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A criterion of the continuous spectrum for elasticity and other self-adjoint systems on sharp peak-shaped domains ☆

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Abstract

The spectra of the elasticity and piezo-electricity systems for a solid with a sharp peak point on the boundary, which is free of traction, are not discrete. An algebraic criterion of non-empty continuous spectrum is found for the Neumann problem for rather arbitrary formally self-adjoint elliptic systems of second-order differential equations on a sharp peak-shaped domain. ***To cite this article: S.A. Nazarov, C. R. Mecanique 335 (2007).***

Résumé

Un critère de spectre continu pour l'élasticité et d'autres systèmes auto-adjoints pour des domaines contenant des pointes. Les spectres de l'élasticité et de systèmes piezo-electriques pour un solide avec une pointe sur la frontière, sans traction, ne sont pas discrets. Un critère algébrique de spectre continu non-vide est établi pour le problème de Neumann pour des systèmes elliptiques formellements auto-adjoints arbitraires d'équations differentielles du deuxième ordre dans un domaine de forme pointue. ***Pour citer cet article : S.A. Nazarov, C. R. Mecanique 335 (2007).***

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1. Peak-shaped elastic bodies

Let $\Omega \in \mathbb{R}^3$ be a domain bounded by the compact surface $\partial\Omega$ which is smooth everywhere except at the origin $\mathcal{O}$ of the Cartesian coordinate system $x = (y, z) = (y_1, y_2, z)$. In a neighborhood $\mathcal{U}$ of the point $\mathcal{O}$ the domain Ω (see Fig. 1) is given by the relations

$$z > 0, \quad z^{-1-\gamma} y \in \omega \tag{1}$$

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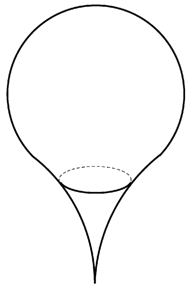

Fig. 1. The peak-shaped domain.

where $\omega \subset \mathbb{R}^2$ is a domain bounded by the simple closed smooth contour $\partial\omega$ and $\gamma > 0$ the peak sharpness exponent. We consider the elasticity problem in matrix notation (see, e.g., [1])

$$L(x, \nabla_x)u(x) := D(-\nabla_x)^\top A(x)D(\nabla_x)u(x) = \lambda\rho(x)u(x), \quad x \in \Omega \tag{2}$$

$$N(x, \nabla_x)u(x) := D\big(\nu(x)\big)^\top A(x)D(\nabla_x)u(x) = 0, \quad x \in \partial\Omega \setminus \mathcal{O} \tag{3}$$

Here $u = (u_1, u_2, u_3)^\top$ is the displacement column, $\top$ stands for transposition, $D(\nabla_x)u$ and $AD(\nabla_x)u$ imply the strain and stress columns of height 6, A and ρ are the Hooke's matrix of elastic moduli and the material density, assumed to be smooth in $\overline{\Omega} = \Omega \cup \partial\Omega$ (for simplifying the presentation), while ν denotes the outward normal and λ the spectral parameter,

$$D(\nabla_x) = \begin{pmatrix} \partial_1 & 0 & 2^{-1/2}\partial_2 & 2^{-1/2}\partial_3 & 0 & 0 \\ 0 & \partial_2 & 2^{-1/2}\partial_1 & 0 & 2^{-1/2}\partial_3 & 0 \\ 0 & 0 & 0 & 2^{-1/2}\partial_1 & 2^{-1/2}\partial_2 & \partial_3 \end{pmatrix}^\top, \quad \partial_j = \frac{\partial}{\partial x_j}, \nabla_x = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \tag{4}$$

Furthermore, $A(x)$ is a symmetric and positive definite matrix-function of size 6×6 and $\rho(x) > 0$ for $x \in \overline{\Omega}$. Hence, in the case of Lipschitz boundary $\partial\Omega$ the Korn inequality

$$\big\|u; H^1(\Omega)\big\| \leqslant c\|u; \mathcal{H}\| := \big(\big(AD(\nabla_x)u, D(\nabla_x)u\big)_\Omega + (\rho u, u)_\Omega\big)^{1/2} \tag{5}$$

is valid where $\mathcal{H}$ denotes the Sobolev space $H^1(\Omega)$ with the specific norm on the right of (5) and $(,)_\Omega$ the inner product in the Lebesgue space $L^2(\Omega)$. Furthermore, the embedding $\mathcal{H} \subset L^2(\Omega)$ is compact and, therefore, the spectrum of problem (2), (3) is discrete and forms the sequence

$$0 = \lambda_1 = \cdots = \lambda_d < \lambda_{d+1} \leqslant \cdots \leqslant \lambda_m \cdots \to +\infty \tag{6}$$

where the eigenvalues are listed according to their multiplicities. In particular, $d = 6$ and the eigenspace of the eigenvalue $\lambda = 0$ consists of rigid motions $a + b \times x$ where $a, b \in \mathbb{R}^3$ and $\times$ stands for the vector product.

In view of (1) the surface $\partial\Omega$ is not Lipschitz and the function space $\mathcal{H}$ is not included into $H^1(\Omega)$ and, moreover, in the case $\gamma \geqslant 1$ the inclusion $\mathcal{H} \subset L^2(\Omega)$ looses the compactness (see [1, § 3.1, p. 123]). To confirm these observations, it suffices to consider the displacement fields Ψ^{pm} with the components

$$\Psi_p^{pm}(x) = \psi_m(z), \qquad \Psi_{3-p}^{pm}(x) = 0, \qquad \Psi_3^{pm}(x) = -y_p\partial_z\psi_m(z) \tag{7}$$

where $p = 1, 2$, $\partial_z = \partial/\partial z$, $\psi_m(z) = \psi(mz)$ and $\psi \in C^\infty(\mathbb{R})$, $\psi(z) = 0$ for $z \notin (1, 2)$ but $\omega(z) > 0$ for $z \in (1, 2)$. A direct calculation leads to the formulas

$$\big\|D(\nabla_x)\Psi^{pm}; L^2(\Omega)\big\|^2 = \mathrm{O}\big(m^{-1-4\gamma}\big), \qquad \big\|\nabla_x\Psi^{pm}; L^2(\Omega)\big\|^2 = \mathrm{O}\big(m^{-1-2\gamma}\big)$$

$$\big\|\Psi^{pm}; L^2(\Omega)\big\|^2 = \mathrm{O}\big(m^{-3-2\gamma}\big) \quad \text{as } m \to +\infty,\ m \in \mathbb{N} := \{1, 2, \ldots\} \tag{8}$$

which readily provides the facts mentioned above. Moreover, the domain of the problem operator is not compactly embedded into $L_2(\Omega)$ and, according to Theorem 10.1.5 [2], for $\gamma \geqslant 1$, the spectrum of problem (2), (3) is not discrete.

2. The spectrum of the elasticity problem

Clearly, the set $\mathbb{C} \setminus \overline{\mathbb{R}}_+$ belongs to the resolvent field of the operator of problem (2), (3) posed on the intrinsic energy space $\mathcal{H}$. The following assertions describe certain properties of its spectrum:

Theorem 2.1. (i) *If $\gamma \in (0,1)$, problem* (2), (3) *has a discrete spectrum* (6).

(ii) *If $\gamma > 1$, the eigenvalue $\lambda = 0$ of multiplicity six belongs to the essential spectrum.*

(iii) *If $\gamma = 1$, there exist two positive numbers $\Lambda_1 \leqslant \Lambda_2$ such that the half-open interval $[0, \Lambda_1)$ on the closed positive semi-axis $\overline{\mathbb{R}}_+ \subset \mathbb{C}$ contains the only point $\lambda = 0$ of the spectrum which is but a normal eigenvalue of multiplicity* 6*, and the ray $[\Lambda_2, +\infty)$ is filled with the continuous spectrum.*

The first assertion in Theorem 2.1 follows from the weighted Korn inequality [3]:

Proposition 2.2. *The inequality $\|r^{-1}u_3; L^2(\Omega)\| + \|r^{\gamma-1}u; L^2(\Omega)\| \leqslant c\|u; \mathcal{H}\|$ is valid with $r = dist(x, \mathcal{O})$ and the constant c, independent of the vector function $u \in \mathcal{H}$.*

The third assertion in Theorem 2.1 can be proved by constructing a singular sequence of type (8) for $\lambda \geqslant \Lambda_2$. The elements of this sequence imitate asymptotic ansätze for elastic fields in a thin rod (cf. [4,5], [1, Ch. 5, 7]), while the (4×4)-system of ordinary differential equations, modeling oscillations of a thin 'rod' (1) is of the Euler type and has an exact solution on the ray $\mathbb{R}_+$. In case $\gamma > 1$ the author cannot derive an exact form of a solution to the corresponding system of ordinary differential equations and, that is why, any detailed information on a structure of the spectrum is not available yet, whilst formulas (7) and (8) prove the second assertion of Theorem 1 due to the Weyl criterion (see, e.g., [2, Thm. 9.1.2]).

To find out the threshold Λ_1 in Theorem 2.1(iii), the following proposition can be employed, which can be checked up in the same way as Proposition 2.2:

Proposition 2.3. *If the vector $u \in \mathcal{H}$ meets the orthogonality condition*

$$\int_\Omega u(x)\,\mathrm{d}x = \int_\Omega x \times u(x)\,\mathrm{d}x = 0 \in \mathbb{R}^3$$

the inequality $(\rho u, u)_\Omega \leqslant (1+c)\|u; \mathcal{H}\|$ is valid with a positive constant c.

Note that in the case of a homogeneous isotropic material with the Young modulus E the threshold Λ_2 satisfies the relation

$$\Lambda_2 \leqslant \frac{49 \cdot 25}{\rho\,\mathrm{mes}_2\,\omega} E\mathbf{I}(\omega), \qquad \mathbf{I}(\omega) = \min\left\{\int_\omega y_1^2\,\mathrm{d}y, \int_\omega y_2^2\,\mathrm{d}y\right\}$$

For $\gamma > 1$, the eigenvalue $\lambda = 0$ belongs to both, the point spectrum and the continuous one. If $\gamma = 1$ and the homogeneous isotropic body Ω is rotationally symmetric, the continuous spectrum of problem (2), (3) contains indefinitely many different eigenvalues of the point spectrum.

The loss of the discreteness by the spectrum can be used to construct filters and dampers of elastic waves. For example, an asymptotic analysis shows that, by a proper choice of physical properties of the rotational symmetric isotropic inclusion

$$\Omega_0 = \{x = (y,z) = (r\sin\varphi, r\cos\varphi, z)\colon\ r < R_0,\ \varphi \in [0, 2\pi),\ |z| < b(R_0 - r)^{1+\gamma}\}, \quad b > 0,\ R_0 > 0,\ \gamma > 1$$

in the homogeneous isotropic cylindrical waveguide $\Omega_1 = \{x = (y,z)\colon\ r < R_1\}$ of radius $R_1 > R_0$, one can fill in any preassigned segment $[0, l]$ of the continuous spectrum with any preassigned number of eigenvalues corresponding to trapped elastic modes.

3. The general setting of the problem

Let $n \geqslant 2$, $\omega \subset \mathbb{R}^{n-1}$, $y = (x_1, \dots, x_{n-1})$ and $z = x_n$ while the domain $\Omega \subset \mathbb{R}^n$ has the peak (1). Let also k and N be positive integers, $N \geqslant k$; A and ρ are symmetric matrices of sizes $N \times N$ and $k \times k$, respectively, measurable, uniformly positive definite and bounded for almost all $x \in \Omega$. The matrix $D(\nabla_x)$ of size $N \times k$ consists of first-order homogeneous differential operators with constant real coefficients. We assume D to be algebraically complete [6], i.e., for a certain $\rho_D \geqslant 1$, any row $P(\xi) = (P_1(\xi), \dots, P_k(\xi))$ of homogeneous in $\xi = (\xi_1, \dots, \xi_n)$ polynomials of degree $\rho > \rho_D$ gives rise to a polynomial row $Q(\xi) = (Q_1(\xi), \dots, Q_N(\xi))$ such that

$$P(\xi) = Q(\xi)D(\xi), \quad \xi \in \mathbb{R}^n \tag{9}$$

An accurate formulation of the problem (2), (3) reads

$$a(u, v; \Omega) := \big(AD(\nabla_x)u, D(\nabla_x)v\big)_\Omega = \lambda(\rho u, v)_\Omega =: \lambda b(u, v; \Omega), \quad v \in \mathcal{H} \tag{10}$$

where the space $\mathcal{H}$ is the completion of $C_c^\infty(\overline{\Omega} \setminus \mathcal{O})$ with respect to the norm $(a(u, u; \Omega) + b(u, u; \Omega))^{1/2}$ (cf. (5)). The requirement (9) provides the operator L in (2) with the formal positivity [6] and the polynomial property [7,8]. The latter means that the quadratic form a degenerates only on the finite-dimensional linear space of the vector polynomials

$$\mathcal{P} = \big\{p = (p_1, \dots, p_k)^\top \colon\ D(\nabla_x)p(x) = 0\big\} \tag{11}$$

Owing to (9), the degree of a scalar polynomial p_j in (11) does not exceed ρ_D. We set $d = \dim \mathcal{P} < \infty$. For the elasticity system, $\mathcal{P}$ is the space of rigid motions and $d = 6$.

4. A criterion for existence of a continuous spectrum in sharp peak-shaped domains

The variational formulation (10) generates a positive, continuous, and symmetric (therefore, self-adjoint) operator $\mathcal{T}$ given by the formula

$$\langle \mathcal{T}u, v\rangle = b(u, v; \Omega), \quad u, v \in \mathcal{H}$$

where $\langle\, ,\rangle$ denotes the inner product in $\mathcal{H}$. The change $\lambda \mapsto \mu = (\lambda + 1)^{-1}$ of the spectral parameter reduces (10) to the abstract equation in the Hilbert space $\mathcal{H}$

$$\mathcal{T}u = \mu u$$

An evident argument ensures that the set $\mathbb{C} \setminus \{\mu \in \mathbb{C}\colon \operatorname{Re}\mu \in [0, 1], \operatorname{Im}\mu = 0\}$ belongs to the resolvent field of the operator $\mathcal{T}$. Moreover, in the theory of elliptic problems with piecewise smooth boundaries it has been shown (cf. [8]) that in a peak-shaped domain (1), the kernel of the elliptic problem operator in a function space with exponential weighted norm is of a finite dimension. Thus, the existence of a singular Weyl sequence at a point $\lambda \in [0, +\infty)$ of the essential spectrum implies that the resolvent of the operator $\mathcal{T}$ cannot be a closed operator and λ falls into the continuous spectrum of $\mathcal{T}$.

Theorem 4.1. (i) *The spectrum on the interval* $\{\mu \in \mathbb{C}\colon \operatorname{Re}\mu \in (0, 1], \operatorname{Im}\mu = 0\}$ *of the operator* $\mathcal{T}$ *is discrete for any* $\gamma > 0$ *if and only if any vector polynomial* p *in the linear space* (11) *does not depend on the variable* $z = x_n$ (*in other words,* $\partial_z\mathcal{P} = \{0\}$).

(ii) *If the linear space* $\mathcal{P}$ *includes the polynomial*

$$p(y, z) = z^J p^0(y) + \cdots + zp^{J-1}(y) + p^J(y) \tag{12}$$

with $J \geqslant 1$ *and* $p^0 \neq 0$, *then the embedding* $\mathcal{H} \subset L_2(\Omega)^k$ *is not compact for* $\gamma \geqslant J^{-1}$ *and the continuous spectrum of* $\mathcal{T}$ *contains a point different from* $\mu = 0$. *In the case* $\gamma > J^{-1}$ *the eigenvalue* $\mu = 1$ *with eigenspace* (12) *belongs to the continuous spectrum of the operator* $\mathcal{T}$.

Since the linear space (11) is invariant with respect to the transformations $x \mapsto x + \tau$ and $x \mapsto \kappa x$ of the Cartesian coordinate system (here $\tau \in \mathbb{R}^n$ and $\kappa \in \mathbb{R}_+$), the polynomial (12) gives rise to the following homogeneous polynomial of degree J in $\mathcal{P}$

$$\mathbf{p}(y,z) = \sum_{j=0}^{J} \mathbf{p}^j(y) \frac{\partial^j z^J}{\partial z^j}, \quad \deg \mathbf{p}^j = j, \mathbf{p}^0 \in \mathbb{R}^k \setminus \{0\}$$

A singular sequence of type (8) consists of the vector functions $\Psi^m(y,z) = m^{n(1+\gamma)-\gamma}\mathbf{p}(y, \partial_z)\psi^m(z)$ and satisfies the relations

$$\|\Psi^m; L_2(\Omega)\|^2 \geqslant C > 0, \qquad \|D(\nabla_x)\Psi^m; L_2(\Omega)\|^2 \leqslant cm^{2(J+1)}m^{-2(1+\gamma)J}$$

which readily confirm the second assertion of Theorem 2. The first assertion follows from the weighted generalized Korn's inequality.

Proposition 4.2. *If the linear space $\mathcal{P}$ does not contain a polynomial depending on the variable z, then any function $v \in \mathcal{H}$ verifies the inequality*

$$\|r^{-1}v; L^2(\Omega)\|^2 \leqslant c\left(\|D(\nabla_x)v; L^2(\Omega)\|^2 + \|v; L_2(\Omega)\|^2\right)$$

We emphasize that Theorem 4.1 delivers only a sufficient condition for the existence of a point of the essential spectrum of the operator $\mathcal{P}$ on the segment $\{\mu \in \mathbb{C}\colon \operatorname{Re}\mu \in (0,1], \operatorname{Im}\mu = 0\}$. The author does not know how to confirm the hypothesis: In the case $\gamma \in (0, J^{-1})$, where J is the maximal degree of polynomials $z \mapsto p(y,z)$ in $\mathcal{P}$, the spectrum on the segment is discrete. However, for a sharp ($\gamma \geqslant 1$) peak the condition becomes also necessary and, thus, Theorem 4.1 provides a criterion of the discrete spectrum.

Example 1. If $\mathcal{P}$ has an anisotropic structure, the existence of the essential spectrum depends on the direction of the peak axis. For instance, replacing matrix (4) by the matrix

$$D(\nabla_x)^\top = \begin{pmatrix} \partial_1 & 0 & 2^{-1/2}\partial_2 & 0 & 0 & 0 & 0 & \partial_z \\ 0 & \partial_2 & 2^{-1/2}\partial_1 & 0 & 0 & 0 & \partial_z & 0 \\ 0 & 0 & 0 & \partial_1 & \partial_2 & \partial_z & 0 & 0 \end{pmatrix}$$

composes the linear space (11) from the linear vector functions of the variables y_1 and y_2 only. Thus, for the sharp peak (1) itself, the spectrum is discrete but, for $\gamma \geqslant 1$ a non-trivial essential spectrum occurs in the case of a positive angle between the axis of the peak and the z-axis (we change the domain Ω but keep the coordinate system, the operator L, and the quadratic form a).

Example 2. Let $n = 2$ and let the matrix $D(\nabla_x)$ be composed from $k+1$ rows of length k:

$$\begin{aligned} &(\partial_y, 0, 0, \ldots, 0, 0), \quad (\partial_z, \partial_y, 0, \ldots, 0, 0), \quad (0, \partial_z, \partial_y, \ldots, 0, 0), \quad \ldots \\ &\ldots, \quad (0,0,0,\ldots,\partial_y,0), \quad (0,0,0,\ldots,\partial_z,\partial_y), \quad (0,0,0,\ldots,0,\partial_z) \end{aligned} \tag{13}$$

A direct calculation demonstrates that the vector polynomial of degree $J = k - 1$

$$P(y,z) = \left(\frac{z^{k-1}}{(k-1)!}, -\frac{z^{k-2}y}{(k-2)!\,1!}, \frac{z^{k-3}y^2}{(k-3)!\,2!}, \ldots, (-1)^{k-1}\frac{y^{k-1}}{(k-1)!}\right)^\top$$

is in the kernel of operator (13). Thus, for any $\gamma > 0$ one can construct a quadratic form and a matrix of second-order differential operators such that the corresponding Neumann problem (2), (3) in a domain with the peak (1) gains non-trivial essential spectrum.

5. Piezo-electric peak-shaped bodies

The piezo-electric three-dimensional body Ω is described with the help of the boundary value problem (2), (3) where $u^{(1)} = (u_1, u_2, u_3)^\top$ is the displacement vector, $u^{(2)} = u_4$ the electric potential and the operators L and N are built out of the following matrices of sizes 4×9 and 9×9:

$$D(\nabla_x)^\top = \begin{pmatrix} D^{(11)}(\nabla_x)^\top & \mathbb{I}_3 \\ (0,0,0,0,0,0) & \nabla_x^\top \end{pmatrix}, \qquad A = \begin{pmatrix} A^{(11)} & A^{(12)} \\ -A^{(21)} & A^{(22)} \end{pmatrix}$$

Here $A^{(11)}$ and $A^{(22)}$ are symmetric and positive definite matrices of sizes 6×6 and 3×3, respectively, $A^{(12)} = (A^{(21)})^\top$, and $D^{(11)}(\nabla_x)^\top$ is the operator $D(\nabla_x)^\top$ in (4). According to the physical interpretation of the problem the matrix A is not symmetric and the matrix $\rho = \operatorname{diag}\{\rho_0, \rho_0, \rho_0, 0\}$ is not positive definite. Nevertheless, the tricks [9, Example 1.13] and [10] that deal with the complex-valued electric potential $\varphi = iu_4$ and detach a scalar problem for φ without a spectral parameter, reduce the piezo-electricity problem to a form which admits an application of the methods and accepts all the result mentioned in Sections 1 and 2 for the elasticity problem. It is interesting that the general properties of the spectrum are indifferent to what boundary conditions, (3) or

$$e_j^\top \mathcal{N}(x, \nabla_x)u(x) = 0, \quad j = 1,2,3, \; u_4(x) = 0, \; x \in \partial\Omega \setminus \mathcal{O} \tag{14}$$

are imposed on the boundary $\partial\Omega \setminus \mathcal{O}$. Note that relations (14) mean that the solid surface is free of traction and earthed electrically while (3) corresponds to a body in the vacuum.

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