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C. R. Mecanique 336 (2008) 329–335

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Oscillations of two superposed fluids in an open and flexible container

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Received 15 August 2007; accepted after revision 30 October 2007

Available online 20 February 2008

Presented by Patrick Huerre

Abstract

The interaction of two superposed inviscid liquids with a flexible side wall of a rectangular container is considered. The governing equations describing the behaviour of the system are analyzed using the concept of normal modes, and their solutions presented in the form of infinite series. The expansion coefficients for the velocity potentials are calculated by employing a new inner product which allows orthogonalizing the fluid shape modes. An eigenfrequency equation is then derived from the requirement for a nontrivial solution exists. The influence of the governing parameters on the coupled frequencies is illustrated in the case of water–mercury system. ***To cite this article: M. Amaouche, B. Meziani, C. R. Mecanique 336 (2008).***

Résumé

Vibrations de deux fluides superposés dans un réservoir avec paroi flexible. On établit une équation permettant de calculer les fréquences de couplage d'une paroi flexible d'un réservoir rectangulaire contenant deux fluides parfaits, non miscibles, avec surface libre. La procédure utilisée est basée sur une décomposition en modes normaux et l'établissement d'un produit scalaire approprié orthogonalisant la suite des modes normaux pour les potentiels des vitesses. Une application numérique est donnée dans le cas du système mercure–eau. ***Pour citer cet article : M. Amaouche, B. Meziani, C. R. Mecanique 336 (2008).***

Keywords: Fluid mechanics; Flexible wall; Coupled frequencies

Mots-clés : Mécanique des fluides ; Paroi flexible ; Fréquences de couplage

1. Introduction

The trend towards thinner and lighter structures leads to the use of high flexibility materials in many engineering applications. This property furthers the interaction of the structure vibrations with the sloshing of the free liquid surface. So, large amplitude motions may take place, often leading to detrimental effects by endangering the integrity of the system. Violent motions may also occur when a hydroelastic system is submitted to an external excitation whose

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1631-0721/$ – see front matter
doi:10.1016/j.crme.2007.10.020

frequency is closed to one of its natural frequencies. Accordingly, the knowledge of the intrinsic vibratory characteristics of a coupled system is determinant to the understanding of how disturbances come into being and the subsequent relationship between their development and the excitation that initiates them. In this context where the viscosity is often neglected, the problem was first considered by Miles [1] who investigated how the eigenfrequencies of a cylindrical flexible container are affected by the liquid motion. Rectangular containers were considered by, Bauer [2], Chai et al. [3] and others by using conventional method of modal analysis. The main results of these investigations can be found in [3].

The focus of the present Note is on the extension of these results to hydroelastic systems with two superposed fluids. Once the boundary conditions for the elastic plate are specified, there are, in addition to the parameters involved in a single fluid system, three other crucial factors, namely the interfacial surface tension and the relative thickness and mass density of the two fluids. Various specific physical mechanisms are then expected to contribute to the occurrence of a richer dynamics than that of a one fluid system.

After setting up the governing equations for the coupled fluid–structure motion in Section 2, we begin by constructing the velocity potentials for the fluid system and the deflection of the plate with the use of the normal mode concept in Section 3. After that, an eigenfrequency equation is established as a condition for the existence of nontrivial motion. Section 4 is concerned with a validation of the newly extended method and a discussion of the numerical results. Concluding remarks are drawn in Section 5.

2. Formulation

We consider the free motion of a two layer medium of mutually immiscible, inviscid and incompressible fluids having constant mass densities ρ_1^* and ρ_2^* and thicknesses h_1^* and h_2^* respectively, filling partially a rectangular container of length l^* and height H^* (asterisks are used to indicate dimensional quantities, dimensionless variables will be asterisk free). The container which has an elastic side wall is assumed to have an infinite extent in the spanwise direction so that all subsequent motion is two dimensional in the vertical plane (x, y). The y axis is pointed upward from the unperturbed state of the interface between the two fluids and the x axis oriented inward from the undeflected position of the elastic wall (see Fig. 1). Assuming both liquids to be in small amplitude irrotational motions, the governing equations may be expressed, after linearization, in terms of velocity potentials Φ_j $(j = 1, 2)$ such that:

$$\Phi_{1xx} + \Phi_{1yy} = 0, \quad 0 < x < 1, \ -h_1 < y < 0 \tag{1}$$

$$\Phi_{2xx} + \Phi_{2yy} = 0, \quad 0 < x < 1, \ 0 < y < h_2 \tag{2}$$

The subscripts appended to Φ_j $(j = 1, 2)$ denote partial differentiation. Eqs. (1) and (2) are subject to the following boundary conditions:

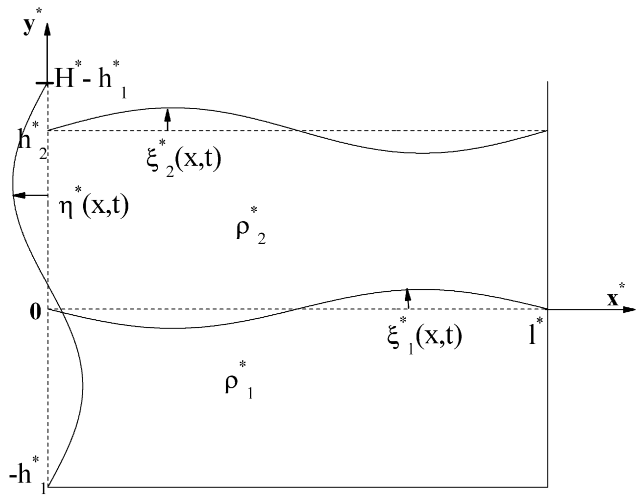

Fig. 1. Schematic representation of the system.

Fig. 1. Schéma représentatif du système.

$$\Phi_{jx} = 0, \quad x = 1 \tag{3}$$

$$\Phi_{jx} = \eta_t, \quad x = 0 \tag{4}$$

$$\Phi_{1y=0}, \quad y = -h_1 \tag{5}$$

$$\Phi_{2tt} + \Phi_{2y} - \sigma_2 \Phi_{2yxx} = 0, \quad y = h_2 \tag{6}$$

$$\Phi_{1tt} - \rho\Phi_{2tt} + (1-\rho)\Phi_{1y} - \sigma_1 \Phi_{1yxx} = 0, \quad y = 0 \tag{7}$$

where η denotes the beam deflection; these equations are in dimensionless form where the space variables are in units of l^*, the time is referred to $(l^*/g^*)^{1/2}$, g^* being the gravity acceleration, σ_j $(j = 1, 2)$ are the dimensionless surface tensions in units of $\rho_j^* g^* l^{*2}$ and ρ is the mass density ratio (ρ_2^*/ρ_1^*). In order to close the problem, the liquid motion equations must be complemented with the equation and associated boundary conditions for the flexible wall. Modelling the latter as a Bernoulli beam for the sake of simplicity, one has:

$$\eta_{tt} + \frac{1}{\delta}\eta_{yyyy} = \begin{cases} \gamma\Phi_{1t}(0, y, t), & -h_1 < y < 0 \\ \rho\gamma\Phi_{2t}(0, y, t), & 0 < y < h_2 \\ 0, & h_2 < y < H - h_1 \end{cases} \tag{8}$$

where $\delta = \rho_s^* g^* l^{*3}/A^*$ and $\gamma = \rho_1^* l^*/\rho_s^*$ are dimensionless parameters with ρ_s^* and A^* being the real mass density of the flat plate and its bending stiffness respectively. In what follows, we limit our attention to the clamped–clamped beam. Other types of conditions may be naturally prescribed at the ends of the beam without specific difficulties. Eq. (8) is then subject to:

$$\eta = \eta_y = 0 \quad \text{at } y = -h_1 \text{ and } y = H - h_1 \tag{9}$$

3. Eigenvalue equation

Assuming time periodic vibrations and using normal mode decomposition, the solution to Eqs. (1)–(7) may be thought in the form:

$$\Phi_j(x, y, t) = e^{i\omega t}\sum_{n=0}^{\infty} C_n \varphi_j(k_n, y)\cos k_n(x-1) + c.c. \tag{10}$$

where $\varphi_1(k_n, y) = \cosh k_n(y + h_1)$, $\varphi_2(k_n, y) = \sinh k_n h_1 . \sinh k_n y + A_n . \cosh k_n h_1 . \cosh k_n y$, A_n and k_n being arbitrary complex constants, ω is the circular frequency (eigenfrequency) and $c.c.$ denotes complex conjugate. Using (10) in the boundary conditions (6) and (7), we obtain two relationships between ω, k_n and A_n. The first one gives the coefficient A_n in terms of ω and k_n, namely:

$$\rho A_n = 1 - \frac{k_n . \tanh k_n h_1}{\omega^2}(1 - \rho - \sigma_1 k_n^2) \tag{11}$$

which naturally gives dispersion relation for a single fluid layer by taking $\rho = 0$. The second one is the dispersion relation that expresses the wave number k_n as an implicit function of the frequency ω, that is to say:

$$(1 + \rho . T_{1n} . T_{2n})\omega^4 - k_n\{T_{1n} . [1 + k_n^2(\sigma_1 + \rho\sigma_2)] + T_{2n} . (1 + \sigma_2 k_n^2)\}\omega^2 + k_n^2 T_{1n} . T_{2n} . (1 + \sigma_2 k_n^2) . (1 - \rho + \sigma_1 k_n^2) = 0 \tag{12}$$

where $T_{1n} = \tanh(k_n h_1)$, $T_{2n} = \tanh(k_n h_2)$. Solving (12) for k_n, ω being fixed, we obtain a countably infinite set $\{k_n\}_{n=0}^{\infty}$ of roots such that $k_n^2 \in \mathbb{R}$ and $|k_n^2| \to \infty$ as $n \to \infty$. Since $k_n^2, n = 0, \ldots, \infty$, are real, so are the amplitudes $\varphi_j(k_n, y)$ of the normal modes for any n. Furthermore, the set of these amplitudes is orthogonal in the sense of the inner product

$$\langle f, g\rangle = \omega^2\left\{\int_{-h_1}^{0} f.g.\mathrm{d}y + \rho\int_{0}^{h_2} f.g\,\mathrm{d}y\right\} + \rho\sigma_2 . \frac{\mathrm{d}f}{\mathrm{d}y}\frac{\mathrm{d}g}{\mathrm{d}y}\bigg|_{y=h_2} + \sigma_1 . \frac{\mathrm{d}f}{\mathrm{d}y}\frac{\mathrm{d}g}{\mathrm{d}y}\bigg|_{y=0} \tag{13}$$

Since the coefficients C_n introduced in (10) are to be determined from the coupling of the fluid flow with the beam motion, one obtains

$$-\omega^2\eta + \frac{1}{\delta}n_{yyyy} = \begin{cases} \mathrm{i}\gamma\omega\sum_{n=0}^{\infty} C_n\varphi_1(k_n, y), & -h_1 < y < 0 \\ \mathrm{i}\gamma\omega\rho\sum_{n=0}^{\infty} C_n\varphi_2(k_n, y), & 0 < y < h_2 \\ 0, & h_2 < y < H - h_1 \end{cases} \tag{14}$$

.

As in (10), the solution to (14) is searched in the form:

$$\eta(y,t) = \mathrm{e}^{\mathrm{i}\omega t}\sum_{l=0}^{\infty} E_l\eta_l(y) \tag{15}$$

where $\eta_l(y)$ is the lth normal mode of the beam oscillations and E_l a sequence of arbitrary constants. In the present case, the normal modes are

$$\eta_l(y) = \cos p_l(y+h_1) - \cosh p_l(y+h_1) + \alpha_l\{\sin p_l(y+h_1) - \sinh p_l(y+h_1)\} \tag{16}$$

where $\alpha_l = \frac{\sin p_l H + \sinh p_l H}{\cos p_l H - \cosh p_l H}$, p_l being the lth root of $\cos p_l H . \cosh p_l H = 1$.

Because of the orthogonality of the normal modes η_l in the sense of $L^2[-h_1, H-h_1]$ and $\|\eta_l\|^2 = H$, (14) yields:

$$E_m = \mathrm{i}\frac{\omega\gamma\delta}{(P_m^4 - \delta\omega^2)H}\sum_{l=0}^{\infty} A_{ml}C_l . \cos k_l \tag{17}$$

with $A_{ml} = \{\int_{-h_1}^{0}\varphi_1(k_l,y)\eta_m(y)\,\mathrm{d}y + \rho\int_0^{H-h_2}\varphi_2(k_l,y)\eta_m(y)\,\mathrm{d}y\}$. Now, using the kinematics condition (4) together with the orthogonality of the functions $\varphi_j(k_n, y)$ is the sense of the inner product (13), one is led to:

$$C_l = \mathrm{i}\frac{\omega}{k_l\sin k_l}\sum_{n=0}^{\infty} B_{\ln}E_n \tag{18}$$

with

$$B_{\ln} = \frac{\omega^2\{\int_{-h_1}^{0}\varphi_1(k_l,y)\eta_n(y)\,\mathrm{d}y + \rho\int_0^{h_2}\varphi_2(k_l,y)\eta_n(y)\,\mathrm{d}y\} + \rho\sigma_2\frac{\mathrm{d}\varphi_{2l}}{\mathrm{d}y}\frac{\mathrm{d}\eta_n}{\mathrm{d}y}|_{y=h_2} + \sigma_1\frac{\mathrm{d}\varphi_{1l}}{\mathrm{d}y}\frac{\mathrm{d}\eta_n}{\mathrm{d}y}|_{y=0}}{\omega^2\{\int_{-h_1}^{0}\varphi_1^2(k_l,y)\,\mathrm{d}y + \rho\int_0^{h_2}\varphi_2^2(k_l,y)\,\mathrm{d}y\} + \rho\sigma_2(\frac{\mathrm{d}\varphi_{2l}}{\mathrm{d}y})^2_{y=h_2} + \sigma_1(\frac{\mathrm{d}\varphi_{1l}}{\mathrm{d}y})^2_{y=0}}$$

Finally, combining Eqs. (17) and (18) yields the eigenfrequency equation which expresses the requirement for the existence of a nontrivial motion, that is:

$$\det\left(\delta_{nm} + \gamma\delta\frac{\omega^2}{P_m^4 - \delta\omega^2}\sum_l \frac{1}{k_l\tan k_l}A_{ml}B_{\ln}\right) = 0 \tag{19}$$

Solving this equation iteratively for ω, with Eq. (12) in mind, gives the coupled eigenfrequencies and hence the shape modes of the system.

4. Numerical results

To simplify matter and in view of comparisons to be made, we neglect surface tension effects and take $\delta = 3.582$, $\gamma = 309.603$, $H = 1.250$ as in Chai et al. [3]. Thus only three parameters namely h_1, h_2 and ρ remain free. The iterative process to solving (19) for ω is implemented by successively decreasing (increasing) ρ from 1 (zero) by a small quantity $\delta\rho$. In order to improve the convergence speed for a given value of ρ, the calculation is initialized by the coupled frequencies associated to the previous value of ρ. The procedure naturally begins by the coupled frequencies of the hydroelastic system with a single fluid. The iterations are stopped when the relative change in the result is less than or equal to 0.1%. The correctness of the present theory and the associated computer program is first verified through comparisons with the limiting case of a single fluid treated in Chai et al. [3] where thirty terms are kept in the series expansion. For the purpose of these comparisons, the infinite series of both the velocity potential and the beam deflection are truncated after only 10 terms. Table 1 gives the first ten coupled eigenfrequencies at $H_f/H = 0.5$

Table 1
First ten coupled frequencies in the simple fluid case; comparison with the results obtained by the method developed in Chai et al. [3] with $H_f = 0.625$, $H = 2H_f$

Tableau 1
Les dix premières fréquences du couplage dans le cas d'un seul fluide, comparaison avec les résultats obtenus par la méthode développée par Chai et al. [3] avec $H_f = 0{,}625$, $H = 2H_f$

Mode number	Chai et al. [3]	Present method			
	(30 terms)	10 terms	20 terms	30 terms	40 terms
1.	1.27610470993569	1.27610605425984	1.27610562184155	1.27610560122281	1.27610559787676
2.	2.22726139837720	2.22726513921768	2.22726168794297	2.22726152259709	2.22726149575041
3.	2.75729615477482	2.75731854154542	2.75731061495927	2.75731023420991	2.75731017234401
4.	3.13813359853608	3.13817489865795	3.13817208151297	3.13817194550561	3.13817192336992
5.	3.54286351418964	3.54289724663794	3.54289714963292	3.54289714507610	3.54289714434222
6.	3.94432321016722	3.94435420568585	3.94435106546953	3.94435091160462	3.94435088654236
7.	4.31822162840264	4.31825614257678	4.31824687735807	4.31824641535131	4.31824633981435
8.	4.71274057731570	4.66480242353380	4.66478340490532	4.66480000000073	4.66480000000073
9.	5.02736222272224	4.98783617388606	4.98780170857161	4.98779991394578	4.98779961788580
10.	5.31700001363177	5.29086248562485	5.29080267554966	5.29079948019236	5.29079895015731

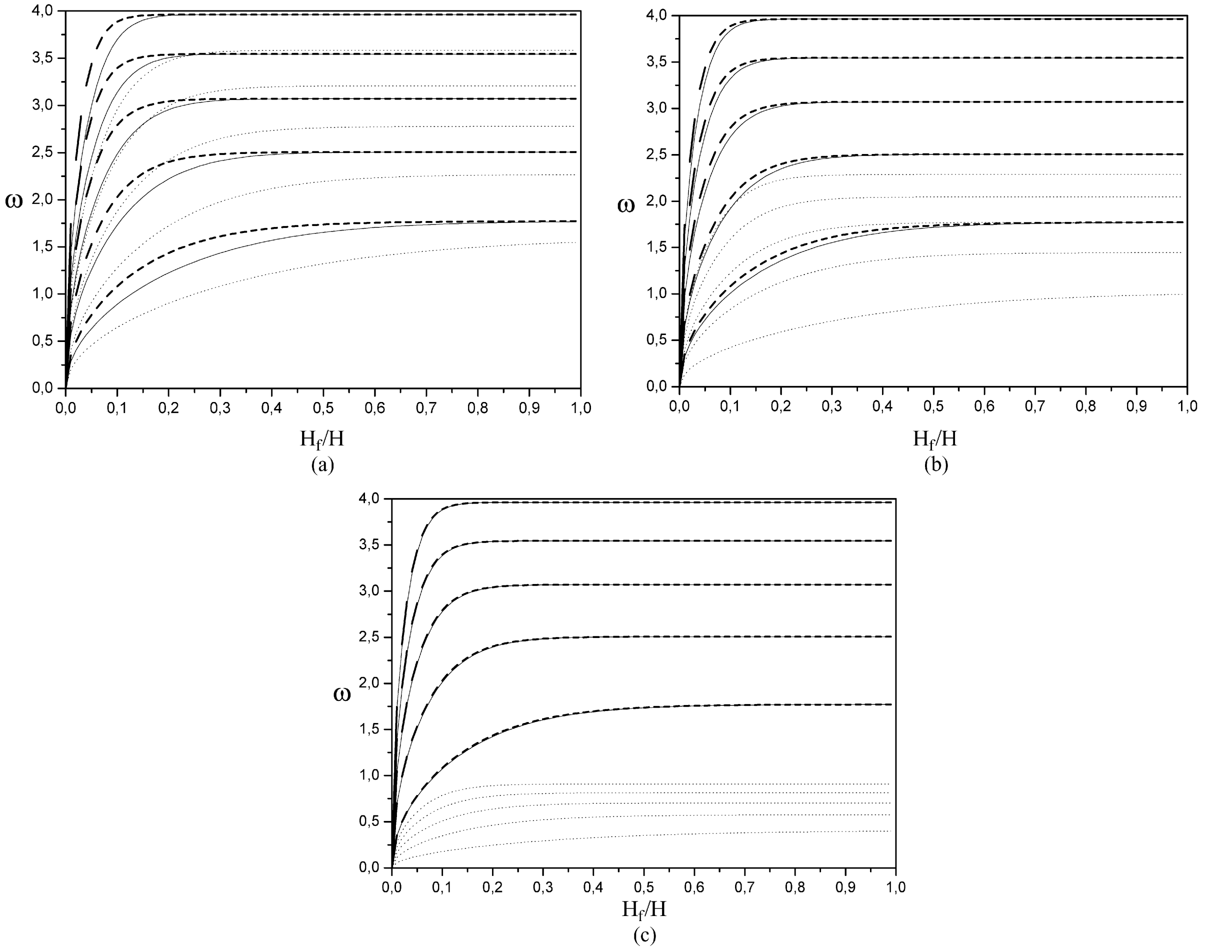

Fig. 2. Influence of the mass density ratio on the variations of the two sets of the sloshing frequencies with the total relative fluid depth ($h_1 = h_2$). (a) $\rho = 0.1$, (b) $\rho = 0.5$, (c) $\rho = 0.9$. (…) ω_{1n}, (—) ω_{2n}, (- - -) ω_{0n}.

Fig. 2. Influence du rapport des masses volumiques sur les variations des fréquences de ballottement en fonction de la hauteur relative totale des deux fluides ($h_1 = h_2$). (a) $\rho = 0{,}1$, (b) $\rho = 0{,}5$, (c) $\rho = 0{,}9$. (…) ω_{1n}, (—) ω_{2n}, (- - -) ω_{0n}.

(H_f being the total relative fluid depth), in comparison with those found by Chai et al. [3]. It can be seen that the agreement is excellent for lower frequencies and slight discrepancies occur at higher frequencies but they do not exceed 2.5%. From the table again we see that increasing the number of modes to successively 20, 30 and 40 does not change the results by more than about 10^{-5}, indicating that convergence had essentially been achieved. Based on these comparisons, the numerics have been carried out by truncating the infinite series after 10 terms, then ensuring a small size eigenfrequency equation to be solved.

Figs. 2(a), (b), (c) illustrate a comparison, for $h_1 = h_2$ and three selective values of ρ, between the variations with the relative total fluid depth of the two sloshing frequency sets, say ω_{1n} (dotted lines) and ω_{2n} (solid thin lines) of a two fluid system and the sloshing frequencies ω_{0n} (dashed lines) of a single fluid having the same relative height. We observe that these sets are such that: $\omega_{1n} < \omega_{2n} < \omega_{0n}$ for all mode numbers. The sets ω_{1n} and ω_{2n} are thus referred to as the lower and higher sets and are respectively related to the sloshing of the interface and the free surface. In the limit $\rho \to 0$, ω_{1n} and ω_{2n} naturally converge to the sloshing frequencies of the lower fluid layer; a relatively important convergence rate is visible with the increase in the fluid depth, especially for higher mode numbers. Increasing ρ from zero causes a decrease (an increase) of lower (higher) sets of sloshing frequencies. As ρ tends to unity, the frequencies of the lower set become vanishingly small while those of the second set converge to the sloshing frequencies of a single fluid layer of thickness H_f. The plots in Fig. 3 show how the first ten frequencies of the two families, ω_{1n} and ω_{2n}, are changed by varying H_f in the practical case of water on the top of a relatively thin mercury layer ($h_2 = 9h_1$). We can see that the two sets of frequencies are clearly distinct and their rates of change with H_f are also different. Most importantly, it is to be noted that the upper set of sloshing frequencies follows quite closely the set ω_{0n}, which is to say that the relatively important mass density of the lower fluid layer has no significant effects on the sloshing of the free surface. The variations of the first ten coupled frequencies for the system water–mercury are displayed in Fig. 4 where the dotted lines represent, for the sake of comparison, the first family of sloshing frequencies. We observe that the first two coupled frequencies of the system follow quite closely the corresponding (same mode number) sloshing frequencies of the first kind for all H_f/H values. This remains true for higher modes provided H_f/H is smaller than a certain critical value $(H_f/H)_1$ which marks the onset of a qualitative change in their behaviours. Indeed, when the value $(H_f/H)_1$ is crossed, a given coupled frequency first undergoes a rapid decrease and then follows the lower order sloshing frequency until a second critical value $(H_f/H)_2$ is reached around which a similar qualitative behaviour is reproduced. It should, however, be mentioned that the drops in the frequencies are less important at the second critical point than the first one. From a global point of view, the coupled frequencies exhibit similar behaviours with the single

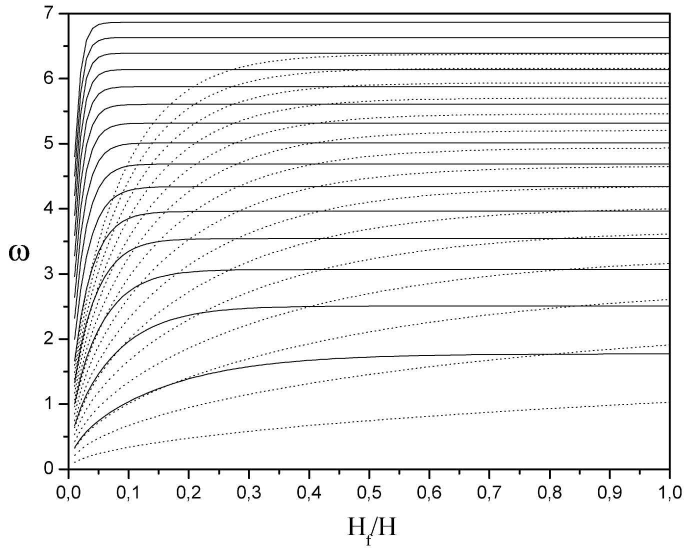

Fig. 3. Variations of the two sets of the sloshing frequencies with the total relative fluid depth for the water–mercury system ($h_2 = 9h_1$, $\rho = 0.0738$). (...) ω_{1n}, (—) ω_{2n}.

Fig. 3. Variations des fréquences de ballottement en fonction de la hauteur relative totale des deux fluides dans le cas du système eau–mercure ($h_2 = 9h_1$, $\rho = 0{,}0738$). (...) ω_{1n}, (—) ω_{2n}.

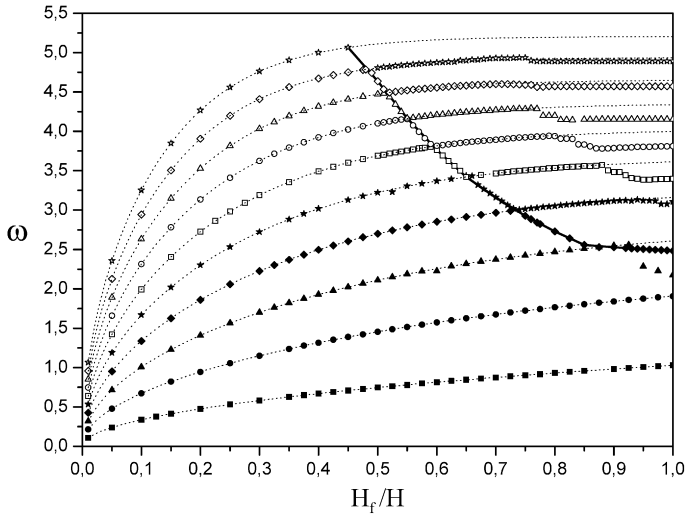

Fig. 4. Variation of the ten first coupled frequencies in the case of water–mercury system with the total relative fluid depth ($h_2 = 9h_1$). (...) ω_{1n}, (■■■) 1st, (●●●) 2nd, (▲▲▲) 3rd, (◆◆◆) 4th, ..., (∗∗∗) 10th coupled frequencies.

Fig. 4. Variation des dix premières fréquences du couplage dans le cas d'un système mercure–eau en fonction de la hauteur relative totale des deux fluides ($h_2 = 9h_1$). (...) ω_{1n}, (■■■) 1ère, (●●●) 2ème, (▲▲▲) 3ème, (◆◆◆) 4ème, ..., (∗∗∗) 10ème fréquences du couplage.

fluid case. Important quantitative differences are, however, to be noticed, as for example the reduction of the growth rate of the coupled frequencies, especially for lower order ones. Another important thing to note is that, as in the single fluid case, the rapid decrease of coupled frequencies is an indicator of the nearness of a coupled structure type frequency.

5. Conclusion

The vibratory characteristics of a hydroelastic system with two superposed, inviscid and immiscible fluids are studied by means of normal mode decomposition. The normal mode shapes are found to be orthogonal with respect to a new inner product. The convergence tests demonstrate the high accuracy and the rapid convergence of this approach. The numerical results indicate that the two sets of coupled frequencies qualitatively have the same global behaviours that those obtained in the single fluid case. The coupled type frequencies are closed to the sloshing frequencies of the first kind for liquid to beam depth ratio less than some critical value that decreases with frequency order. The case of a hydroelastic system with a continuous density stratification is a natural extension of this work; this is being examined at the present time.

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