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Duality, inverse problems and nonlinear problems in solid mechanics

Application of invariant integrals to elastostatic inverse problems

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Abstract

A problem of parameters identification for embedded defects in a linear elastic body using results of static tests is considered. A method, based on the use of invariant integrals is developed for solving this problem. A problem for the spherical inclusion parameters identification is considered as an example of the proposed approach application. It is shown that a radius, elastic moduli and coordinates of a spherical inclusion center are determined from one uniaxial tension (compression) test. The explicit formulae, expressing the spherical inclusion parameters by means of the values of corresponding invariant integrals are obtained. The values of the integrals can be calculated from the experimental data if both applied loads and displacements are measured on the surface of the body in the static test. A numerical analysis of the obtained explicit formulae is fulfilled. It is shown that the formulae give a good approximation of the spherical inclusion parameters even in the case when the inclusion is located close enough to the surface of the body. ***To cite this article: R. Goldstein et al., C. R. Mecanique 336 (2008).***

Résumé

Application d'intégrales invariantes à des problèmes élastostatiques inverses. On considère un problème d'identification de paramètres pour des défauts inclus dans un corps linéaire élastique à partir de résultats d'expériences statiques. Une méthode fondée sur l'utilisation d'intégrales invariantes est dévelopée pour résoudre ce problème. A titre d'exemple d'application de l'approche proposée, on considère un problème d'identification de paramètres pour une inclusion sphérique. On montre que le rayon, les modules d'élasticité et les coordonnées du centre de cette inclusion peuvent être déterminés à partir d'une expérience uniaxiale de traction ou de compression. Des formules explicites exprimant les paramètres de l'inclusion sphérique grâce aux valeurs des intégrales invariantes correspondantes sont obtenues. Les valeurs des intégrales peuvent être calculées à partir des données expérimentales si l'on mesure à la fois les tractions et les déplacements sur la surface du corps. Une analyse numérique des formules explicites obtenues est réalisée. On montre que ces formules fournissent une bonne approximation des paramètres de l'inclusion sphérique même dans le cas où l'inclusion est située près de la surface du corps. ***Pour citer cet article : R. Goldstein et al., C. R. Mecanique 336 (2008).***

Keywords: Computational solid mechanics; Defect identification; Invariant integrals; Spherical inclusion

Mots-clés : Mécanique des solides numérique ; Identification de défaut ; Intégrales invariantes ; Inclusion sphérique

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1. Introduction

The problems of defects, mainly cracks and cavities, identification were considered in a number of publications. The most of defect identification methods use the surface measurements for bodies subjected to dynamic forces [1–5]. The data of static tests are used for the defect detection also often [6–9]. A review of different approaches for solving elastostatic and elastodynamic inverse problems is presented in [10]. The usual way for solving the inverse problems is the following:

- a defect and its location are described by some parameters;
- a direct problem is solved by one of the numerical methods for the prescribed parameters of the defect and its location;
- an error function, describing the difference between calculated and experimental data, is constructed;
- one of the optimization methods is used for the determination of the unknown defect parameters, giving an extremum for the error function.

Since the error function can have several extrema, the realization of optimization methods becomes a difficult problem. In this connection the methods which enable to determine some defect parameters without using of the error function optimization are of great interest [7,8]. In particular, a reciprocity gap principle was used in [7] for a plane crack identification.

The aims of the present publication are as follows:

- to supplement the reciprocity gap principle approach with other types of invariant integrals;
- to develop an approach for obtaining explicit formulae for the defect parameters in the case when the sizes of a defect are small as compared to the distance between the defect and the body boundary.

2. Statement of the problem

Let V be a simply connected domain in a three-dimensional space R^3. $G \subset V$ is an embedded subdomain, $\Omega = V \setminus G$. Let us suppose that Ω is an isotropic linear elastic body with a shear modulus μ_M and Poisson ratio ν_M. The defect G can be a cavity, a crack or an isotropic linear elastic inclusion. Let us introduce Cartesian coordinates $OX_1X_2X_3$. The stress–strain state in the matrix Ω we'll mark with the superscript (f): $\sigma_{ij}^{(f)}$ is the stress tensor, $e_{ij}^{(f)}$ is the strain tensor and $u^{(f)} = (u_1^{(f)}, u_2^{(f)}, u_3^{(f)})$ is the displacement vector. According to our suppositions the following equalities are valid:

$$
\begin{aligned}
&e_{ij}^{(f)}(X) = \big(u_{i,j}^{(f)}(X) + u_{j,i}^{(f)}(X)\big)/2 \quad (i = 1, 2, 3;\ j = 1, 2, 3)\\
&\sigma_{ij}^{(f)}(X) = 2\mu_M\left[\frac{\nu_M}{1-2\nu_M}\theta^{(f)}(X)\delta_{ij} + e_{ij}^{(f)}(X)\right], \quad \theta^{(f)}(X) = \sum_{k=1}^{3} e_{kk}^{(f)}(X)\\
&\sigma_{ij,j}^{(f)}(X) = 0, \quad X = (X_1, X_2, X_3) \in \Omega
\end{aligned}
\tag{1}
$$

where the convention of summation for repeated indexes is used, δ_{ij} is the Kronecker delta.

It is supposed that the loads $t^{(f)} = (t_1^{(f)}, t_2^{(f)}, t_3^{(f)})$ are applied to the external boundary of Ω, coinciding with the boundary of the domain V—∂V

$$
\sigma_{ij}^{(f)}(X)n_j(X) = t_i^{(f)}(X), \quad X \in \partial V \tag{2}
$$

where $n(X) = (n_1(X), n_2(X), n_3(X))$ is a unit outward normal to the boundary ∂V at the point X.

The applied loads are self-balanced

$$
\int_{\partial V} t_i^{(f)}(X)\,\mathrm{d}S = 0, \qquad \int_{\partial V} X \wedge t^{(f)}(X)\,\mathrm{d}S = 0 \tag{3}
$$

where $\wedge$ is a vector product.

If the defect G is a cavity or a crack we suppose that the boundary ∂G is unloaded

$$\sigma_{ij}^{(f)}(X)N_j(X) = 0, \quad X \in \partial G \tag{4}$$

where $N(X) = (N_1(X), N_2(X), N_3(X))$ is a unit normal to ∂G at the point X.

If the defect G is an inclusion, we suppose that G is an isotropic and linear elastic body with unknown shear modulus μ_I and Poisson ratio ν_I. It is supposed also complete bonding between the matrix and inclusion. Let us denote by σ_{ij}^I, e_{ij}^I and $u^I = (u_1^I, u_2^I, u_3^I)$ the stresses, strains and displacements of the inclusion. The mentioned suppositions lead to the following equations

$$\begin{aligned}
&e_{ij}^I(X) = \big(u_{i,j}^I(X) + u_{j,i}^I(X)\big)/2, \quad X \in G \ (i = 1,2,3;\ j = 1,2,3)\\
&\sigma_{ij}^I(X) = 2\mu_I\left[\frac{\nu_I}{1-2\nu_I}\theta^I(X)\delta_{ij} + e_{ij}^I(X)\right], \quad \theta^I(X) = \sum_{k=1}^{3} e_{kk}^I(X)\\
&\sigma_{ij,j}^I(X) = 0
\end{aligned} \tag{5}$$

The bonding conditions have the following form

$$u^I(X) = u^{(f)}(X), \quad \sigma_{ij}^I(X)N_j(X) = \sigma_{ij}^{(f)}(X)N_j(X), \quad X \in \partial G \tag{6}$$

We suppose that overdetermined boundary data (the applied loads $t^{(f)}(X)$ and displacements $u^{(f)}(X)$) are available on the whole boundary ∂V. The problem consists in searching for the shape, location and elastic moduli (in the case of inclusion) of the defect G using the available data.

3. Invariant integrals and their use in elastostatic inverse problems

The idea of the reciprocity gap principle, used in [7] for the plane crack identification, is as follows. Let us suppose that isotropic linear elastic body with shear modulus μ_M and Poisson ratio ν_M occupies the domain V. A regular elastic field in the body we'll mark by a superscript (r) $(\sigma_{ij}^{(r)}, e_{ij}^{(r)}, u^{(r)} = (u_1^{(r)}, u_2^{(r)}, u_3^{(r)}))$. Consider an integral

$$RG^{(f)}(r) = \int_S \big(t_i^{(f)}(X)u_i^{(r)}(X) - t_i^{(r)}(X)u_i^{(f)}(X)\big)\,\mathrm{d}S \tag{7}$$

where $S \subset \Omega$ is a closed surface, $t_i^{(r)}(X) = \sigma_{ij}^{(r)}(X)n_j(X)$, $n(X) = (n_1(X), n_2(X), n_3(X))$ is a unit outward normal to S.

If the surface S does not contain the domain G inside then $RG^{(f)}(r) = 0$, otherwise the values $RG^{(f)}(r)$ can differ from zero and give some information about the defect G. In the case when the loads $t^{(f)}$ and displacements $u^{(f)}$ are available on the surface of the body ∂V, it is possible to take $S = \partial V$ and for all known regular fields the values $RG^{(f)}(r)$ can be calculated. It was shown in [7] that it is possible to reconstruct a plane crack using the appropriate regular fields.

It is well known that for isotropic linear elastic solids the following invariant integrals are valid (see, [11]):

$$\begin{aligned}
&J_i = \int_S (Wn_i - t_j u_{j,i})\,\mathrm{d}S, \quad i = 1,2,3\\
&L_i = \int_S \varepsilon_{ijk}(WX_k n_j + t_j u_k - t_p u_{p,j} X_k)\,\mathrm{d}S, \quad i = 1,2,3\\
&M = \int_S \left(WX_i n_i - t_j u_{j,i} X_i - \frac{1}{2}t_i u_i\right)\mathrm{d}S
\end{aligned} \tag{8}$$

where S as above is a closed surface; σ_{ij}, e_{ij} and $u = (u_1, u_2, u_3)$ are stress and strain tensors and displacement vector corresponding to some stress–strain state of elastic body; $W = \sigma_{ij}e_{ij}/2$; $t_i = \sigma_{ij}n_j$; ε_{ijk} is the alternating tensor.

All these integrals are equal to zero if there are no defects inside S. If a defect is located inside S then the integrals can differ from zero and the values of the integrals give some information about the defect. Due to this property all invariant integrals (8) can be used for the defect detection analogously to the reciprocity gap principle given by Eq. (7).

Let us mark the invariant integrals (8) for the elastic field $u^{(f)}$ by the superscript (f): $J_i^{(f)}$, $L_i^{(f)}$, $M^{(f)}$. Consider invariant integrals for the sum of the applied and regular elastic fields and mark these integrals by the superscript $(f)+(r)$. Because the invariant integrals for the regular elastic fields are equal to zero the following equalities are valid

$$
\begin{aligned}
&J_i^{(f)+(r)} = J_i^{(f)} + J_{i\,\mathrm{int}}^{(f)}(r)\\
&L_i^{(f)+(r)} = L_i^{(f)} + L_{i\,\mathrm{int}}^{(f)}(r)\\
&M^{(f)+(r)} = M^{(f)} + M_{\mathrm{int}}^{(f)}(r)
\end{aligned}
\tag{9}
$$

where the integrals describing the interaction between the applied and regular elastic fields have the following form

$$
\begin{aligned}
&J_{i\,\mathrm{int}}^{(f)}(r) = \int_S \big(\sigma_{kl}^{(f)} e_{kl}^{(r)} n_i - t_j^{(f)} u_{j,i}^{(r)} - t_j^{(r)} u_{j,i}^{(f)}\big)\,\mathrm{d}S\\
&L_{i\,\mathrm{int}}^{(f)}(r) = \int_S \varepsilon_{ijk}\big(\sigma_{mn}^{(f)} e_{mn}^{(r)} X_k n_j + t_j^{(f)} u_k^{(r)} + t_j^{(r)} u_k^{(f)} - t_p^{(f)} u_{p,j}^{(r)} X_k - t_p^{(r)} u_{p,j}^{(f)} X_k\big)\,\mathrm{d}S\\
&M_{\mathrm{int}}^{(f)}(r) = \int_S \left(\sigma_{kl}^{(f)} e_{kl}^{(r)} X_i n_i - t_j^{(f)} u_{j,i}^{(r)} X_i - t_j^{(r)} u_{j,i}^{(f)} X_i - \frac{1}{2} t_i^{(f)} u_i^{(r)} - \frac{1}{2} t_i^{(r)} u_i^{(f)}\right)\mathrm{d}S
\end{aligned}
\tag{10}
$$

Integrals (10) are also invariant.

If we suppose that applied loads $t^{(f)}$ and displacements $u^{(f)}$ are known on the boundary ∂V then all stresses $\sigma_{ij}^{(f)}$, strains $e_{ij}^{(f)}$ and distortion tensor $u_{j,i}^{(f)}$ can be calculated on ∂V. So for $S = \partial V$ and known regular fields invariant integrals (10) can be calculated.

In the case when the sizes of the defect are small as compared to the distance between the defect and the boundary ∂V, the values of the integrals (7) and (10) only slightly differ from the integrals for the defect G, located in an infinite elastic solid. Integrals (7) and (10) for the infinite elastic body with a defect G can be expressed by means of the defect parameters and coordinates of its location. Equating the values of integrals in Eqs. (7) and (10) calculated by using the experimental data and their expressions by means of the defect parameters and coordinates one obtains a system of equations relative to the defect parameters and coordinates.

4. Identification of a spherical inclusion using one static uniaxial tension test

Let us suppose that applied loads are related to uniform uniaxial tension (compression) in the direction of the axis X_3, $t^{(f)}(X) = (0, 0, \sigma n_3(X))$, $X \in \partial V$. To emphasize the form of applied loads we'll mark below the stress–strain state of the body outside the defect G by superscript (3) instead the superscript (f).

Let a defect G be a spherical inclusion of a radius a, whose center is located at the point $M^0(x_1^0, x_2^0, x_3^0)$. Consider Cartesian coordinates $M^0 x_1 x_2 x_3$ with the origin at M^0

$$
X_i = x_i + x_i^0, \quad i = 1, 2, 3 \tag{11}
$$

Introduce the spherical coordinates with the origin at M^0

$$
x_1 = r\sin\theta\cos\varphi, \qquad x_2 = r\sin\theta\sin\varphi, \qquad x_3 = r\cos\theta \tag{12}
$$

Solution of the problem for a spherical inclusion in an infinite elastic solid under uniaxial tension (compression) in the direction of the axis x_3 was obtained in [12]. According to [12] solution of the problem outside the inclusion has the following form

$$
u_r^{(3)} = \frac{\sigma a^3}{r^2}\left\{-A - \frac{3a^2 B}{r^2} + \left[5(5-4\nu_M) - \frac{9a^2}{r^2}\right] B\cos 2\theta\right\} + \frac{\sigma r}{4\mu_M(1+\nu_M)}\big[(1-\nu_M) + (1+\nu_M)\cos 2\theta\big]
$$

$$u_\theta^{(3)} = \frac{-2B\sigma a^3 \sin 2\theta}{r^2}\left[5(1-2\nu_M) + \frac{3a^2}{r^2}\right] - \frac{\sigma r}{4\mu_M}\sin 2\theta$$
$$u_\varphi^{(3)} = 0 \tag{13}$$

$$\sigma_{rr}^{(3)} = \frac{2\mu_M\sigma a^3}{r^3}\left\{2A + \left(-10\nu_M + \frac{12a^2}{r^2}\right)B + \left[10(\nu_M - 5) + \frac{36a^2}{r^2}\right]B\cos 2\theta\right\} + \frac{\sigma}{2}(1+\cos 2\theta)$$
$$\sigma_{\theta\theta}^{(3)} = \frac{2\mu_M\sigma a^3}{r^3}\left\{-A - \left(10\nu_M + \frac{3a^2}{r^2}\right)B + \left[5(1-2\nu_M) - \frac{21a^2}{r^2}\right]B\cos 2\theta\right\} + \sigma\sin^2\theta$$
$$\sigma_{\varphi\varphi}^{(3)} = \frac{2\mu_M\sigma a^3}{r^3}\left\{-A - \left(10(1-\nu_M) + \frac{9a^2}{r^2}\right)B + 15\left[(1-2\nu_M) - \frac{a^2}{r^2}\right]B\cos 2\theta\right\}$$
$$\sigma_{r\theta}^{(3)} = \frac{2\mu_M\sigma a^3 B}{r^3}\left[-10(1+\nu_M) + \frac{24a^2}{r^2}\right]\sin 2\theta$$
$$\sigma_{r\varphi}^{(3)} = 0, \qquad \sigma_{\theta\varphi}^{(3)} = 0 \tag{14}$$

where

$$B = \frac{\mu_M - \mu_I}{8\mu_M[(7-5\nu_M)\mu_M + (8-10\nu_M)\mu_I]} \tag{15}$$
$$A = -B\frac{(1-2\nu_I)(6-5\nu_M)2\mu_M + (3+19\nu_I - 20\nu_I\nu_M)\mu_I}{2(1-2\nu_I)\mu_M + (1+\nu_I)\mu_I} + D$$
$$D = \frac{(1-\nu_M-2\nu_I\nu_M)\mu_I - (1-2\nu_I)(1+\nu_M)\mu_M}{4\mu_M(1+\nu_M)[2(1-2\nu_I)\mu_M + (1+\nu_I)\mu_I]} \tag{16}$$

For the spherical inclusion identification we'll use the following regular elastic fields with constant, linear and quadratic stresses. Below the stress tensors and displacements vectors are presented in the initial Cartesian coordinates $OX_1X_2X_3$

$$\sigma^{(C1)} = \begin{pmatrix}\sigma & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad u^{(C1)} = \frac{\sigma}{2\mu_M(1+\nu_M)}\begin{pmatrix}X_1\\ -\nu_M X_2\\ -\nu_M X_3\end{pmatrix}$$
$$\sigma^{(C2)} = \begin{pmatrix}0 & 0 & 0\\ 0 & \sigma & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad u^{(C2)} = \frac{\sigma}{2\mu_M(1+\nu_M)}\begin{pmatrix}-\nu_M X_1\\ X_2\\ -\nu_M X_3\end{pmatrix}$$
$$\sigma^{(C3)} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \sigma\end{pmatrix}, \qquad u^{(C3)} = \frac{\sigma}{2\mu_M(1+\nu_M)}\begin{pmatrix}-\nu_M X_1\\ -\nu_M X_2\\ X_3\end{pmatrix} \tag{17}$$

$$\sigma^{(L1)} = \begin{pmatrix}\frac{\sigma X_1}{L} & \frac{-\sigma X_2}{L} & 0\\ \frac{-\sigma X_2}{L} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad u^{(L1)} = \frac{\sigma}{4L\mu_M(1+\nu_M)}\begin{pmatrix}X_1^2 - (2+\nu_M)X_2^2 + \nu_M X_3^2\\ -2\nu_M X_1X_2\\ -2\nu_M X_1X_3\end{pmatrix}$$
$$\sigma^{(L2)} = \begin{pmatrix}0 & \frac{-\sigma X_1}{L} & 0\\ \frac{-\sigma X_1}{L} & \frac{\sigma X_2}{L} & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad u^{(L2)} = \frac{\sigma}{4L\mu_M(1+\nu_M)}\begin{pmatrix}-2\nu_M X_1X_2\\ -(2+\nu_M)X_1^2 + X_2^2 + \nu_M X_3^2\\ -2\nu_M X_2X_3\end{pmatrix} \tag{18}$$
$$\sigma^{(L3)} = \begin{pmatrix}0 & 0 & \frac{-\sigma X_1}{L}\\ 0 & 0 & 0\\ \frac{-\sigma X_1}{L} & 0 & \frac{\sigma X_3}{L}\end{pmatrix}, \qquad u^{(L3)} = \frac{\sigma}{4L\mu_M(1+\nu_M)}\begin{pmatrix}-2\nu_M X_1X_3\\ -2\nu_M X_2X_3\\ -(2+\nu_M)X_1^2 + \nu_M X_2^2 + X_3^2\end{pmatrix}$$

$$\sigma^{(Q)} = \begin{pmatrix}\frac{\sigma X_3^2}{L^2} & 0 & 0\\ 0 & \frac{\nu_M\sigma(X_3^2 - X_1^2)}{L^2} & 0\\ 0 & 0 & \frac{-\sigma X_1^2}{L^2}\end{pmatrix}, \qquad u^{(Q)} = \frac{\sigma}{2\mu_M L^2}\begin{pmatrix}(1-\nu_M)X_1X_3^2 + \nu_M X_1^3/3\\ 0\\ -[(1-\nu_M)X_1^2X_3 + \nu_M X_3^3/3]\end{pmatrix} \tag{19}$$

where L is a typical linear size of the domain V.

All the interaction integrals (7) and (10) for the applied elastic field (3) and regular elastic fields (Ci), (Li) and (Q) can be calculated analytically. For the calculation of the integrals they are written in the coordinates $M^0x_1x_2x_3$ and the sphere ∂G is taken as a surface S. In the expressions (17)–(19) the coordinates X_i are replaced by $x_i + x_i^0$ according to (11) and the applied elastic field (13), (14) is transformed from the spherical to Cartesian coordinates. The analytical expressions for some of interaction integrals (7), (10) are as follows

$$M_{\text{int}}^{(3)}(C1) = \frac{-6\pi(1-\nu_M)a^3\sigma^2(A+15B)}{1+\nu_M}, \qquad M_{\text{int}}^{(3)}(C2) = M_{\text{int}}^{(3)}(C1)$$

$$M_{\text{int}}^{(3)}(C3) = \frac{-6\pi(1-\nu_M)a^3\sigma^2[A-5(1+4\nu_M)B]}{1+\nu_M} = M^{(3)} \tag{20}$$

$$M_{\text{int}}^{(3)}(L1) = \frac{-6\pi(1-\nu_M)a^3\sigma^2(A+15B)}{1+\nu_M}\left(\frac{x_1^0}{L}\right)$$

$$M_{\text{int}}^{(3)}(L2) = \frac{-6\pi(1-\nu_M)a^3\sigma^2(A+15B)}{1+\nu_M}\left(\frac{x_2^0}{L}\right) \tag{21}$$

$$M_{\text{int}}^{(3)}(L3) = \frac{-6\pi(1-\nu_M)a^3\sigma^2[A-5(1+4\nu_M)B]}{1+\nu_M}\left(\frac{x_3^0}{L}\right)$$

$$RG^{(3)}(Ck) = -\frac{2}{3}M_{\text{int}}^{(3)}(Ck), \quad RG^{(3)}(Lk) = -\frac{2}{3}M_{\text{int}}^{(3)}(Lk), \quad k = 1,2,3 \tag{22}$$

Let us note that invariant J and RG integrals do not depend on the location of the Cartesian coordinates origin, but L and M integrals depend on the origin location. The presented in the formulae (20)–(22) and below M integrals are calculated in the coordinates $M^0x_1x_2x_3$. Their expressions by means of the invariant integrals calculated in the coordinates $OX_1X_2X_3$ are considered in a separate publication.

It follows from (20) and (21) that in the case when $A + 15B \neq 0$ and $A - 5(1 + 4\nu_M)B \neq 0$ the coordinates of the center M^0 of the ball G can be expressed by means of invariant integrals

$$\frac{x_1^0}{L} = \frac{M_{\text{int}}^{(3)}(L1)}{M_{\text{int}}^{(3)}(C1)}, \qquad \frac{x_2^0}{L} = \frac{M_{\text{int}}^{(3)}(L2)}{M_{\text{int}}^{(3)}(C2)}, \qquad \frac{x_3^0}{L} = \frac{M_{\text{int}}^{(3)}(L3)}{M_{\text{int}}^{(3)}(C3)} \tag{23}$$

From (22) and (23) it follows that the coordinates can be expressed also by the integrals following from the reciprocity gap principle

$$\frac{x_1^0}{L} = \frac{RG^{(3)}(L1)}{RG^{(3)}(C1)}, \qquad \frac{x_2^0}{L} = \frac{RG^{(3)}(L2)}{RG^{(3)}(C2)}, \qquad \frac{x_3^0}{L} = \frac{RG^{(3)}(L3)}{RG^{(3)}(C3)} \tag{24}$$

In the formulae (23) and (24) the coordinates of the defect center are expressed by means of one type of invariant integrals and different types of regular fields. The calculations show that the coordinates can be expressed also by different types of invariant integrals and one type of the regular fields. In particular

$$x_1^0 = \frac{2M_{\text{int}}^{(3)}(L1)}{3J_{1\,\text{int}}^{(3)}(L1)}, \qquad x_2^0 = \frac{2M_{\text{int}}^{(3)}(L2)}{3J_{2\,\text{int}}^{(3)}(L2)}, \qquad x_3^0 = \frac{2M_{\text{int}}^{(3)}(L3)}{3J_{3\,\text{int}}^{(3)}(L3)} \tag{25}$$

Now consider some special cases of the considered problem. Let us suppose that G is a spherical cavity ($\mu_I = 0$). In this case the formulae (15) and (16) have the following form

$$B = \frac{1}{8\mu_M(7-5\nu_M)}, \qquad A = \frac{10\nu_M - 13}{8\mu_M(7-5\nu_M)} \tag{26}$$

Using (26) in (20) one has

$$M_{\text{int}}^{(3)}(C3) = M^{(3)} = \frac{3\pi(1-\nu_M)(9+5\nu_M)a^3\sigma^2}{2\mu_M(1+\nu_M)(7-5\nu_M)} \tag{27}$$

$$M_{\text{int}}^{(3)}(C1) = M_{\text{int}}^{(3)}(C2) = \frac{-3\pi(1-\nu_M)(1+5\nu_M)a^3\sigma^2}{2\mu_M(1+\nu_M)(7-5\nu_M)} \tag{28}$$

It follows from (27) and (28) that if $a \neq 0$ then $M_{\text{int}}^{(3)}(Ci) \neq 0$, $i = 1, 2, 3$. In this case the radius of the defect can be calculated from (27)

$$a^3 = \frac{2\mu_M(1+\nu_M)(7-5\nu_M)M_{\text{int}}^{(3)}(C3)}{3\pi(1-\nu_M)(9+5\nu_M)\sigma^2} \tag{29}$$

The coordinates of the cavity center are calculated by means of the formulae (23) or (24) and the problem of the spherical cavity identification is completely solved.

In the case of a rigid inclusion ($\mu_I \to +\infty$) the values A and B according to (15) and (16) have the form

$$B = \frac{-1}{16\mu_M(4-5\nu_M)}, \qquad A = \frac{19-33\nu_M+20\nu_M^2}{16\mu_M(4-5\nu_M)(1+\nu_M)} \tag{30}$$

It follows from (20) and (30)

$$M_{\text{int}}^{(3)}(C1) = M_{\text{int}}^{(3)}(C2) = \frac{-3\pi(1-\nu_M)(5\nu_M^2-12\nu_M+1)a^3\sigma^2}{2\mu_M(4-5\nu_M)(1+\nu_M)^2} \tag{31}$$

$$M_{\text{int}}^{(3)}(C3) = \frac{-3\pi(1-\nu_M)(5\nu_M^2-\nu_M+3)a^3\sigma^2}{\mu_M(4-5\nu_M)(1+\nu_M)^2} \tag{32}$$

The radius of the rigid inclusion is determined from (32)

$$a^3 = \frac{-\mu_M(4-5\nu_M)(1+\nu_M)^2 M_{\text{int}}^{(3)}(C3)}{3\pi(1-\nu_M)(5\nu_M^2-\nu_M+3)\sigma^2} \tag{33}$$

Let us note that according to (31) the values $M_{\text{int}}^{(3)}(C1)$ and $M_{\text{int}}^{(3)}(C2)$ can become zero when ν_M is a root of the quadratic equation $5\nu_M^2 - 12\nu_M + 1 = 0$ in the interval $(0, 1/2)$, more precisely $\nu_M = \nu_M^0 \approx 0.0864$. So if $\nu_M \neq \nu_M^0$ then the coordinates of the rigid inclusion center are calculated by formulae (23) or (24). If $\nu_M = \nu_M^0$ then the value x_3^0/L can be calculated by the formulae (23) or (24) as before. For the calculation of the values x_1^0/L and x_2^0/L it is necessary to use some other regular elastic fields with linear stresses. Let us take for an example

$$\sigma^{(L4)} = \begin{pmatrix} \frac{\sigma X_1}{L} & \frac{-\sigma X_2}{L} & 0 \\ \frac{-\sigma X_2}{L} & 0 & 0 \\ 0 & 0 & \frac{\nu_M \sigma X_1}{L} \end{pmatrix}, \qquad u^{(L4)} = \frac{\sigma}{4\mu_M L}\begin{pmatrix} (1-\nu_M)X_1^2 + (\nu_M-2)X_2^2 \\ -2\nu_M X_1 X_2 \\ 0 \end{pmatrix}$$

$$\sigma^{(L5)} = \begin{pmatrix} 0 & \frac{-\sigma X_1}{L} & 0 \\ \frac{-\sigma X_1}{L} & \frac{\sigma X_2}{L} & 0 \\ 0 & 0 & \frac{\nu_M \sigma X_2}{L} \end{pmatrix}, \qquad u^{(L5)} = \frac{\sigma}{4\mu_M L}\begin{pmatrix} -2\nu_M X_1 X_2 \\ (\nu_M-2)X_1^2 + (1-\nu_M)X_2^2 \\ 0 \end{pmatrix} \tag{34}$$

The calculations lead to the following expressions

$$M_{\text{int}}^{(3)}(L4) = \frac{-3\pi(1-\nu_M)(1-2\nu_M)(1-5\nu_M)a^3\sigma^2}{2\mu_M(1+\nu_M)(4-5\nu_M)}\left(\frac{x_1^0}{L}\right)$$

$$M_{\text{int}}^{(3)}(L5) = \frac{-3\pi(1-\nu_M)(1-2\nu_M)(1-5\nu_M)a^3\sigma^2}{2\mu_M(1+\nu_M)(4-5\nu_M)}\left(\frac{x_2^0}{L}\right) \tag{35}$$

It follows from (32) and (35) that in the case when ν_M is equal or close to ν_M^0 the coordinates x_1^0 and x_2^0 can be calculated from the following formulae

$$\frac{x_1^0}{L} = \frac{2(5\nu_M^2-\nu_M+3)}{(1-2\nu_M)(1-5\nu_M)(1+\nu_M)}\frac{M_{\text{int}}^{(3)}(L4)}{M_{\text{int}}^{(3)}(C3)}$$

$$\frac{x_2^0}{L} = \frac{2(5\nu_M^2-\nu_M+3)}{(1-2\nu_M)(1-5\nu_M)(1+\nu_M)}\frac{M_{\text{int}}^{(3)}(L5)}{M_{\text{int}}^{(3)}(C3)} \tag{36}$$

To solve the inverse problem in a general case of spherical elastic inclusion we'll use the regular elastic field (19) with quadratic stresses. Consider some interaction integrals for the applied elastic field (3) and the regular field (Q) given by Eq. (19)

$$M_{\text{int}}^{(3)}(Q) = \frac{2(1-\nu_M)\pi a^3\sigma^2}{L^2}\left[-28Ba^2 + 3(A-5B)(x_1^0)^2 - 3(A+15B)(x_3^0)^2\right] \tag{37}$$

$$RG^{(3)}(Q) = \frac{-4(1-\nu_M)\pi a^3\sigma^2}{L^2}\left[-4Ba^2 + (A-5B)(x_1^0)^2 - (A+15B)(x_3^0)^2\right] \tag{38}$$

It follows from (37) and (38)

$$2M_{\text{int}}^{(3)}(Q) + 3RG^{(3)}(Q) = \frac{-64(1-\nu_M)\pi a^5\sigma^2 B}{L^2} \tag{39}$$

From (20) one has

$$M_{\text{int}}^{(3)}(C3) - M_{\text{int}}^{(3)}(C1) = 120(1-\nu_M)\pi a^3\sigma^2 B \tag{40}$$

It is interesting to note that the sign of the expression $M_{\text{int}}^{(3)}(C3) - M_{\text{int}}^{(3)}(C1)$ (40), coinciding with the sign of B (Eq. (15)), indicates the matrix or the inclusion is more stiff. If $M_{\text{int}}^{(3)}(C3) - M_{\text{int}}^{(3)}(C1) > 0$ then $\mu_M > \mu_I$, if $M_{\text{int}}^{(3)}(C3) - M_{\text{int}}^{(3)}(C1) < 0$ then $\mu_M < \mu_I$.

Let us suppose that $M_{\text{int}}^{(3)}(C3) - M_{\text{int}}^{(3)}(C1) \neq 0$. In this case from (39) and (40) one has

$$\frac{a^2}{L^2} = -\frac{15[2M_{\text{int}}^{(3)}(Q) + 3RG^{(3)}(Q)]}{8[M_{\text{int}}^{(3)}(C3) - M_{\text{int}}^{(3)}(C1)]} \tag{41}$$

From (40) and (41) it is possible to obtain an expression for the constant B

$$B = \frac{M_{\text{int}}^{(3)}(C3) - M_{\text{int}}^{(3)}(C1)}{120(1-\nu_M)\pi a^3\sigma^2} \tag{42}$$

where a is expressed by means of (41).

After the calculation of the value B one can calculate the shear modulus μ_I using (15)

$$\frac{\mu_I}{\mu_M} = \frac{1 - 8\mu_M(7-5\nu_M)B}{1 + 16\mu_M(4-5\nu_M)B} \tag{43}$$

It follows from (20)

$$(1+4\nu_M)M_{\text{int}}^{(3)}(C1) + 3M_{\text{int}}^{(3)}(C3) = -24\pi(1-\nu_M)a^3\sigma^2 A \tag{44}$$

From (42) and (44) one has

$$A = \frac{5B[(1+4\nu_M)M_{\text{int}}^{(3)}(C1) + 3M_{\text{int}}^{(3)}(C3)]}{M_{\text{int}}^{(3)}(C1) - M_{\text{int}}^{(3)}(C3)} \tag{45}$$

Finally one obtains an expression for the Poisson ratio ν_I of the inclusion

$$\nu_I = R/S \tag{46}$$

where

$$R = 4\mu_M(1+\nu_M)\left\{(2\mu_M + \mu_I)A + \left[2\mu_M(6-5\nu_M) + 3\mu_I\right]B\right\} + (1+\nu_M)\mu_M - (1-\nu_M)\mu_I$$

$$S = -4\mu_M(1+\nu_M)\left\{(\mu_I - 4\mu_M)A + \left[-4\mu_M(6-5\nu_M) + \mu_I(19-20\nu_M)\right]B\right\} + 2\left[(1+\nu_M)\mu_M - \nu_M\mu_I\right]$$

Table 1
Identification of a spherical cavity

Coordinates of the cavity center	The values obtained by approximate formulae			
(x_1^0, x_2^0, x_3^0)	a	x_1^0	x_2^0	x_3^0
(0, 0, 0)	0.9995	−0.0028	−0.0015	0.0019
(0, 3, 0)	0.9995	−0.0076	3.0047	0.0006
(0, 6, 0)	1.0028	−0.0119	6.0705	0.0071
(0, 8, 0)	1.0162	0.0005	8.5417	−0.0005
(0, 0, 3)	0.9991	0.0092	−0.0006	2.9968
(0, 0, 6)	1.0102	−0.0193	0.0035	5.9759
(0, 0, 8)	1.0077	0.0006	−0.0019	8.6567
(3, 3, 3)	1.0010	3.0067	3.0004	2.9963
(6, 6, 6)	1.0010	6.1185	6.1197	6.0664
(8, 8, 8)	1.1205	8.4437	8.4729	8.0198

Table 2
Identification of a spherical rigid inclusion

Coordinates of the rigid inclusion center	The values obtained by approximate formulae			
(x_1^0, x_2^0, x_3^0)	a	x_1^0	x_2^0	x_3^0
(0, 0, 0)	0.9999	0.0027	−0.0033	0.0050
(0, 3, 0)	0.9994	0.0109	3.0112	0.0094
(0, 6, 0)	1.0013	−0.0037	5.9933	−0.0166
(0, 8, 0)	1.0054	−0.0095	8.0358	0.0015
(0, 0, 3)	0.9996	0.0054	−0.0120	3.0067
(0, 0, 6)	1.0035	−0.0005	−0.0140	5.9949
(0, 0, 8)	1.0275	−0.0081	0.0015	8.0275
(3, 3, 3)	1.0004	2.9841	3.0074	2.9977
(6, 6, 6)	1.0050	5.9610	5.9753	5.9666
(8, 8, 8)	1.0387	7.9138	7.9201	8.0081

5. Numerical analysis of the obtained formulae

The obtained formulae (23), (24), (29), (33) and others are exact for a spherical defect in an infinite elastic solid, but in the case of a bounded domain they can be considered only as approximate ones. It is clear that the formulae give a good approximation in the case when the sizes of the defect are small as compared to the distance between the defect and the boundary of the body. The aim of the numerical analysis is to determine how close to the boundary of the body can be a defect so that the formulae will still be applicable.

Let $OX_1X_2X_3$ is the Cartesian coordinate system. As an example consider the cube domain V: $|X_i| \leqslant 10$, $i = 1, 2, 3$. The Poisson ratio of the matrix is $\nu_M = 0.25$. We consider below two types of the defects G: (1) G is a spherical cavity of the radius 1; (2) G is a rigid inclusion of the radius 1. The applied loads correspond to the uniform uniaxial tension in the direction of the axis X_3: $t^{(3)}(X_1, X_2, 10) = \sigma$, $t^{(3)}(X_1, X_2, -10) = -\sigma$, $t^{(3)}(\pm 10, X_2, X_3) = t^{(3)}(X_1, \pm 10, X_3) = 0$. For different locations of the defect center the direct problem was solved by the FEM and the elastic field on the surface ∂V was calculated. After that the invariant integrals were calculated and the defect parameters were determined by the formulae (29) and (23), (24) for a cavity and by the formulae (33) and (23), (24) for a rigid inclusion. We took $L = 10$ for the linear regular elastic fields.

The results of the calculations are presented in Tables 1 and 2, respectively. The numerical results show that obtained explicit formulae give a good approximation to the inverse problem solution for a spherical cavity and a rigid inclusion even when the defect is located close enough to the boundary of the body.

6. Conclusion

A method, based on the use of invariant integrals, is proposed for the defect identification. The method extends the possibilities of the reciprocity gap principle.

Explicit formulae for determination of the parameters of the spherical cavities and inclusions by means of the results of one uniaxial tension (compression) static test are obtained. These formulae are exact for the infinite elastic solids and approximate for the bounded elastic bodies.

Numerical analysis of the formulae is fulfilled. The results of the analysis show that the formulae give a good approximation of the spherical defect parameters even in a case when a defect is close enough to the body boundary.

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