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Duality, inverse problems and nonlinear problems in solid mechanics

Solitary SH waves in two-layered traction-free plates

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Abstract

A solitary wave, resembling a soliton wave, is observed when analyzing the linear problem of polarized shear (SH) surface acoustic waves propagating in elastic orthotropic two-layered traction-free plates. The analysis is performed by applying a special complex formalism and the Modified Transfer Matrix (MTM) method. Conditions for the existence of solitary SH waves are obtained. Analytical expressions for the phase speed of the solitary wave are derived. ***To cite this article: I. Djeran-Maigre, S. Kuznetsov, C. R. Mecanique 336 (2008).***

Résumé

Ondes SH isolées en plaques bicouches libres de contraintes. Une onde isolée ressemblant à un soliton est décrite mathématiquement en analysant les problèmes linéaires pour des ondes de cisaillement de surface se propageant dans les plaques élastiques, anisotropes et stratifiées. Le modèle mathématique est basé sur un formalisme particulier dans un espace complexe et sur la méthode de la matrice modifiée de transfert (MTM). Les conditions d'existence des ondes isolées sont obtenues. La vitesse de phase de ces ondes est décrite par des solutions analytiques. ***Pour citer cet article : I. Djeran-Maigre, S. Kuznetsov, C. R. Mecanique 336 (2008).***

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Mots-clés : Ondes ; Ondes SH ; Onde de cisaillement ; Onde de surface ; Soliton ; Onde isolée

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L'onde découverte par Scott Russell [1], connue par la suite sous le nom de soliton est une onde (i) isolée ; (ii) parcourant des distances importantes sans perturbation, ni atténuation ; (iii) qui ne se joint pas aux autres ondes, ayant (iv) une vitesse constante qui peut dépendre de la taille de l'onde, de la largeur du canal et de la profondeur de l'eau [2]. Une telle onde est décrite mathématiquement comme solution de l'équation non-linéaire KdV [3–5]. Nous présentons un model mathématique basé sur la combinaison du formalisme de Stroh et de la méthode de la matrice modifiée de transfert, qui permet d'analyser les ondes de cisaillement de surface, de polarisation horizontale (SH) se propageant dans les plaques élastiques, anisotropes et stratifiées ; sous certaines conditions peuvent constituer des solutions res-

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semblant aux solitons en vérifiant l'ensemble des conditions de (i)–(iv). Contrairement aux véritables solitons pour l'hydrodynamique satisfaisant l'équation non-linéaire de KdV, la solution présentée satisfait l'équation différentielle linéaire, connue comme équation de Christoffel pour les ondes acoustiques de surface. La méthode principale utilisée pour la construction des solutions des solitons pour les ondes SH, est basée sur la combinaison du formalisme dans un espace complexe avec la méthode de la matrice modifiée de transfert (MTM) [7,9,10]. Les ondes isolées présentent des similitudes avec les modes inférieurs des ondes de Love et des ondes de Sezawa [11].

1. Introduction

Since its first discovery by Scott Russell [1], the traveling wave (known later as the soliton) is a wave that is (i) solitary; (ii) traveling along large distances without disturbance or attenuation; (iii) not merging with other waves; and (iv) having constant speed that can depend upon the size of the wave, its width and the depth of water; see also [2]. Mathematically, such a wave is described as the solution of the nonlinear KdV equation [3–5].

Here, we present a mathematical model based on a combination of the six-dimensional formalism and the modified transfer matrix method, allowing us to analyze horizontally polarized shear (SH) surface acoustic waves propagating in anisotropic elastic laminated plates. It will be demonstrated that at certain conditions there can be solutions resembling solitons in that they satisfy all the conditions (i)–(iv) solitons must satisfy. In contrast to the genuine solitons, for hydrodynamic solitons that satisfy the nonlinear KdV equation, the solution presented satisfies a linear differential equation, known as the Christoffel equation for surface acoustic waves.

Below, the surface acoustic waves propagating in an infinite plate with the unit normal ν to the median plane and coordinate $x' = \nu \cdot \mathbf{x}$ belonging to the interval $[-h, h]$, are analyzed. Horizontally polarized shear surface waves (SH waves) propagating in multilayered plates resemble Love waves [6] in polarization, but differ in the absence of a contacting half-space (substrate), and, hence exclude the necessity to impose Sommerfield's emission condition:

$$\mathbf{u}(\mathbf{x}, t) = \mathrm{O}\big(|x'|^{-1}\big), \quad |x'| \to \infty \tag{1}$$

where $\mathbf{u}$ is the displacement field, $\mathbf{x}$ is a three dimensional coordinate vector, $x' = \mathbf{x} \cdot \nu$ is a coordinate along depth of the half-space, ν is the unit normal to the plane boundary of the half-space. The absence of condition (1) for plates leads to the appearance of solitary wave solutions for layered traction-free plates, but, as will be shown later, these waves cannot propagate in homogeneous (single-layered) plates.

The main method used for constructing solitary wave solutions for SH waves is based on a combination of the complex formalism and the modified transfer matrix (MTM) method [7,9,10]. The solitary waves observed have similarity with the leakage lower mode Love and Sezawa [11] waves.

2. Basic notations

All the regarded layers in a plate are assumed to be homogeneous, anisotropic and linearly hyperelastic. Equations of motion for a homogeneous anisotropic elastic medium can be written in the form:

$$\mathbf{A}(\partial_x, \partial_t)\mathbf{u} \equiv \operatorname{div}_x \mathbf{C} \cdot \cdot \nabla_x \mathbf{u} - \rho \ddot{\mathbf{u}} = 0 \tag{2}$$

where ρ is the material density, and $\mathbf{C}$ is the elasticity tensor assumed to be *positive definite*:

$$\mathbf{A} \cdot \cdot \mathbf{C} \cdot \cdot \mathbf{A} > \mathbf{0} \tag{3}$$

for any symmetric non-trivial tensor of the second rank $\mathbf{A}$.

Remark 2.1. (a) The other assumption concerns symmetry of the elasticity tensor. It will be assumed that all the materials considered possess planes of elastic symmetry coinciding with the sagittal plane $\mathbf{m} \cdot \mathbf{x} = 0$, where vector $\mathbf{m}$ is the polarization vector of the SH wave. This is achieved by the elasticity tensor belonging to the *monoclinic* system, see [12]. The monoclinic system is equivalent to the vanishing of all of the decomposable components of the tensor $\mathbf{C}$ having an odd number of entries of the vector $\mathbf{m}$ (in the orthogonal basis in R^3 generated by vector $\mathbf{m}$ and any two orthogonal vectors belonging to the sagittal plane).

(b) It will be shown later that assuming monoclinic symmetry provides a sufficient condition for the surface tractions acting on any plane $\nu \cdot \mathbf{x} = const$ to be collinear with polarization vector $\mathbf{m}$. This ensures existence of an SH or Love wave propagating in the corresponding direction. If an elasticity tensor does not belong to the monoclinic system or if the direction of propagation does not coincide with the axis of elastic symmetry, then no genuine SH or Love waves can propagate.

Following [7,10], we will seek a horizontally polarized shear wave in a particular layer in the form:

$$\mathbf{u}(\mathbf{x}) = \mathbf{m} f(\mathrm{i}rx')\mathrm{e}^{\mathrm{i}r(\mathbf{n}\cdot\mathbf{x}-ct)} \tag{4}$$

where f is the unknown scalar complex-valued function; the exponential multiplier $\mathrm{e}^{\mathrm{i}r(\mathbf{n}\cdot\mathbf{x}-ct)}$ in (4) corresponds to propagation of the plane wave front along direction $\mathbf{n}$ with the phase speed c; r is the wave number.

Substituting representation (4) into Eq. (2) and taking into account Remark 2.1(a), yields the following differential equation:

$$\begin{pmatrix} (\mathbf{m}\otimes\nu\cdot\cdot\mathbf{C}\cdot\cdot\nu\otimes\mathbf{m}) f''_{x'} + 2\big(\mathbf{m}\cdot sym(\nu\cdot\mathbf{C}\cdot\mathbf{n})\cdot\mathbf{m}\big) f'_{x'} + \\ \big(\mathbf{m}\otimes\mathbf{n}\cdot\cdot\mathbf{C}\cdot\cdot\mathbf{n}\otimes\mathbf{m} - \rho c^2\big) f \end{pmatrix} = 0 \tag{5}$$

It should be noted that function f satisfying Eq. (5), represents a solution in a particular layer. Imposing interface conditions between layers will be discussed in Section 4. The characteristic equation for the differential equation (5), known also as the Christoffel equation, has the form:

$$(\mathbf{m}\otimes\nu\cdot\cdot\mathbf{C}\cdot\cdot\nu\otimes\mathbf{m})\gamma^2 + 2\big(\mathbf{m}\cdot sym(\nu\cdot\mathbf{C}\cdot\mathbf{n})\cdot\mathbf{m}\big)\gamma + \big(\mathbf{m}\otimes\mathbf{n}\cdot\cdot\mathbf{C}\cdot\cdot\mathbf{n}\otimes\mathbf{m} - \rho c^2\big) = 0 \tag{6}$$

The left-hand side of Eq. (6) represents a polynomial of degree 2 with respect to the Christoffel parameter γ. Thus, for monoclinic elastic symmetry only two partial waves form the SH wave considered in a layer.

The following lemma flows from solving the Cauchy problem for Eq. (5):

Lemma 2.2. *A necessary and sufficient condition for the real-analytic solution of Eq.* (5)*, to be a non-zero function, is the simultaneous non-vanishing f and its first derivative at some x'.*

Remark 2.3. (a) For an orthotropic medium and the SH wave propagating in a direction of the principle elasticity, Eq. (6) is simplified:

$$(\mathbf{m}\otimes\nu\cdot\cdot\mathbf{C}\cdot\cdot\nu\otimes\mathbf{m})\gamma^2 + \big(\mathbf{m}\otimes\mathbf{n}\cdot\cdot\mathbf{C}\cdot\cdot\mathbf{n}\otimes\mathbf{m} - \rho c^2\big) = 0 \tag{7}$$

The solution for Eq. (7) is:

$$\gamma_{1,2} = \pm\sqrt{\frac{\rho c^2 - \mathbf{m}\otimes\mathbf{n}\cdot\cdot\mathbf{C}\cdot\cdot\mathbf{n}\otimes\mathbf{m}}{\mathbf{m}\otimes\nu\cdot\cdot\mathbf{C}\cdot\cdot\nu\otimes\mathbf{m}}} \tag{8}$$

Thus, the general solution of Eq. (5) can be represented in the form:

$$f(\mathrm{i}rx') = C_1\sinh(\mathrm{i}r\gamma x') + C_2\cosh(\mathrm{i}r\gamma x') \tag{9}$$

where γ is the positive root in (8).

(b) Supposing that the roots of the Christoffel equation (6) are multiple, we arrive at the necessity to modify the solution of Eq. (5) by placing a logarithmic term (this corresponds to the appearance of one Jordan block in Eq. (5), that is reduced to a system of first order). However, here the case of multiple roots will not be studied; see [7,8] for the solutions related to multiple roots.

(c) Representation (4) allows us to express surface tractions acting on any plane $x' = const$ in the form:

$$\mathbf{t}_\nu(\mathbf{x}) = \mathrm{i}r\nu\cdot\big(\mathbf{C}\cdot\nu f'(\mathrm{i}rx') + \mathbf{C}\cdot\mathbf{n} f'(\mathrm{i}rx')\big)\cdot\mathbf{m}\mathrm{e}^{\mathrm{i}r(\mathbf{n}\cdot\mathbf{x}-ct)} \tag{10}$$

The assumption of monoclinic symmetry ensures surface tractions (10) to be collinear with vector $\mathbf{m}$. For an orthotropic material with the axes of elastic symmetry coinciding with vectors $\mathbf{m}$, $\mathbf{n}$, and ν expression reduces to

$$\mathbf{t}_\nu(\mathbf{x}) = \mathrm{i}r\nu\cdot\mathbf{C}\cdot\cdot\nu\otimes\mathbf{m} f'(\mathrm{i}rx')\mathrm{e}^{\mathrm{i}r(\mathbf{n}\cdot\mathbf{x}-ct)} \tag{11}$$

As before, vector $\mathbf{t}_\nu(\mathbf{x})$ is necessarily collinear with vector $\mathbf{m}$. Expression (11) shows that on a boundary plane x'_0 of a *traction-free orthotropic plate* the following condition must be satisfied: $f'(\mathrm{i}rx'_0) = 0$.

From now, it will be assumed that vectors ν, $\mathbf{m}$, and $\mathbf{n}$ coincide with the axes of elastic symmetry of an orthotropic medium.

Remark 2.4. It can be shown (see [6]) that regardless of boundary conditions and at the imaginary roots of Eq. (7), no SH wave can propagate in the directions of elastic symmetry of an orthotropic single-layered plate. Thus, the following inequality

$$c > \sqrt{\frac{\mathbf{m}\otimes\mathbf{n}\cdot\cdot\mathbf{C}\cdot\cdot\mathbf{n}\otimes\mathbf{m}}{\rho}} \tag{12}$$

naturally arising from (8), delivers a necessary condition for an existing surface SH wave. Thus, for the plate considered, all surface SH waves are necessarily supersonic, since the radicand in the right-hand side of (12) defines speed of the corresponding shear bulk wave $c^T_{\mathbf{nm}}$. In this section we assume condition (12) to hold.

3. Homogeneous traction-free plate

Now, we consider a single-layered plate with the traction-free boundary conditions:

$$\begin{cases} \mathbf{t}_\nu(h/2) = 0 \\ \mathbf{t}_\nu(-h/2) = 0 \end{cases} \tag{13}$$

where h is the thickness of the plate (we choose origin of coordinates at the median plane).

For such a plate, finding function f from (7)–(11) and (13), yields:

$$f(\mathrm{i}rx') = \begin{cases} \cos(r\gamma x'), & \text{at } r = \frac{2n\pi}{\gamma h} \\ \sin(r\gamma x'), & \text{at } r = \frac{(2n-1)\pi}{\gamma h} \end{cases} \quad n = 1, 2, \ldots \tag{14}$$

where γ is defined by (8).

Proposition 3.1. (a) *On planes $x' = const$, where*

$$x' = \begin{cases} \frac{\frac{1}{2}+k}{2n} h & at\ r = \frac{2n\pi}{\gamma h},\ -n \leqslant k \leqslant n \\ \frac{k}{2n-1} h & at\ r = \frac{(2n-1)\pi}{\gamma h},\ -n \leqslant k \leqslant n \end{cases} \quad n, k \in Z \tag{15}$$

both the displacement field and specific kinetic energy vanish. This is equivalent to the existence of the internal immovable layers under propagating SH wave on a traction-free plate.

(b) *At any finite phase speed satisfying inequality* (12)*, there are no waves propagating at vanishing frequency* (*both phase speed and frequency are delimited from zero*).

Proof. (a) Follows from considering zeroes of the function, defined by (14). Proof (b) also follows from analyzing expressions (14). Indeed, in view of (12) and (14) condition $r \to 0$ (or $\omega \to 0$) requires $c \to \infty$. Thus, no non-trivial solution exists at $\omega = 0$ and the finite phase speed c.

Thus, there are no solitary SH waves propagating in the plane of elastic symmetry of homogeneous orthotropic traction-free plates. □

4. Two-layered traction-free plate

It is assumed that (i) both layers are orthotropic with axes of elastic symmetry coincident with vectors $\mathbf{n}$, ν, and $\mathbf{m}$; and (ii) the corresponding shear bulk waves differ:

$$\left(c^T_{\mathbf{nm}}\right)_1 \neq \left(c^T_{\mathbf{nm}}\right)_2 \tag{16}$$

Remark 4.1. In this section, and beyond, we assume that each layer has its own reference coordinate system with the origin lying in the median plane of a layer. Boundary conditions for a traction-free two-layered plate are:

$$\begin{cases} t_\nu(h_1/2) = 0 \\ t_\nu(-h_2/2) = 0 \end{cases} \tag{17}$$

where lower indices are referred to the corresponding layers.

Applying the MTM method [7,10], functions f_k, $k = 1, 2$, that define the displacement field in the corresponding layers, can be represented in the form:

$$f_k(\mathrm{i}rx') = \cos\left(r\gamma_k\left(x' + (-1)^k \frac{h_k}{2}\right)\right), \quad k = 1, 2 \tag{18}$$

at the wave number r satisfying the following equation [7]:

$$\frac{(\mathbf{m}\otimes\nu\cdot\cdot\mathbf{C_1}\cdot\cdot\nu\otimes\mathbf{m})\gamma_1}{(\mathbf{m}\otimes\nu\cdot\cdot\mathbf{C_1}\cdot\cdot\nu\otimes\mathbf{m})\gamma_2}\sin(r\gamma_1 h_1)\cos(r\gamma_2 h_2) + \cos(r\gamma_1 h_1)\sin(r\gamma_2 h_2) = 0 \tag{19}$$

Proposition 4.2. (a) *Suppose that*

$$\min\left(\left(c^T_{\mathbf{nm}}\right)_1; \left(c^T_{\mathbf{nm}}\right)_2\right) < c < \max\left(\left(c^T_{\mathbf{nm}}\right)_1; \left(c^T_{\mathbf{nm}}\right)_2\right) \tag{20}$$

where $(c^T_{\mathbf{nm}})_k$ *is the bulk wave speed in the corresponding layer, then on planes* $x' = const$,

$$x' = \frac{\pi(1+2n)}{2r\gamma_k} - (-1)^k \frac{h_k}{2} \tag{21}$$

$$-\mathrm{Ent}\left(\left(1-(-1)^k\right)\frac{r\gamma_k h_k}{2\pi} + \frac{1}{2}\right) \leqslant n \leqslant \mathrm{Ent}\left(\left(1+(-1)^k\right)\frac{r\gamma_k h_k}{2\pi} - \frac{1}{2}\right) \tag{22}$$

and r *satisfies Eq.* (19)*, then both the displacement field and specific kinetic energy vanish in a layer with the minimal bulk wave speed* $c^T_{\mathbf{nm}}$.

(b) *Suppose that*

$$c > \max\left(\left(c^T_{\mathbf{nm}}\right)_1; \left(c^T_{\mathbf{nm}}\right)_2\right) \tag{23}$$

(*the phase speed is transonic in both layers*)*, then on planes* $x' = const$ *where* x' *satisfies Eq.* (21) *and* n *satisfies Eq.* (22)*, the displacement field and specific kinetic energy vanish in both layers.*

(c) *At the phase speed* $c \to c_s - 0$*, where*

$$c_s = \sqrt{\frac{(\mathbf{m}\otimes\mathbf{n}\cdot\cdot\mathbf{C_1}\cdot\cdot\mathbf{n}\otimes\mathbf{m})h_1 + (\mathbf{m}\otimes\mathbf{n}\cdot\cdot\mathbf{C_2}\cdot\cdot\mathbf{n}\otimes\mathbf{m})h_2}{\rho_1 h_1 + \rho_2 h_2}} \tag{24}$$

there is a lower mode solitary SH wave propagating with vanishing wave number $r \to 0$.

Proofs. (a) and (b) flow out from expression (18) for functions f_k. Values for x' defined by (21) and (22), are zeroes of these functions. To prove (c) we need to consider Eq. (19) at small r:

$$\left((\mathbf{m}\otimes\nu\cdot\cdot\mathbf{C_1}\cdot\cdot\nu\otimes\mathbf{m})\gamma_1^2 h_1 + (\mathbf{m}\otimes\nu\cdot\cdot\mathbf{C_2}\cdot\cdot\nu\otimes\mathbf{m})\gamma_2^2 h_2\right)r + O\left(r^3\right) = 0 \tag{25}$$

Equating to zero the coefficient at r in the left-hand side of Eq. (25), we arrive at the solution for the phase speed given by (24). □

Proposition 4.2 ensures:

Corollary 4.3. (a) *At sufficiently small* r *and real* γ_k *in both planes there can be no planes with vanishing displacement field* (*all SH waves in the vicinity of the anomalous SH wave do not have planes at which the displacement field vanishes*).

(b) *Direct analysis reveals that the wave speed* c_s *satisfies the inequalities*:

$$\min\left(\left(c^T_{\mathbf{nm}}\right)_1; \left(c^T_{\mathbf{nm}}\right)_2\right) \leqslant c_s \leqslant \max\left(\left(c^T_{\mathbf{nm}}\right)_1; \left(c^T_{\mathbf{nm}}\right)_2\right) \tag{26}$$

Inequality (26) demands that the solitary wave speed to be subsonic for one of the layers and supersonic for the other.

Some of the lower branches of the dispersion curves (these curves show dependence of frequency on the phase speed) are presented in Fig. 1, where the solitary waves correspond to the lowest branch in the vicinity of c_s. The presented results are obtained for a two-layered traction-free plate with layers of the equal depths $h_1 = h_2 = 1$, equal densities $\rho_1 = \rho_2 = 1$ and $(c^T_{\mathbf{nm}})_1 = 1$, $(c^T_{\mathbf{nm}})_2 = 4$.

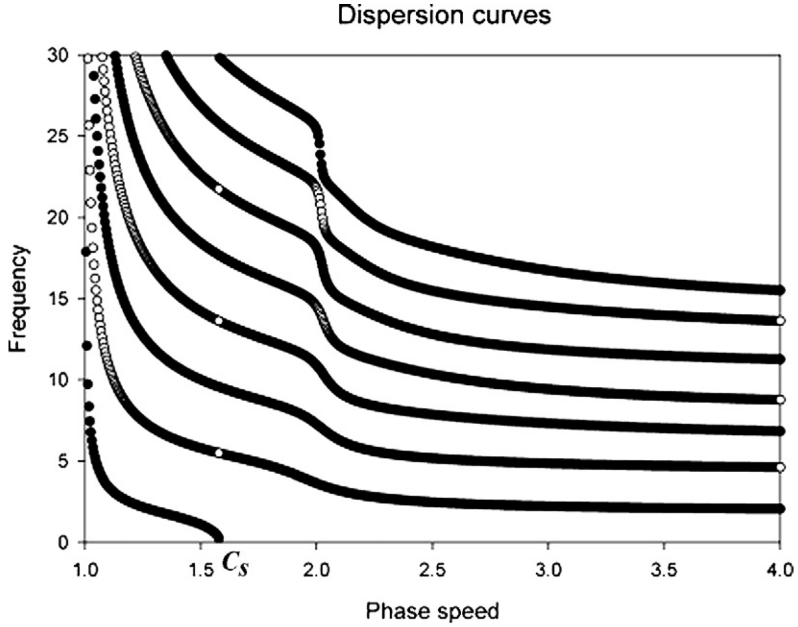

Fig. 1. Dispersion curve for a two-layered traction-free plate.

5. Conclusion

As we have already observed, the solitary wave solutions of the linear differential equation, known as the Christoffel equation for SH waves, appear in any two-layered plates with orthotropic (and henceforth, *isotropic* and *transversely isotropic*) layers, provided axes of elastic symmetry in both layers are the same, and direction of propagation of the SH wave coincides with one of these axes. The analysis carried out revealed that these SH waves are (i) solitary; (ii) travel along large distances without disturbance or attenuation (as the corresponding phase speed is real); (iii) not merge with other waves; and (iv) have a constant phase speed that depends upon only physical and geometrical properties of layers.

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