r^n S_+^{-1}(S_-\lambda^n - \chi) + (1 - r^n)\lambda^n \tag{9}$$

where, in the iteration n, the relaxation parameter r^n is updated as the unique real minimizing the functional

$$\phi(r) = \left\|\lambda^n(r) - \lambda^{n-1}(r)\right\|$$

given by

$$r^n = \frac{\langle \lambda^n - \lambda^{n-1}, \lambda^n - \lambda^{n-1} - S_+^{-1}S_-(\lambda^{n-1} - \lambda^{n-2})\rangle}{\|\lambda^n - \lambda^{n-1} - S_+^{-1}S_-(\lambda^{n-1} - \lambda^{n-2})\|_{L^2(\Gamma_i)}} \tag{10}$$

Here $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(\Gamma_i)$ and $\|\cdot\|$ is the associated norm.

Fig. 1 illustrates the decimal logarithm of the gap between the exact and the reconstructed solutions according to the iterations number. The coefficient q is fixed to 10. It is noted that the algorithm converges and preserves the same behavior for very large orders of iterations.

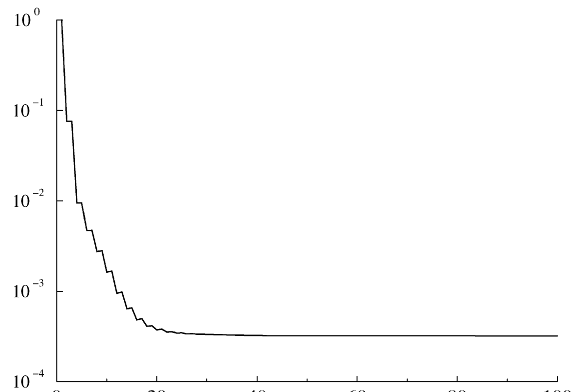

Fig. 1. Convergence of preconditioned Richardson algorithm with dynamic relaxation.

Fig. 1. Courbe de convergence de l'algorithme de Richardson préconditionné avec relaxation dynamique.

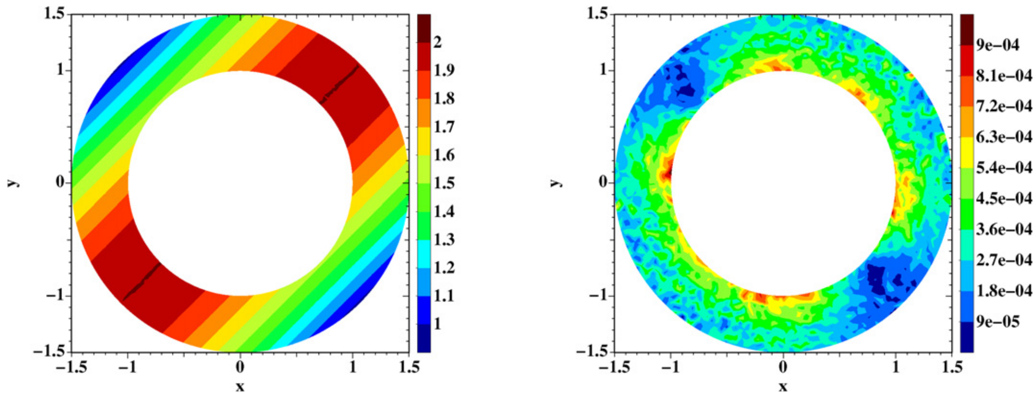

Fig. 2. Reconstruction on the internal boundary of a tube: the reconstructed pressure (left panel) and the gap between the exact and the reconstructed solutions for noise free data (right panel).

Fig. 2. Reconstruction de la solution à la frontière interne du tube : la reconstruction de la pression (à gauche) et la différence entre la pression exacte et la pression calculée (à droite).

Table 1
The convergence speed and the accuracy of the preconditioned Richardson algorithm with relaxation for polluted Dirichlet data with different noise levels

Tableau 1
La vitesse et l'erreur de la convergence de l'algorithme (Richardson préconditionné) pour différents niveaux de bruit de type Dirichlet sur la frontière Γ_c

Noise level	n	$\frac{\|u_{ex}-u_{cal}\|_{L^2(\Gamma_i)}}{\|u_{ex}\|_{L^2(\Gamma_i)}}$	$\frac{\|u_{ex}-u_{cal}\|_{L^\infty(\Gamma_i)}}{\|u_{ex}\|_{L^\infty(\Gamma_i)}}$
2.5%	11	2.7×10^{-3}	4.9×10^{-3}
5%	9	4.92×10^{-3}	7.75×10^{-3}
10%	7	7.51×10^{-3}	1.07×10^{-2}

We start by reconstructing a pressure field in a annular shaped thick domain with radii $r_1 = 1$ and $r_2 = 1.5$. The left plot of Fig. 2 depicts the computed pressure recovered in the domain and obtained after 35 iterations, that of the right illustrates the gap between the exact solution u and the computed solution. Observe that the maximum value of the error is reached around the internal wall.

To emphasize the reliability of this algorithm and to attest the stabilizing effect, we performed a reconstruction of the pressure from some noisy data. Table 1 illustrates the convergence speed and the accuracy of this algorithm for polluted Dirichlet data with different white noise levels. In the legend, u_{ex} denotes the exact pressure, u_{cal} is the computed one and n is the iterations number in convergence.

In Fig. 3 are reported the curves of the recovered pressure and the flux with respect to the angular abscissa $\theta/2\pi$ which measures the angle between the x-axis and the line joining the center of the domain-for noise free and noisy Dirichlet data. In the legend, the abbreviation N.L. stands for noise level.

Remark. The Robin coefficient q has a significant impact on the convergence rate of the preconditioned Richardson algorithm with relaxation. There exists an optimal coefficient q (for our test case the optimal value of q is 10) which produces a remarkably fast and convergent solution in comparison with that obtained for a different value of q.

4. Conclusions

This Note has dealt with a method to solve the Cauchy problem for the Helmholtz equation. A new adaptation of the preconditioned Richardson algorithm with dynamic relaxation is used for this data reconstruction problem. It does not seen that the reconstruction problem for Helmholtz equation has been treated in the literature except for the case

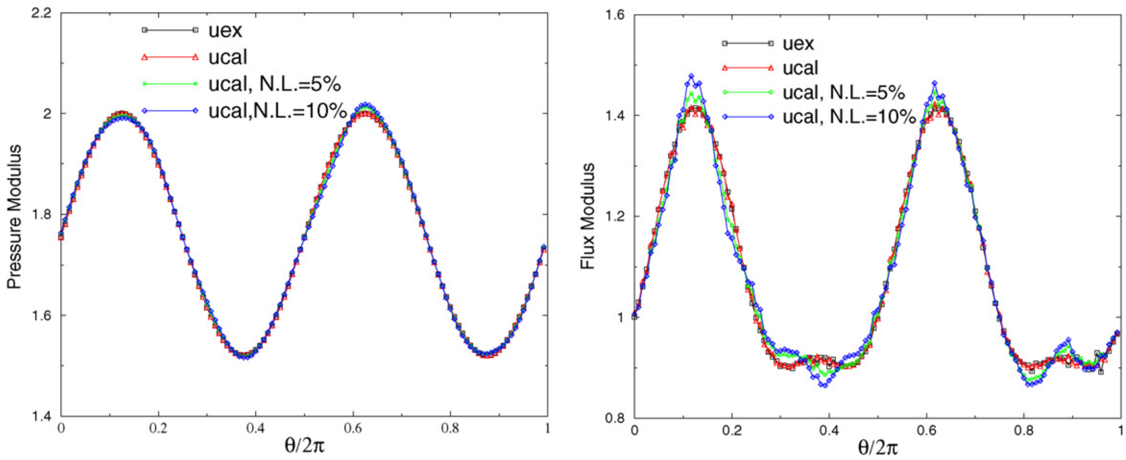

Fig. 3. Reconstruction on the internal boundary of a tube: the pressure (left panel) and the flux (right panel).

Fig. 3. Reconstruction de la solution à la frontière interne du tube : la pression (à gauche) et le flux (à droite).

where the data are lacking on a flat boundary [10]. Presently further numerical experiments are continuing, as well as some practical applications such as the interfacial crack identification problem.

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