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C. R. Mecanique 336 (2008) 163–169

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Duality, inverse problems and nonlinear problems in solid mechanics

Is incompressible elasticity a conformal field theory?

Markus Lazar *, Charalampos Anastassiadis

Emmy Noether Research Group, Department of Physics, Darmstadt University of Technology, Hochschulstr. 6, 64289 Darmstadt, Germany

Available online 8 January 2008

Abstract

In this work, we investigate the theory of linear isotropic incompressible elasticity as a conformal field theory. We calculate the conformal currents, the conservation laws and the balance laws of incompressible elasticity. We investigate the Euler–Lagrange symmetries, variational and divergence symmetries. If the pressure $p = 0$, the conformal group is the symmetry group for homogeneous isotropic linear incompressible elasticity without external forces. The additional symmetry is the special conformal transformation. We also discuss the symmetry breaking terms of special conformal transformations in elasticity. ***To cite this article: M. Lazar, C. Anastassiadis, C. R. Mecanique 336 (2008).***

Résumé

L'élasticité incompressible est-elle une théorie de champ conforme ? Dans ce travail, nous examinons la théorie de l'élasticité linéaire incompressible isotrope en tant que théorie conforme. Nous calculons les courants conformes, les lois de conservation et les lois de bilan de l'élasticité incompressible. Nous examinons les symétries d'Euler–Lagrange, ainsi que les symétries variationnelles et les symétries de divergence. Si la pression est nulle, le groupe conforme est le groupe de symétrie pour l'élasticité homogène linéaire isotrope incompressible en l'absence de forces extérieures. La symétrie additionnelle est la transformation spéciale conforme. Nous discutons aussi les termes des transformations spéciales conformes de l'élasticité qui brisent la symétrie. ***Pour citer cet article : M. Lazar, C. Anastassiadis, C. R. Mecanique 336 (2008).***

Keywords: Incompressible elasticity; Conservation laws; Eshelby stress tensor; Conformal field theory

Mots-clés : Élasticité incompressible ; Lois de conservation ; Tenseur des contraintes d'Eshelby ; Théorie de champ conforme

1. Introduction

Conformal symmetries are very important for field theories (see, e.g, [1]). Lagrangian field theories which are formally invariant under dilatation are often invariant also under special conformal transformations. Conformal transformations are non-linear. The special conformal transformations can be thought as product of inversion × translation × inversion. Special conformal transformations may be interpreted as space (or space–time) dependent dilatations. A conformal field theory has more symmetries then a usual field theory. For example the Lagrangian of

* Corresponding author.
E-mail address: lazar@fkp.tu-darmstadt.de (M. Lazar).

1631-0721/$ – see front matter
doi:10.1016/j.crme.2007.11.006

the free Maxwell field, the massless Dirac-field and their gauge-covariant coupling are invariant under the conformal group. Also pure Yang–Mills theory is conformally invariant.

Elasticity can also be considered as a field theory [2]. Elasticity is invariant under the symmetry of translations, rotations and dilatations [3,4]. But elasticity is not a conformal field theory, because it is not invariant under special conformal transformations. From the field theoretical point of view the theory of elasticity is a massless field theory.

In this article we ask the question: what about conformal symmetry in elasticity? For some special cases such a symmetry exists in elasticity. In two dimensional elasticity Li [5] found for the case $\nu = 1/2$ (the incompressible limit) that special conformal symmetry exists if one uses a stress function of second order instead of a displacement vector. He found a dual conservation law of special conformal symmetry. Does a special conformal symmetry exist in three-dimensional elasticity? Kienzler and Herrmann [6] ask the question: do conservation laws, quadratic in x_k, exist in three-dimensional elasticity? Olver [4] found such a symmetry for the condition $7\mu + 3\lambda = 0$, i.e. $\nu = 7/8$. He used a displacement vector and found a conservation law for special conformal transformation. Unfortunately this condition violates the requirement of positive strain energy. Thus, this case is unphysical. Does the special conformal symmetry exist in three-dimensions for a physical situation? What are the symmetries of incompressible elasticity?

2. Equations of linear isotropic incompressible elasticity

Incompressible elasticity is a special case of classical theory of elasticity. Materials that deform without a volume change are called incompressible. In isotropic linear elasticity the incompressibility condition is usually expressed in terms of Poisson's ratio ν. The limiting value $\nu = 1/2$ defines incompressibility. In this limit the Hooke's law is no longer valid since the Lamé constant $\lambda = 2\mu\nu/(1-2\nu)$ tends to infinity, the pressure cannot be determined from the gradient of the displacement vector and there is a need for another constitutive equation (see, e.g., [7]). In absence of body forces the elastic energy of linear homogeneous isotropic incompressible elasticity is given by

$$W = \frac{1}{2}\mu(u_{\alpha,j} + u_{j,\alpha})u_{\alpha,j} - p u_{\alpha,\alpha}, \quad \mu > 0 \tag{1}$$

where $\boldsymbol{u}$ is the displacement vector, p is the pressure and μ is the shear modulus. In this formulation the stress tensor reads

$$t_{\alpha j} = \frac{\partial W}{\partial u_{\alpha,j}} = \mu(u_{\alpha,j} + u_{j,\alpha}) - \delta_{\alpha j} p \tag{2}$$

The corresponding Euler–Lagrange equations are

$$E_\alpha(W) := \mathrm{D}_j \frac{\partial W}{\partial u_{\alpha,j}} - \frac{\partial W}{\partial u_\alpha} = t_{\alpha j,j} = \mu \Delta u_\alpha - p_{,\alpha} = 0 \tag{3}$$

$$E(W) := \mathrm{D}_j \frac{\partial W}{\partial p_{,j}} - \frac{\partial W}{\partial p} = u_{\alpha,\alpha} = 0 \tag{4}$$

where Δ is the Laplacian. Here D_j denotes the so-called total derivative (see [8]). Eq. (3) is an inhomogeneous Laplace equation for the vector filed $\boldsymbol{u}$. The gradient of the pressure gives the inhomogeneous part. The additional equation (4) is the kinematic condition of incompressibility. It has the form like the Coulomb gauge of the magnetic vector potential in Maxwell's theory. The pressure p can also be understood as the Lagrange multiplier associated with the constraint (4). Applying D_α to Eq. (3) and taking into account the relation (4), one finds that the pressure p satisfies the Laplace equation

$$\Delta p = 0 \tag{5}$$

On the other hand, the trace of the stress tensor (2) is

$$t_{\alpha\alpha} = 2\mu u_{\alpha,\alpha} - np = -np \tag{6}$$

where $n = \delta_{\alpha\alpha}$.

3. Conformal Lie algebra ($n \geqslant 3$)

If $p = 0$, the Euler–Lagrange equation (3) is invariant under the Lie groups generated by the operators (see for a scalar field, e.g., Ibragimov [9]):

$$T_j = \frac{\partial}{\partial x_j} \qquad \text{(translations)} \tag{7}$$

$$L_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u_j} - u_j \frac{\partial}{\partial u_i} \qquad \text{(rotations)} \tag{8}$$

$$D_1 = x_i \frac{\partial}{\partial x_i} \qquad \text{(dilatations of } \boldsymbol{x}) \tag{9}$$

$$D_2 = u_i \frac{\partial}{\partial u_i} \qquad \text{(dilatations of } \boldsymbol{u}) \tag{10}$$

$$C_j = 2x_j x_l \frac{\partial}{\partial x_l} - x^2 \frac{\partial}{\partial x_j} + 2d_u x_j u_l \frac{\partial}{\partial u_l} + 2x_l \left(u_j \frac{\partial}{\partial u_l} - u_l \frac{\partial}{\partial u_j} \right) \quad \text{(special conformal transformations)} \tag{11}$$

We note that the 'spinorial part' of L_{ij} is

$$S_{ij} = u_i \frac{\partial}{\partial u_j} - u_j \frac{\partial}{\partial u_i} \tag{12}$$

The special conformal transformations act nontrivially on both $\boldsymbol{x}$ and $\boldsymbol{u}$. We combine the dilatations as follows:

$$D = D_1 + d_u D_2 \tag{13}$$

with the (scale) dimension of the vector field u_α:

$$d_u = -\frac{n-2}{2} \tag{14}$$

Here n denotes the space dimension (for $n = 3$ we have $d_u = -1/2$). These generators span the Lie algebra of the conformal group in n dimensions, $so(n+1, 1)$. The nontrivial commutation relations are [10]:

$$\begin{aligned} &[L_{ij}, L_{kl}] = -(\delta_{ik} L_{jl} - \delta_{il} L_{jk} - \delta_{jk} L_{il} + \delta_{jl} L_{ik}), && [L_{ij}, T_k] = -(\delta_{ik} T_j - \delta_{jk} T_i) \\ &[D, C_j] = C_j, && [D, T_j] = -T_j \\ &[C_j, T_k] = -2(\delta_{jk} D + L_{jk}), && [L_{ij}, C_l] = -(\delta_{il} C_j - \delta_{jl} C_i) \end{aligned} \tag{15}$$

If $p \neq 0$, the Euler–Lagrange equations (3) and (4) are invariant under the Lie groups generated by (7)–(10) and

$$D_3 = p \frac{\partial}{\partial p} \qquad \text{(dilatation of } p) \tag{16}$$

Thus, in this case the special conformal transformation (11) is not a symmetry of the Euler–Lagrange equations.

4. Conformal currents

In order to calculate the conformal currents, we have used the mathematical technique of prolongation [9,8,4,6]. It is well known that every variational and divergence symmetry of the elastic energy is also a symmetry of the associated Euler–Lagrange equations. On the other hand, not every symmetry of the Euler–Lagrange equations is a variational or divergence symmetry. Thus, we will use the symmetries of the Euler–Lagrange equations found in the previous section in order to construct the associated currents or fluxes. If they are divergence-less, we deal with a variational or divergence symmetry.

The static energy momentum tensor, called Eshelby stress tensor [11], is the translational flux. It corresponds to the generator of translations (7). The Eshelby stress tensor has the form

$$P_{lj} = W \delta_{lj} - u_{\alpha,l} t_{\alpha j} \tag{17}$$

To calculate the total angular momentum tensor we used the generator of rotations (8). The total angular momentum tensor is given as

$$M_{[mi]j} = x_m P_{ij} - x_i P_{mj} + u_m t_{ij} - u_i t_{mj} \\ = x_m P_{ij} - x_i P_{mj} + u_\alpha \Sigma_{mi}^{\alpha\beta} t_{\beta j} \tag{18}$$

with the representation matrices of infinitesimal generator of the rotational group ($SO(n)$) for a vector field:

$$\Sigma_{mi}^{\alpha\beta} = \delta_m^\alpha \delta_i^\beta - \delta_i^\alpha \delta_m^\beta \tag{19}$$

It can be decomposed into

$$M_{[mi]j} = M_{[mi]j}^{(\mathrm{o})} + M_{[mi]j}^{(\mathrm{i})} \tag{20}$$

with the orbital angular momentum tensor

$$M_{[mi]j}^{(\mathrm{o})} = x_m P_{ij} - x_i P_{mj} \tag{21}$$

and the spin (intrinsic) angular momentum tensor

$$M_{[mi]j}^{(\mathrm{i})} = u_\alpha \Sigma_{mi}^{\alpha\beta} t_{\beta j} \tag{22}$$

The spin part is connected with the polarization properties of a field.

The scaling flux corresponds to the generator of scaling symmetry (13). It reads

$$Y_j = x_l P_{lj} + d_u u_\alpha t_{\alpha j} \tag{23}$$

In order to calculate the flux of special conformal transformations we have used the generator (11). The flux of special conformal transformations is given by

$$I_{lj} = 2x_l x_i P_{ij} - x^2 P_{lj} + 2t_{\alpha j} x_i \big(d_u\, \delta_{il} u_\alpha + \Sigma_{il}^{\alpha\beta} u_\beta\big) - B_{lj} \tag{24}$$

where B_{lj} is a tensor which we have to determine from the conservation law of special conformal transformations. If I_{lj} is divergence-free in the second index and B_{lj} is non-zero, the special conformal symmetry is a divergence symmetry.

5. Divergence of the currents—conservation laws

We now turn to the discussion of the properties of the conformal currents. The information of a symmetry defined by the transformation law of the fields lies in the properties of the divergence of the corresponding currents. If the divergence is zero, we speak of a conservation law. If it is not zero, we have a balance law.

We start with the divergence of the translational current. The divergence of the Eshelby stress tensor is zero:

$$\mathrm{D}_j P_{lj} = 0 \tag{25}$$

Thus, it is a conservation law.

The angular momentum conservation reads

$$\mathrm{D}_j M_{[mi]j} = x_m \mathrm{D}_j P_{ij} - x_i \mathrm{D}_j P_{mj} + P_{im} - P_{mi} + \mathrm{D}_j \big(u_\alpha \Sigma_{mi}^{\alpha\beta} t_{\beta j}\big) \tag{26}$$

With Eqs. (25) and $\mathrm{D}_j M_{[mi]j} = 0$, we get

$$\mathrm{D}_j M_{[mi]j}^{(\mathrm{i})} - (P_{mi} - P_{im}) = 0 \tag{27}$$

and with (3) and (22), we obtain

$$P_{mi} - P_{im} = u_{\alpha,j} \Sigma_{mi}^{\alpha\beta} t_{\beta j} \tag{28}$$

Using (17), Eq. (26) can be rewritten:

$$\mathrm{D}_j M_{[mi]j} = u_{l,m} t_{li} - u_{l,i} t_{lm} + u_{m,l} t_{il} - u_{i,l} t_{ml} \tag{29}$$

which is nothing but the so-called isotropy condition. Thus, for isotropic incompressible elasticity it is zero: $\mathrm{D}_j M_{[mi]j} = 0$ and it is a conservation law. It is, of course, the total angular momentum which is conserved.

From Eq. (23) it is seen that the dilatation current Y_j depends on x explicitly. We find its divergence

$$\mathrm{D}_j Y_j = x_l \mathrm{D}_j P_{lj} + P_{jj} + \mathrm{D}_j(d_u u_\alpha t_{\alpha j}) \tag{30}$$

It can be rewritten in the form

$$\mathrm{D}_j Y_j = nW + d_u u_\alpha \frac{\partial W}{\partial u_\alpha} + (d_u - 1) u_{\alpha,j} t_{\alpha j} + x_l \mathrm{D}_j P_{lj} \tag{31}$$

which is nothing but the condition for scale invariance. Because the strain energy density (1) is bilinear in $u_{\alpha,j}$, it does not depend on u_α and if the momentum conservation (25) is valid, such a strain energy is scale invariant. Thus, it holds: $\mathrm{D}_j Y_j = 0$ and it is a conservation law.

Next we turn to the divergence for the entire conformal current (24). So we find from Eq. (24)

$$\begin{aligned} \mathrm{D}_j I_{lj} = {} & (2x_l x_i - x^2 \delta_{il}) \mathrm{D}_j P_{ij} + 2x_l \big(P_{jj} + \mathrm{D}_j [d_u u_\alpha t_{\alpha j}]\big) + 2x_i \big(P_{il} - P_{li} + \mathrm{D}_j [t_{\alpha j} \Sigma_{il}^{\alpha\beta} u_\beta]\big) \\ & + 2 d_u t_{\alpha l} u_\alpha + 2 t_{\alpha j} \Sigma_{jl}^{\alpha\beta} u_\beta - \mathrm{D}_j B_{lj} \end{aligned} \tag{32}$$

The first part of the first line is equal to the momentum conservation, the second part in the first line is related to the scale invariance (30) and the third part in the first line is related to the angular momentum conservation (26). Hence, we can rewrite it in the following form

$$\mathrm{D}_j I_{lj} = -\big(2x_l x_i - x^2 \delta_{il}\big) \mathrm{D}_j P_{ij} + 2x_l \mathrm{D}_j Y_j + 2x_i \mathrm{D}_j M_{[li]j} + R_l \tag{33}$$

where R_l is a local vector field defined by

$$R_l = 2 d_u t_{\alpha l} u_\alpha + 2 t_{\alpha j} \Sigma_{jl}^{\alpha\beta} u_\beta - \mathrm{D}_j B_{lj} \tag{34}$$

If translational invariance, scale invariance and angular momentum conservation are fulfilled, we have the following condition of entire conformal invariance:

$$R_l = 0, \qquad 2 d_u t_{\alpha l} u_\alpha + 2 t_{\alpha j} \Sigma_{jl}^{\alpha\beta} u_\beta - \mathrm{D}_j B_{lj} = 0 \tag{35}$$

Thus, if $R_l = 0$, the theory is invariant under special conformal transformations.

Let us now calculate R_l for linear isotropic incompressible elasticity to prove if it is invariant under special conformal transformations. We start with

$$R_l = -n t_{\alpha l} u_\alpha + 2 t_{\alpha\alpha} u_l - \mathrm{D}_j B_{lj} \tag{36}$$

Using Eqs. (2) and (6), we obtain

$$R_l = -\mathrm{D}_j \left[n\mu \left(\frac{1}{2} \delta_{jl} u_i u_i + u_l u_j \right) \right] - npu_l - \mathrm{D}_j B_{lj} \tag{37}$$

Therefore, it is not possible to rewrite it as a divergence only. Anyway, we can determine the field B_{lj} as follows:

$$B_{lj} = -n\mu \left(\frac{1}{2} \delta_{lj} u_i u_i + u_l u_j \right) \tag{38}$$

and R_l,

$$R_l = -npu_l \tag{39}$$

Thus, in general, the condition of entire conformal invariance is not valid for linear incompressible elasticity due to $\mathrm{D}_j I_{lj} = -npu_l$. Iff $p = 0$, isotropic linear incompressible elasticity is a conformal field theory.

Using the conservation laws (25), (26), (30) and (33), we can define path-independent integrals:

$$J_l = \int_S P_{lj} n_j \, \mathrm{d}S = 0 \tag{40}$$

$$L_{[mi]} = \int_S M_{[mi]j} n_j \, \mathrm{d}S = 0 \tag{41}$$

$$M = \int_S Y_j n_j \, \mathrm{d}S = 0 \tag{42}$$

$$K_l = \int_S I_{lj} n_j \, \mathrm{d}S = -\int_V n p u_l \, \mathrm{d}V \tag{43}$$

Eqs. (40), (41) and (42) are the so-called J, L and M integrals for incompressible elasticity as in elasticity (see, e.g., [2,6]). The additional integral (43), which appears in incompressible elasticity, we call the K integral. It corresponds to the symmetry of special conformal transformation. It is only zero if $p = 0$.

Let us now investigate the condition of special conformal invariance for isotropic linear elasticity. Because the special conformal transformation is not a symmetry in elasticity apart from the (unphysical) case $7\mu + 3\lambda = 0$ found by Olver [4], we want the determine the symmetry breaking terms. For isotropic linear elasticity the stress tensor has the form

$$t_{\alpha j} = \mu(u_{\alpha,j} + u_{j,\alpha}) + \lambda \delta_{\alpha j} u_{l,l}, \qquad 2\mu + n\lambda > 0 \tag{44}$$

If we substitute it into Eq. (34), we can rewrite it as a divergence and a term violating $R_l = 0$:

$$R_l = -\mathrm{D}_j\left[n\mu\left(\frac{1}{2}\delta_{jl} u_i u_i + u_l u_j\right)\right] + \left[(4+n)\mu + n\lambda\right] u_{j,j} u_l - \mathrm{D}_j B_{lj} \tag{45}$$

So, we see that B_{lj} has the same form as given in Eq. (38). It is not possible to rewrite it as a total divergence. The symmetry breaking term is calculated as

$$R_l = \left[(4+n)\mu + n\lambda\right] u_{j,j} u_l \tag{46}$$

We obtain the result that the current of special conformal transformations is not divergence-less

$$\mathrm{D}_j I_{lj} = \left[(4+n)\mu + n\lambda\right] u_{j,j} u_l \tag{47}$$

Finally $R_l = 0$ iff $(4+n)\mu + n\lambda = 0$ or $u_{j,j} = 0$. The first possibility is for $n = 3$, the case $7\mu + 3\lambda = 0$ found by Olver [4]. The second possibility corresponds to the case $p = 0$ in incompressible elasticity. In both cases the special conformal current (24) is expressed with (38) and (44) and the special conformal transformation is a divergence symmetry of the elastic energy. In addition, for both cases the K integral is zero.

6. Balance laws

Up to now, we have examined conservation laws for homogeneous incompressible elasticity without external sources. Now we want to investigate balance laws for nonhomogeneous incompressible elasticity with external forces. We postulate that the strain energy density to be of the form

$$W = \frac{1}{2} t_{\alpha j} u_{\alpha,j} + V, \quad V = u_\alpha \frac{\partial V}{\partial u_\alpha} \tag{48}$$

where V is the potential of external forces. Let us assume that it depends explicitly on x_i:

$$W = W(x_i, u_\alpha, u_{\alpha,i}, p) \tag{49}$$

In this case, the material force (or inhomogeneity force) is defined by

$$f_i^{\mathrm{inh}} := -\frac{\partial W}{\partial x_i} \tag{50}$$

which is caused by material inhomogeneities. External body forces are defined by

$$F_\alpha := -\frac{\partial V}{\partial u_\alpha} \tag{51}$$

The Euler–Lagrange equation (3) with external forces (51) has now the form:

$$\mathrm{D}_j t_{\alpha j} = -F_\alpha \tag{52}$$

Finally, we obtain the following balance law:

$$\mathrm{D}_j P_{lj} = -f_l^{\mathrm{inh}} \tag{53}$$

$$\mathrm{D}_j M_{[mi]j} = u_{l,m}t_{li} - u_{l,i}t_{lm} + u_{m,l}t_{il} - u_{i,l}t_{ml} + x_i f_m^{\mathrm{inh}} - x_m f_i^{\mathrm{inh}} + u_i F_m - u_m F_i \tag{54}$$

$$\mathrm{D}_j Y_j = -x_j f_j^{\mathrm{inh}} - \frac{n+2}{2} u_\alpha F_\alpha \tag{55}$$

$$\begin{aligned}\mathrm{D}_j I_{lj} = {} & 2x_i(u_{j,l}t_{ji} - u_{j,i}t_{jl} + u_{l,j}t_{ij} - u_{i,j}t_{lj}) + 2x_i(u_i F_l - u_l F_i) - (n+2)x_l u_\alpha F_\alpha \\ & - (2x_l x_i - x^2\delta_{il}) f_i^{\mathrm{inh}} - npu_l\end{aligned} \tag{56}$$

It can be seen how the external force and the inhomogeneity force break the conservation laws.

Acknowledgements

The authors have been supported by an Emmy-Noether grant of the Deutsche Forschungsgemeinschaft (Grant No. La1974/1-2).

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