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Asymptotic modeling of assemblies of thin linearly elastic plates

Christian Licht

Laboratoire de mécanique et génie civil, UMR 5508 CNRS–UMII, université Montpellier II, c.c. 48, place Eugène-Bataillon, 34095 Montpellier cedex 05, France

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Presented by Évariste Sanchez-Palencia

Abstract

We derive various models of assemblies of thin linearly elastic plates by abutting or superposition through an asymptotic analysis taking into account small parameters associated with the size and the stiffness of the adhesive. They correspond to the linkage of two Kirchhoff–Love plates by a mechanical constraint which strongly depends on the magnitudes of the previous parameters. ***To cite this article: C. Licht, C. R. Mecanique 335 (2007).***

Résumé

Modélisation asymptotique d'assemblages de plaques minces linéairement élastiques. On obtient divers modèles d' assemblages de plaques minces par aboutage ou superposition à partir d'une analyse asymptotique prenant en considération de petits paramètres associés à la taille et à la rigidité de l'adhésif. Ils correspondent au couplage de deux plaques de Kirchoff–Love par une liaison mécanique dont la nature dépend fortement des ordres de grandeur des paramètres précédents. ***Pour citer cet article : C. Licht, C. R. Mecanique 335 (2007).***

Keywords: Solids and structures; Assemblies of plates; Composite plates; Asymptotic analysis

Mots-clés : Solides et structures ; Assemblages de plaques ; Plaques composites ; Analyse asymptotique

1. Setting the problem

We consider the structure constituted by two linearly elastic thin plates linked by a soft elastic adhesive, the assembly being done by abutting (case $p=1$) or superposition (case $p=2$). We make no difference between $\mathbb{R}^3$ and the Euclidean physical space whose orthonormal basis is denoted by $\{e_1, e_2, e_3\}$, Greek coordinate indexes will run in $\{1,2\}$ and Latin ones in $\{1,2,3\}$; for all $\xi=(\xi_1,\xi_2,\xi_3)$ of $\mathbb{R}^3$, $\hat{\xi}$ stands for (ξ_1,ξ_2). Let η and ε two small positive real numbers, ω a domain of $\mathbb{R}^2$ with a Lipschitz-continuous boundary $\partial\omega$, and $\Omega^\varepsilon := \omega\times(-\varepsilon,\varepsilon)$. We assume that the intersection l of ω with $\{x_2=0\}$ is of positive length. The reference configurations of the adhesive layer and of the two plates are respectively $B_p^{\eta,\varepsilon}=\{x\in\Omega^\varepsilon;\ |x_{p+1}|<\eta\varepsilon^{p-1}\}$ and $\Omega_p^{\eta,\varepsilon}=\Omega^\varepsilon\setminus\overline{B_p^{\eta,\varepsilon}}$. We denote the strain energy

E-mail address: licht@lmgc.univ-montp2.fr.

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density of the adhesive layer and of the two plates by $W^{\lambda,\mu}$ and W^{ε} respectively. For almost all x in $\Omega_p^{\eta,\varepsilon}$, $W^{\varepsilon}(x,.)$ is a positive-definite quadratic form of the linearized strain tensor e and $W^{\lambda,\mu}$ corresponds to the homogeneous Hook law: $W^{\lambda,\mu}(e)=\lambda/2(\operatorname{tr} e)^2+\mu|e|^2$, $\lambda,\mu>0$. The structure, clamped on $\Gamma_0^{\varepsilon}:=\gamma_0\times(-\varepsilon,\varepsilon)$, is subjected to body forces f^{ε} and surface forces g^{ε} on $\Gamma_1^{\varepsilon}=\partial\Omega^{\varepsilon}\setminus\Gamma_0^{\varepsilon}$; we assume that $(f^{\varepsilon},g^{\varepsilon})\in L^2(\Omega^{\varepsilon})^3\times L^2(\Gamma_1^{\varepsilon})^3$ and that the length of the part γ_0 of $\partial\omega$ is positive. The plates being perfectly stuck to the adhesive layer, the problem of finding an equilibrium configuration involves a quadruplet $s=(\varepsilon,\eta,\lambda,\mu)$ and reads as:

$$(\mathcal{P}_p^s)\quad \operatorname{Min}\left\{F_p^s(v)-L^{\varepsilon}(v);\ v\in H^1(\Omega^{\varepsilon})^3,\ v=0 \text{ on } \Gamma_0^{\varepsilon}\right\}$$

with

$$F_p^s(v):=\int_{\Omega_p^{\eta,\varepsilon}} W^{\varepsilon}\big(x,e(v)(x)\big)\,\mathrm{d}x+\int_{B_p^{\eta,\varepsilon}} W^{\lambda,\mu}\big(x,e(v)(x)\big)\,\mathrm{d}x,\qquad L^{\varepsilon}(v):=\int_{\Omega^{\varepsilon}} f^{\varepsilon}\cdot v\,\mathrm{d}x+\int_{\Gamma_1^{\varepsilon}} g^{\varepsilon}\cdot v\,\mathrm{d}\sigma$$

Clearly, $(\mathcal{P}_p^s)$ has a unique solution u_p^s, but determining numerical approximations of u_p^s may be tricky because of the large number of degrees of freedom implied by the meshing of the very thin adhesive layer and the ill-conditioned system due to the low stiffness of the glue. Thus, it is of interest to propose a simpler but accurate enough modeling of this structure. For that purpose, we will consider s as a quadruplet of small *parameters* and derive our models (see Section 3) through a rigorous mathematical study of the asymptotic behavior of u_p^s when s goes to zero.

2. A convergence result

Classically [1], we come down to a fixed open set $\Omega:=\omega\times(-1,1)$ through the mapping $x=(x_1,x_2,x_3)\in\overline{\Omega}\mapsto x^{\varepsilon}=\pi^{\varepsilon}(x)=(x_1^{\varepsilon},x_2^{\varepsilon},\varepsilon x_3^{\varepsilon})\in\overline{\Omega^{\varepsilon}}$. Let Γ_0,Γ_1 the images by $(\pi^{\varepsilon})^{-1}$ of Γ_0^{ε}, Γ_1^{ε}, $\Gamma_{\pm}=\omega\times\{\pm1\}$, $\Gamma_{\mathrm{lat}}=\partial\omega\times(-1,1)$, $B_p^{\eta,1}=\{x\in\Omega;\ |x_{p+1}|<\eta\}$, $\Omega_p^{\eta,1}=\Omega\setminus\overline{B_p^{\eta,1}}$, $S_p=\{x\in\Omega;\ x_{p+1}=0\}$, $\Omega_p=\Omega\setminus S_p$, $\Omega_p^{\pm}=\{x\in\Omega;\ \pm x_{p+1}>0\}$. In the sequel, for any open set G of $\mathbb{R}^n$, $H_g^1(G)$ denotes the subset of $H^1(G)$ whose elements vanish on $g\subset\partial G$.

The magnitude of the external loading is chosen as follows:

$$\begin{cases} f_{\alpha}^{\varepsilon}(\pi^{\varepsilon}x)=\varepsilon f_{\alpha}(x), & f_3^{\varepsilon}(\pi^{\varepsilon}x)=\varepsilon^2 f_3(x), \quad \forall x\in\Omega\\ g_{\alpha}^{\varepsilon}(\pi^{\varepsilon}x)=\varepsilon^2 g_{\alpha}(x), & g_3^{\varepsilon}(\pi^{\varepsilon}x)=\varepsilon^3 g_3(x), \quad \forall x\in\Gamma_1\cap\Gamma_{\pm}\\ g_{\alpha}^{\varepsilon}(\pi^{\varepsilon}x)=\varepsilon g_{\alpha}(x), & g_3^{\varepsilon}(\pi^{\varepsilon}x)=\varepsilon^2 g_3(x), \quad \forall x\in\Gamma_1\cap\Gamma_{\mathrm{lat}} \end{cases}\tag{1}$$

where (f,g) is an element (independent of ε) of $L^2(\Omega)^3\times L^2(\Gamma_1)^3$ and there exists $\eta_0>0$ such that $\operatorname{support}(f,g)\cap\overline{B_p^{\eta_0,1}}=\emptyset$. Let S^n the space of symmetric $n\times n$ matrices, we denote the space of linear operators on S^n by $\mathcal{L}(S^n)$ and $\xi\otimes_s\zeta$ stands for the symmetrized tensor product of $\xi\in\mathbb{R}^n$ by $\zeta\in\mathbb{R}^n$. We assume that the bulk energy density W^{ε} satisfies:

$$\begin{cases} \exists a\in L^{\infty}\big(\Omega,\mathcal{L}(S^n)\big); \quad W^{\varepsilon}(\pi^{\varepsilon}x,e)=W(x,e) \quad \forall e\in S^3, \text{ a.e. } x\in\Omega\\ \exists\alpha>0; \quad W(x,e)\geqslant\alpha|e|^2 \quad \forall e\in S^3,\ \text{a.e. } x\in\Omega \end{cases}\tag{2}$$

For all $e\in S^3$, $\hat{e}$ is the element of S^2 such that $\hat{e}_{\alpha\beta}=e_{\alpha\beta}$, so that a strictly convex quadratic form W_{KL} is well-defined on S^2 by:

$$W_{\mathrm{KL}}(x,q)=\operatorname{Min}\left\{W(x,e);\ e\in S^3,\hat{e}=q\right\}\quad \forall e\in S^3,\ \text{a.e. } x\in\Omega\tag{3}$$

With any displacement v defined on Ω^{ε} is associated a *scaled* displacement $S(\varepsilon)v$ defined on Ω by:

$$\widehat{S(\varepsilon)v}(x)=\varepsilon^{-1}\hat{v}(\pi^{\varepsilon}x),\quad \big(S(\varepsilon)v\big)_3(x)=v_3(\pi^{\varepsilon}x),\quad \text{a.e. } x\in\Omega\tag{4}$$

then $u(s)_p:=S(\varepsilon)u_p^s$ is the unique solution of the problem:

$$(\mathcal{P}_p(s))\quad \operatorname{Min}\left\{F_p(s)(v)-L(v);\ v\in H_{\Gamma_0}^1(\Omega)^3\right\}$$

with

$$F_p(s)(v) := \int_{\Omega_p^{\eta,1}} W\big(x, d_\varepsilon e(v)(x) d_\varepsilon\big)\,\mathrm{d}x + \int_{B_p^{\eta,1}} W^{\lambda,\mu}\big(d_\varepsilon e(v)(x) d_\varepsilon\big)\,\mathrm{d}x, \qquad L(v) := \int_\Omega f\cdot v\,\mathrm{d}x + \int_{\Gamma_1} g\cdot v\,\mathrm{d}\sigma$$

and d_ε the diagonal element of S^3 such that $(d_\varepsilon)_{ii} = \varepsilon^{\min(2-i,0)}$.

Henceforth, we assume that $(\lambda_p, \mu_p, \mu_{p'}) := (\lambda/4\eta\,\varepsilon^{4(p-1)}, \mu/2\eta\,\varepsilon^{4(p-1)}, \mu/4\eta\,\varepsilon^2)$ has a limit $(\overline{\lambda_p}, \overline{\mu_p}, \overline{\mu_{p'}})$ in $[0,+\infty]^3$ with, moreover, $\varepsilon^{4(2p-3)}\eta^3/\mu$ goes to zero if $\overline{\lambda_p} = +\infty$. Let X_p defined by:

(i) $\limsup \eta^2/\mu = 0$: $X_p = L^2(\Omega)^3$,
(ii) $\limsup \eta^2/\mu < +\infty$: $X_1 = L^q(\Omega)^3$, $X_2 = L^q(\Omega)^2 \times L^2(\Omega)$, q arbitrary in $[1,2)$,
(iii) $\limsup \eta^2/\mu = +\infty$: $X_1 = \bigcup_h L^2(\Omega^{h,1})^3$:
  – $\limsup \eta^2\varepsilon^4/\mu = 0$: $X_2 = \bigcup_h L^2(\Omega^{h,1})^2 \times L^2(\Omega)$,
  – $\limsup \eta^2\varepsilon^4/\mu < +\infty$: $X_2 = \bigcup_h L^2(\Omega^{h,1})^2 \times L^q(\Omega)$, q arbitrary in $[1,2)$,
  – $\limsup \eta^2\varepsilon^4/\mu = +\infty$: $X_2 = \bigcup_h L^2(\Omega^{h,1})^3$.

Its topology τ is the strong topology unless one previous lim sup is not finite where τ then involves the strong topology on $L^2(\Omega^{h,1})$ for every positive h. The asymptotic behavior of the scaled displacement depends strongly on the relative behavior of the parameters but can be described in an unified way as follows. Let

$$V_{\mathrm{KL}}(\Omega_p) = \big\{v \in H^1_{\Gamma_0}(\Omega_p)^3;\ e_{i3}(v) = 0 \text{ in the sense of distributions on } \Omega_p\big\}$$

whose an equivalent characterization is:

$$p = 1: \quad \exists!(v^M, v^F) \in H^1_{\gamma_0}(\omega\setminus l)^2 \times H^2_{\gamma_0}(\omega\setminus l); \quad v_\alpha(x) = v_\alpha^M(\hat{x}) - x_3\,\partial_\alpha v^F(\hat{x}),\ v_3(x) = v^F(\hat{x})\ \forall x \in \Omega_1$$

$$p = 2: \quad \exists!(v^{M\pm}, v^{F\pm}) \in H^1_{\gamma_0}(\omega)^2 \times H^2_{\gamma_0}(\omega); \quad v_{\alpha\restriction_{\Omega_2^\pm}}(x) = v_\alpha^{M\pm}(\hat{x}) - x_3\partial_\alpha v^{F\pm}(\hat{x}),$$
$$v_{3\restriction_{\Omega_2^\pm}}(x) = v^{F\pm}(\hat{x})\ \forall x \in \Omega_2.$$

We denote the jump across S_p in the direction of e_{p+1} of the membrane and flexural parts v^M, v^F of $v \in V_{\mathrm{KL}}(\Omega_p)$ by $[v^M], [v^F]$: when $p = 1$, $[v^M], [v^F]$ are the differences of the traces on l of v^M, v^F, whereas $[v^M](\hat{x}) = v^{M+}(\hat{x}) - v^{M-}(\hat{x})$, $[v^F](\hat{x}) = v^{F+}(\hat{x}) - v^{F-}(\hat{x})\ \forall \hat{x} \in \omega$ when $p = 2$. The functional

$$v \in V_{\mathrm{KL}}(\Omega_p) \mapsto G_p(v) := \int_{\Omega_p} W_{\mathrm{KL}}\big(x, \widehat{e(v)}(x)\big)\,\mathrm{d}x$$

will supply the variational limit of the total strain energy of the adherents while the variational limit of the total strain energy of the adhesive will be:

$$v \in V_{\mathrm{KL}}(\Omega_p) \mapsto H(v; \overline{\lambda_p}, \overline{\mu_p}, \overline{\mu_{p'}}) := \int_{S_p} h\big([v]x; \overline{\lambda_p}, \overline{\mu_p}, \overline{\mu_{p'}}\big)\,\mathrm{d}\sigma$$

h being the mapping $h : \mathbb{R}^3 \mapsto [0,\infty]$ well-defined by $h(w; \overline{\lambda_p}, \overline{\mu_p}, \overline{\mu_{p'}}) := \lim_{s\to 0} 1/2\eta W^{\lambda,\mu}(d_\varepsilon w \otimes_s e_{p+1} d_\varepsilon)$ and $[v]$ the jump of v across S_p in the direction of e_{p+1}. Of course, if some coefficients $\overline{\lambda_p}, \overline{\mu_p}, \overline{\mu_{p'}}$ equal $+\infty$, H involves the indicator function I_{V_q} of a suitable subspace V_q of $V_{\mathrm{KL}}(\Omega_p)$. Eight cases indexed by q can be distinguished and we will use the notation $H_q(v)$ in place of $H(v; \overline{\lambda_p}, \overline{\mu_p}, \overline{\mu_{p'}})$:

$$q = 1: \quad p = 1, \quad \overline{\mu_{1'}} < \infty, \quad \overline{\lambda_1} < \infty, \quad \overline{\mu_1} = 0, \quad H_1(v) = 2\int_l \overline{\mu_{1'}}[v^F]^2 + \overline{\lambda_1}\big([v_2^M]^2 + [\partial_2 v^F]^2/3\big)\,\mathrm{d}x_1$$

$$V_1 = V_{\mathrm{KL}}(\Omega_1)$$

$$q = 2: \quad p = 1, \quad \overline{\mu_{1'}} < \infty, \quad \overline{\lambda_1} = \infty, \quad \overline{\mu_1} = 0, \quad H_2(v) = 2\int_l \overline{\mu_{1'}}[v^F]^2\,\mathrm{d}x_1 + I_{V_2}$$

$$V_2 = \big\{v \in V_1; [v_2^M] = [\partial_2 v^F] = 0\big\}$$

$$q=3:\quad p=1,\quad \overline{\mu_{1'}}=\infty,\quad \overline{\lambda_1}<\infty,\quad \overline{\mu_1}<\infty$$

$$H_3(v)=2\int_l (\overline{\mu_1}+\overline{\lambda_1})\big([v_2^M]^2+[\partial_2 v^F]^2/3\big)+\overline{\mu_1}[v_1^M]^2\,\mathrm{d}x_1+I_{V_3},\quad V_3=\{v\in V_1;\ [v^F]=0\}$$

$$q=4:\quad p=1,\quad \overline{\mu_{1'}}=\infty,\quad \overline{\lambda_1}=\infty,\quad \overline{\mu_1}<\infty,\quad H_4(v)=2\int_l \overline{\mu_1}[v_1^M]^2\,\mathrm{d}x_1+I_{V_4}$$

$$V_4=\{v\in V_2;\ [v^F]=0\}$$

$$q=5:\quad p=1,\quad \overline{\mu_{1'}}=\infty,\quad \overline{\mu_1}=\infty,\quad H_5(v)=I_{V_5},\quad V_5=\{v\in V_1;\ [v^F]=[\partial_2 v^F]=[v^M]=0\}$$

$$q=6:\quad p=2,\quad \overline{\lambda_2},\overline{\mu_2}<\infty,\quad H_6(v)=(\overline{\lambda_2}+\overline{\mu_2})\int_\omega [v^F]^2\,\mathrm{d}\hat{x},\quad V_6=V_{\mathrm{KL}}(\Omega_2)$$

$$q=7:\quad p=2,\quad \{\overline{\lambda_2},\overline{\mu_2}\}\ni\infty,\quad \overline{\mu_{2'}}<\infty,\quad H_7(v)=\overline{\mu_{2'}}\int_\omega \big|[v^M]\big|^2\,\mathrm{d}\hat{x}+I_{V_7};\quad V_7=\{v\in V_6;\ [v^F]=0\}$$

$$q=8:\quad p=2,\quad \{\overline{\lambda_2},\overline{\mu_2}\}\ni\infty,\quad \overline{\mu_{2'}}=\infty,\quad H_8(v)=I_{V_8},\quad V_8=\{v\in V_6;\ [v^M]=[v^F]=0\}$$

We extend $\overline{F}_q:=G_p+H_q$ by $+\infty$ on $X_p\setminus V_{\mathrm{KL}}(\Omega_p)$, then the limit behavior of the scaled structure is given by:

Theorem 2.1. *When s goes to 0, the unique solution $u(s)_p$ of the problem $(\mathcal{P}_p(s))$ converges in X_p to the unique solution $\bar{u}_q$ of the problem $(\overline{\mathcal{P}}_q(s))$: $\mathrm{Min}\{\overline{F}_q(v)-L(v);\ v\in V_q\}$ and $\overline{F}_q(\bar{u}_q)-L(\bar{u}_q)=\lim_{s\to 0}(F_p(s)(u(s)_p)-L(u(s)_p)$. Moreover, the restriction to $\Omega_p^{\eta,1}$ of $e(u(s)_p)$ converges strongly to $e(\bar{u}_q)$ in $L^2(\Omega_p;S^3)$.*

Proof. Due to the very structure of the problem, and thus of the functional $F_p(s)$, the proof juxtaposes classical arguments of the mathematical analysis of adhesively bonded joints [2,3] and an extension to the heterogeneous and anisotropic cases of the mathematical derivation of the Kirchhoff–Love plate theory [4]. To simplify, we confine to the case $p=2$, the arguments being similar if $p=1$.

First step. Let v_s a sequence such that $F_2(s)(v_s)$ is bounded. The standard estimates

$$|v|^2_{L^2(B_2^{\eta,1})^3}\leqslant C\eta\big(|e(v)|^2_{L^2(\Omega_2^{\eta,1};S^3)}+\eta|e(v)|^2_{L^2(B_2^{\eta,1};S^3)}\big)\quad \forall v\in H^1_{\Gamma_0}(\Omega)^3$$

$$|w|^2_{L^2(B_2^{\eta,1})}\leqslant C\eta\big(|\partial_3 w|^2_{L^2(\Omega_2^{\eta,1})}+\eta|\partial_3 w|^2_{L^2(B_2^{\eta,1})}\big)\quad \forall w\in H^1_{\Gamma_0}(\Omega)$$

and a combination of the arguments of [2,3] imply that there exist v in $H^1_{\Gamma_0}(\Omega_2)^3$ and a not relabelled subsequence such that $v_s\xrightarrow{\tau}v$, the restriction of $e(v_s)$ to $\Omega_2^{\eta,1}$ weakly converges to $e(v)$ in $L^2(\Omega_2;S^3)$ and $v_{s\restriction x_3=\eta}-v_{s\restriction x_3=-\eta}\to[v]$ in $L^2(\omega)^3$. Moreover, (2) implies $v\in V_{\mathrm{KL}}$.

Second step. We show that for all $u\in V_q$, there exists $u_s\in H^1_{\Gamma_0}(\Omega)^3$ such that

$$u_s\xrightarrow{\tau}u,\quad \limsup_{s\to 0}F_2(s)(u_s)\leqslant\overline{F}_q(u)\tag{5}$$

From the very definition of W_{KL} there exists q_s in $C_0^\infty(\Omega_2^{\eta,1})^3$ such that $\int_\Omega W(x,e(u)+q_s\otimes_s e_3)\,\mathrm{d}x\leqslant G_2(u)+|s|$ and we define u_s^1 in $H^1_{\Gamma_0}(\Omega)^3$ by: $(u_s^1)_\alpha(x)=\varepsilon\int_0^{x_3}\{2(q_s)_\alpha(\hat{x},y)-\varepsilon\int_0^y\partial_\alpha(q_s)_3(\hat{x},z)\,\mathrm{d}z\}\,\mathrm{d}y$, $(u_s^1)_3(x)=\varepsilon^2\int_0^{x_3}(q_s)_3(\hat{x},y)\,\mathrm{d}y$. Let u_s^2 the field whose components are those of u except $(u_s^2)_\alpha$ equal to $(R_\eta u)_\alpha$ if $q=7$ and $(u_s^2)_3$ equal to $(R_\eta u)_3$ if $q=6$, R_η being the smooth operator:

$$v\in H^1_{\Gamma_0}(\Omega_2)^3\mapsto R_\eta v=1/2\big(v(\hat{x},x_3)+v(\hat{x},-x_3)+\min\big(1,|x_3|/\varepsilon\big)\big(v(\hat{x},x_3)-v(\hat{x},-x_3)\big)\big)\in H^1_{\Gamma_0}(\Omega)^3$$

clearly, $u_s:=u_s^1+u_s^2$ satisfies (5).

Third step. We establish that for all sequence v_s in $H^1_{\Gamma_0}(\Omega)^3$ which τ-converges toward u in X_2

$$\overline{F}_q(u)\leqslant\liminf_{s\to 0}F_2(s)(v_s)\tag{6}$$

The first step and the weak sequential $L^2(\Omega; S^3)$ lower semicontinuity of $q \mapsto \int_\Omega W_{\mathrm{KL}}(x, q(x))\,\mathrm{d}x$ imply

$$\liminf_{s\to 0} \int_{B_2^{\eta,1}} W^{\lambda,\mu}\big(d_\varepsilon e(v_s)(x)d_\varepsilon\big)\,\mathrm{d}x + G_2(u) \leqslant \liminf_{s\to 0} F_2(s)(v_s)$$

When $q = 7, 8$, we obtain $u \in V_q$ by noticing that, for all (φ, ψ) in $C_0^\infty(\omega) \times C_0^\infty(\omega)^2$, the first step and an integration by parts in $B_2^{\eta,1}$ yield:

$$\left|\int_\omega \varphi\,[u^F]\,\mathrm{d}\hat{x}\right| \leqslant C \lim_{s\to 0}(\eta^{1/2}\varepsilon^2/\mu^{1/2})\,|\varphi|_{L^\infty(\omega)}$$

$$\left|\int_\omega \varphi\,[u^F]\,\mathrm{d}\hat{x}\right| \leqslant C\Big\{\lim_{s\to 0}(\eta^{1/2}\varepsilon^2/\lambda^{1/2})\,|\varphi|_{L^\infty(\omega)} + \lim_{s\to 0}(\eta^{3/2}\varepsilon^2/\mu^{1/2})\,|\nabla\varphi|^2_{L^\infty(\omega)}\Big\}$$

$$\left|\int_\omega \psi\cdot\widehat{[u]}\,\mathrm{d}\hat{x}\right| \leqslant C\Big\{\lim_{s\to 0}(\eta^{1/2}\varepsilon^2/\mu^{1/2})|\psi|_{L^\infty(\omega)^2} + \lim_{s\to 0}(\eta^{3/2}\varepsilon^2/\mu^{1/2})|\nabla\psi|_{L^\infty(\omega;S^2)}\Big\}$$

Eventually, when $q = 6, 7$, the sub-differential inequality and the second step imply:

$$\liminf_{s\to 0} F_2(s)(v_s) \geqslant \overline{F}_q(u) + \liminf_{s\to 0} \int_{B_2^{\eta,1}} DW^{\lambda,\mu}\big(d_\varepsilon e(u_s^2)(x)d_\varepsilon\big)\cdot\big(d_\varepsilon e(v_s - u_s^2)(x)d_\varepsilon\big)\,\mathrm{d}x$$

which establishes (6), because the last term vanishes (see [2]).

Last step. Classically [5], the first two assertions of the theorem are a consequence of the first step and (5), (6), while the last one stems from the convergences obtained in the first step and from

$$\limsup_{s\to 0} \int_{\Omega_2^{\eta,1}} W_{\mathrm{KL}}\big(x, e\widehat{\big(u(s)_2\big)}(x)\big)\,\mathrm{d}x \leqslant \limsup_{s\to 0} \int_{\Omega_2^{\eta,1}} W\big(x, d_\varepsilon e\big(u(s)_2\big)(x)d_\varepsilon\big)\,\mathrm{d}x$$

$$= \overline{F}_q(\bar{u}_q) - L(\bar{u}_q) - \liminf_{s\to 0} \int_{B_2^{\eta,1}} W^{\lambda,\mu}\big(d_\varepsilon e\big(u(s)_2\big)(x)d_\varepsilon\big)\,\mathrm{d}x \leqslant G_2(\bar{u}_q)$$

as established in the third step. □

Remark 1. The previous arguments about the variational convergence of $\int_{\Omega_p^{\eta,1}} W(x, e(v)(x))\,\mathrm{d}x$ toward $\int_{\Omega_p} W_{\mathrm{KL}}(x, \widehat{e(v)}(x))\,\mathrm{d}x$ allows one to derive the Kirchhoff–Love theory of thin linearly elastic plates by Γ-convergence, which generalizes [4] to the case of anisotropic and heterogeneous materials (see [8,9] also).

3. A proposal of a simplified but accurate model

Let $\Omega_p^\varepsilon = (\pi^\varepsilon)^{-1}(\Omega_p)$, $\Omega_p^{\varepsilon\pm} = (\pi^\varepsilon)^{-1}(\Omega_p^\pm)$, the de-scaled field of displacement $\bar{u}_q^s := S(\varepsilon)^{-1}\bar{u}_q$ solves the minimization problem

$$(\overline{\mathcal{P}}_q^s)\quad \mathrm{Min}\Big\{\int_{\Omega_p^\varepsilon} W_{\mathrm{KL}}\big(x, \widehat{e(v)(x)}\big)\,\mathrm{d}x + H_q^\varepsilon(v) - L^\varepsilon(v);\ v \in V_q^\varepsilon\Big\}$$

the expression of $H_q^\varepsilon(v)$ being deduced from the one of $H_q(v)$ by replacing v^M by $\varepsilon^{1/2}v^M$ and v^F by $\varepsilon^{3/2}v^F$ while the spaces V_q^ε are the analogues of V_q with Ω_p replaced by Ω_p^ε. This problem models the linkage of two Kirchhoff–Love plates, occupying $\Omega_p^{\varepsilon\pm}$, by a mechanical constraint along $S_p^\varepsilon := (\pi^\varepsilon)^{-1}(S_p)$. This constraint which takes place of the thin adhesive layer depends strongly on the relative behaviors of $\varepsilon, \eta, \lambda, \mu$. When $p = 2$, it can be pure adhesion ($q = 8$), isoflexion with membrane pull-back or membrane separation without resistance ($q = 7$), separation with or without flexural pull-back ($q=6$). From a material stand-point, it can be said that when the strength of the glue increases transversal adhesion and longitudinal adhesion occur successively. Conversely, from the debonding point of view, when the strength of the glue decreases the plates first separate longitudinally and next transversally. When $p=1$, the panel of constraints is wider. With increasing values of $\overline{\mu_1\prime}$, three main stages occur successively: free flexural separation, flexural elastic pull-back, isoflexion. In the first stage and with increasing values of $\overline{\lambda_1}$, free separation, e_2-membrane and rotation around l pull-backs, longitudinal adhesion appear successively. The same occurs in the second stage except that free separation is replaced by free longitudinal separation. In the last stage, increasing values of $\overline{\lambda_1}$ supply free longitudinal separation, e_2-membrane and rotation around l pull-backs, adhesion in the e_1 and e_3 directions (but not in the e_2 direction) while increasing values of $\overline{\mu_1}$ yield free longitudinal separation, membrane and rotation around l pull-backs, full adhesion.

Actually, the problem $(\overline{\mathcal{P}}^s_q)$ is a two-dimensional problem, it reduces to a minimization problem in (v^M, v^F) on $H^1_{\gamma_0}(\omega \setminus l)^2 \times H^2_{\gamma_0}(\omega \setminus l)$ when $p=1$ or in $(v^{M\pm}, v^{F\pm})$ on $H^1_{\gamma_0}(\omega)^2 \times H^2_{\gamma_0}(\omega)$ when $p=2$. When $p=1$ and $\int_{-1}^1 x_3 W_{\mathrm{KL}}(.,x_3)\,\mathrm{d}x_3 = 0$ (which is implied by W is an even function of x_3), the problem breaks down into two independent problems, one satisfied by the membrane displacement, the other by the flexural one. When $p=2$, generally a coupling occurs except, of course, if $q=8$ and $\int_{-1}^1 x_3 W_{\mathrm{KL}}(.,x_3)\,\mathrm{d}x_3 = 0$. Clearly, this model is simpler and easier to implement numerically than the genuine three-dimensional one. It is also accurate because of the previous convergence result: $\bar{u}^s_q$ is asymptotically equivalent to u^s_p in the sense that $\lim_{s\to 0} \varepsilon^{-2} \int_{\Omega_p^{\eta,\varepsilon}} |\hat{e}(\bar{u}^s_q - u^s_p)|^2\,\mathrm{d}x = 0$ and $\varepsilon^{-3}\int_{\Omega_p^{\eta,\varepsilon}} |e_{i3}(u^s_p)|^2$ is bounded.

In practice, s does not tend to 0 and we believe that a rational proposal for a simpler and efficient model of the genuine structure is to replace $\overline{\lambda_p}, \overline{\mu_p}, \overline{\mu_p\prime}$ by their actual values $\lambda/4\eta\,\varepsilon^{4(p-1)}$, $\mu/2\eta\,\varepsilon^{4(p-1)}$, $\mu/4\eta\,\varepsilon^2$ in the formulae giving H^ε_q in order to obtain the problem:

$$(\overline{\mathcal{P}}^s)\quad \mathrm{Min}\left\{ \int_{\Omega^\varepsilon_p} W_{\mathrm{KL}}\big(x, \widehat{e(v)(x)}\big)\,\mathrm{d}x + 1/2\eta\,\varepsilon^{p-1} \int_{S^\varepsilon_p} W^{\lambda,\mu}\big([v]\otimes_s e_{p+1}\big)\,\mathrm{d}x \;-L^\varepsilon(v);\; v \in V_{\mathrm{KL}}(\Omega^\varepsilon_p)\right\}$$

In this model, the strain energies of the adherents are replaced by Kirchhoff–Love plates strain energies while the strain energy of the adhesive is replaced by a classical surface constraint energy (see [2,3]).

Remark 2. In these questions of modeling of adhesively bonded joints, the use of the framework of small strains, which does not account for the impenetrability of the adherents, is questionable because it does not supply any unilateral condition in the constraint along S^ε_p. A first remedy is to include the condition $v_{p+1}(x+\eta\,\varepsilon^{p-1}e_{p+1}) - v_{p+1}(x - \eta\varepsilon^{p-1}e_{p+1}) \geqslant 0$ in the genuine problem, it is easy to show that the conclusions of Theorem 1 still hold but with the vectorial spaces V_q replaced by the convex cones $C_q := \{v \in V_q;\ [v]_{p+1} \geqslant 0 \text{ on } S_p\}$ and the same for the models proposed in this section.

Eventually, the present study may be considered as a framework to asses the models of soft abutting of thin plates proposed in [6,7].

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