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Duality, inverse problems and nonlinear problems in solid mechanics

Stress inversion method and analysis of GPS array data

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Abstract

The stress inversion method is developed to find a stress field which satisfies the equation of equilibrium for a body in a state of plane stress. When one stress–strain relation is known and data on the strain distribution on the body and traction along the boundary are provided, the method solves a well-posed problem, which is a linear boundary value problem for Airy's stress function, with the governing equation being the Poisson equation and the boundary conditions being of the Neumann type. The stress inversion method is applied to the Global Positioning System (GPS) array data of the Japanese Islands. The stress increment distribution, which is associated with the displacement increment measured by the GPS array, is computed, and it is found that the distribution is not uniform over the islands and that some regions have a relatively large increment. The elasticity inversion method is developed as an alternative to the stress inversion method; it is based on the assumption of linear elastic deformation with unknown elastic moduli and does not need boundary traction data, which are usually difficult to measure. This method is applied to the GPS array data of a small region in Japan to which the stress inversion method is not applicable. ***To cite this article: M. Hori et al., C. R. Mecanique 336 (2008).***

Résumé

Méthode d'inversion des contraintes et analyse des données GPS. La méthode d'inversion est développée pour trouver un champ de contraintes satisfaisant les équations d'équilibre pour un corps en situation de contrainte plane. La relation entre contrainte et déformation étant connue et les données de distribution de déformation sur le corps et de traction sur le bord étant fournies, la méthode résout un problème bien posé, qui consiste en un problème aux limites linéaire pour la fonction de contrainte d'Airy, comprenant l'équation de Poisson et des conditions aux limites de type Neumann. La méthode d'inversion est appliquée aux données GPS (Global Positioning System) concernant les îles japonaises. Les incréments de contraintes associés aux incréments de déplacements mesurés par le système GPS sont calculés, et l'on trouve que leur distribution n'est pas uniforme sur les îles, certaines régions présentant un incrément relativement grand. La méthode d'inversion élastique est développée comme alternative à la méthode d'inversion des contraintes ; elle est fondée sur l'hypothèse d'une déformation linéaire élastique avec des coefficients d'élasticité inconnus, et ne requiert pas de données concernant les tractions au bord, généralement difficilement mesurables. La méthode est appliquée aux données GPS d'une petite région du Japon pour laquelle la méthode d'inversion des contraintes n'est pas utilisable. ***Pour citer cet article : M. Hori et al., C. R. Mecanique 336 (2008).***

Keywords: Dynamical systems; Inverse analysis; Crustal deformation; GPS (Global Positioning System); Identification of local constitutive relation

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Mots-clés : Systèmes dynamiques ; Analyse inverse ; Déformation de la croûte ; GPS (Global Positioning System) ; Identification de la relation constitutive locale

1. Introduction

Due to the advancement of computational mechanics, numerical analysis of a structure has reached the level that commercial applications are used in most practical problems. For more advanced analysis, a better understanding of the material properties which are associated with fracture is required. A new material test is needed for such an understanding. While conventional material tests measure overall stress and strain of the sample, which is assumed to be uniform, a new test ought to seek local stress and strain; a fracture always stems from a local point of the material. Recent measurement technology, such as image analysis, enables us to measure the spatial distribution of displacement or strain. However, it is not possible to estimate local stress from strain measurements, when the stress–strain relation is not known, and a device which measures local stress is not yet developed.

The identification of local stress is also needed in the geological length scale. For the prediction of an earthquake, it is essential to estimate the regional stress state which triggers the occurrence of an earthquake as a fracture of the crust. An array of Global Positioning System (GPS) stations has been installed in Japan to monitor the crustal deformation of the Japanese Islands [1]. Regional displacement and strain increments during the observation period are measured. However, the stress increment, which is associated with the strain increment, cannot be estimated since regional stress–strain relations are not known.

According to Bui's celebrated work [2], we regard the identification of local stress as an ill-posed problem; while data for strain are available, stress–strain relations are not known. It is true that stress is determined from strain if the stress–strain relations are known. However, this does not imply that stress cannot be determined from strain if the stress–strain relations are not known. Indeed, a mathematical problem can be constructed in order to identify local stress even if the stress–strain relation is only partially known. This mathematical problem is stated as follows:

> *For a body in a state of plane stress, identify stress components which satisfy the equation of equilibrium, using measured strain components and one stress–strain relation.*

The stress inversion takes advantage of the equilibrium equation that the stress field satisfies. Since the number of the stress components and the equilibrium equation is three and two, respectively, one equation is needed to determine the stress components. One stress–strain relation (or partial information of the constitutive relation) serves as the last equation. The mathematical problem stated above is well posed in the sense that the stress components are uniquely determined when data of strain distribution in the body are provided; see a list of related references [3–5]; see also references [6–8].

We name the method of solving the above problem as a stress inversion method [9,10]. This article briefly explains the stress inversion method and presents several examples of applying this method to the GPS array data. The content of the present paper is as follows: In Section 2, the stress inversion method is formulated. The formulation uses the Airy's stress function which produces self-equilibrating stress components for a body in a state of plane stress. Some assumptions must be made in applying the stress inversion method to the GPS array data, and these assumptions are explained in Section 3. Results of numerical analysis of the GPS array data by means of the stress inversion method are presented, too. In Section 4, another inversion method, which does not need assumptions of the stress inversion method, but is applicable to linear elastic bodies, is explained. Results of analyzing the GPS array data using this method are presented. Concluding remarks are made in Section 5.

This paper uses two-dimensional Cartesian coordinates, denoted by x_i. Index notation is used for a vector or tensor quantity, the summation convention is employed, and indices following a comma denote partial differentiation with respect to the corresponding coordinates.

2. Formulation of stress inversion method

We consider a thin material sample in a state of plane stress, and denote the surface and the boundary of the sample by S and ∂S, respectively; see Fig. 1. While the material properties are not known for this sample, it is assumed that

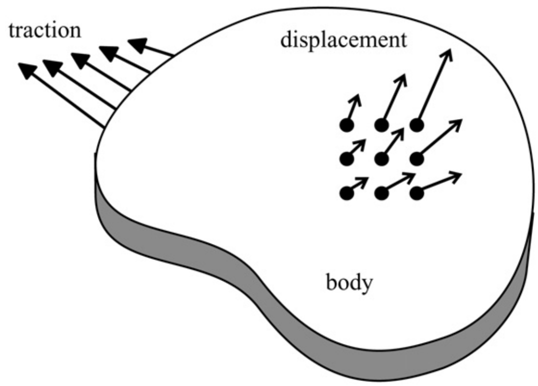


Fig. 1. A body in the plane stress state, and data input to the stress inversion method.

one relation between the two-dimensional dilatant or volumetric stress, $\sigma = \sigma_{ii}$, and strain, is known; the relation is given as

$$\sigma = f(\boldsymbol{\epsilon}) \tag{1}$$

where f is a function of strain tensor, $\boldsymbol{\epsilon}$ or ϵ_{ij}.

The equation of equilibrium for the three stress components is automatically satisfied by introducing Airy's stress function, denoted by a. The three components are given as

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_{,22} \\ a_{,11} \\ -a_{,12} \end{bmatrix} \tag{2}$$

The assumed stress–strain relation, Eq. (1), is used as a governing equation for a, when the local strain is measured and the value of f is determined. Indeed, the governing equation for a is

$$a_{,ii}(\mathbf{x}) = f\big(\boldsymbol{\epsilon}(\mathbf{x})\big) \quad \text{for } \mathbf{x} \text{ in } S \tag{3}$$

It should be recalled that a governing equation for Airy's stress function is usually derived from the compatibility condition of strain when the stress–strain relations are fully known; for instance, when the stress–strain relations are linearly isotropic, the governing equation for Airy's stress function becomes a homogeneous bi-harmonic equation. In the present case, however, strain cannot be expressed in terms of Airy's stress function since the stress–strain relations are only partially known. One stress–strain relation, Eq. (1), is used to derive the governing equation, Eq. (3), when strain, which is measured on the surface S, is used as a non-homogeneous term.

The governing equation for a is Poisson's equation. If suitable boundary conditions are given, we can uniquely determine a by solving the resulting boundary value problem. Such boundary conditions are obtained when traction is measured on the boundary ∂S. Denoting the boundary traction and the unit outer normal by t_i and n_i, respectively, we have $t_j = n_i \sigma_{ij}$ or

$$\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} n_1 a_{,22} - n_2 a_{,12} \\ -n_1 a_{,12} + n_2 a_{,11} \end{bmatrix} \tag{4}$$

Since the unit tangential vector, denoted by s_i, is given as $[s_1, s_2]^T = [-n_2, n_1]^T$, we can rewrite the left side of Eq. (4) as

$$\begin{bmatrix} n_1 a_{,22} - n_2 a_{,12} \\ -n_1 a_{,12} + n_2 a_{,11} \end{bmatrix} = \begin{bmatrix} \frac{\partial a_{,2}}{\partial s} \\ -\frac{\partial a_{,1}}{\partial s} \end{bmatrix}$$

Thus, integrating the boundary tractions, we obtain the resultant force, $r_i = \int t_i \, \mathrm{d}s$, which is expressed in terms of a as

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} a_{,2} \\ -a_{,1} \end{bmatrix}$$

As is seen, the derivative of a is given by the resultant force. Therefore, we can prescribe Neumann boundary conditions for a, using the resultant force, as follows:

$$n_i(\mathbf{x})a_{,i}(\mathbf{x}) = g\big(\mathbf{r}(\mathbf{x})\big) \quad \text{for } \mathbf{x} \text{ in } S \tag{5}$$

where $g(\mathbf{r}) = -n_1 r_2 + n_2 r_1$. The resultant force is determined by integrating the boundary traction when the boundary traction is measured.

The governing equation and the boundary conditions, Eqs. (3) and (5), form a linear boundary value problem for a. That is,

$$\begin{cases} a_{,ii}(\mathbf{x}) = f(\bar{\boldsymbol{\epsilon}}(\mathbf{x})), & \mathbf{x} \text{ in } S \\ n_i(\mathbf{x})a_{,i}(\mathbf{x}) = g(\bar{\mathbf{r}}(\mathbf{x})), & \mathbf{x} \text{ on } \partial S \end{cases} \tag{6}$$

Here, the bar on strain and resultant force emphasizes that they are the measured data. It should be emphasized that the information about the form of f and the measured data of strain distribution on S and traction distribution along ∂S are needed to form Eq. (6).

We must pay attention to the *consistency* of Eq. (6). Since the governing equation is Poisson's equation, the surface integration of it coincides with the line integration of $n_i a_{,i}$, i.e.,

$$\int_S a_{,ii}\,\mathrm{d}s = \int_{\partial S} n_i(\mathbf{x})a_{,i}(\mathbf{x})\,\mathrm{d}\ell$$

It follows from Eq. (6) that the right side of the governing equation and the boundary conditions must satisfy

$$\int_S f\big(\bar{\boldsymbol{\epsilon}}(\mathbf{x})\big)\,\mathrm{d}s = \int_{\partial S} g\big(\bar{\mathbf{r}}(\mathbf{x})\big)\,\mathrm{d}\ell \tag{7}$$

This is the consistency condition for the measured data of strain and traction. When the measured data satisfies Eq. (7), the stress components are uniquely determined by solving Eq. (6). Due to the nature of the Neumann boundary conditions, an arbitrary constant can be added to the solution of a. The constant, however, does not change the stress components since they are computed by taking the derivative of a; see Eq. (2).

The solution of Eq. (6), and hence the local stress components, are sensitive to measured local strain, but the dependence of the local stress on the measured strain is *stable* in the sense the stress does not change abruptly unless the stress does. As is seen, f, which provides dilatant stress from strain, does not have to be linear. The boundary value problem for a remains linear since f gives a non-homogeneous term of Eq. (6). This linear boundary value problem is applicable to any arbitrary material sample which is in a quasi-static state. The authors have applied this linear boundary value problem to a non-linear elasto-plastic material sample in order to estimate the plastic deformation [9].

It is interesting to extend the stress inversion method such that it can be applied to a material sample at a dynamic state. The equation of equilibrium is replaced by the equation of motion, i.e.,

$$\sigma_{ij,i}(\mathbf{x},t) = \rho\ddot{u}_i(\mathbf{x},t)$$

where $\ddot{u}_i = \frac{\partial^2 u_i}{\partial t^2}$ and ρ is density of the material. Thus, the distribution of acceleration must be measured in order to use the equation of motion to determine the stress components. When they are measured, the governing equations for the stress components are written as

$$\begin{cases} \sigma_{ij,i}(\mathbf{x},t) = \rho\ddot{\bar{u}}_i(\mathbf{x},t) \quad (i = 1,2) \\ \sigma_{ii}(\mathbf{x},t) = f(\bar{\boldsymbol{\epsilon}}(\mathbf{x},t)) \end{cases} \tag{8}$$

The boundary conditions are the same as Eq. (5). Suitable initial conditions must be found; for instance, when no loading is applied to the material sample, we may assume $\sigma_{ij} = 0$ as the initial conditions. Together with these boundary and initial conditions, the three equations of Eq. (8) form a linear initial-boundary value problem for the three stress components. This linear problem is applicable to non-linear materials, just as Eq. (6) is applicable to non-linear materials in quasi-static state.

3. GPS array analysis by the stress inversion method

A nation-wide GPS array that consists of more than 1000 stations has been operating in Japan since 1994; see [1] and Fig. 2. Each station measures daily the location, with the accuracy being better than 1 cm. While the displacement distribution itself provides vital data for the crustal deformation, further analysis is needed for these measured data in order to understand the mechanism of the crustal deformation that triggers an earthquake. The authors are applying the stress inversion method to the GPS array data, to develop a system which monitors the change in regional strain and strain. This system makes use of the stress inversion method.

It is necessary to make the assumption of the plane stress state for the Japanese Islands, in applying the stress inversion method to the GPS array data. This assumption is not correct since the islands are a part of the Eurasia Plate under which Pacific and Philippine Sea Plates are subducting; the North America Plate is pushing the Eurasia Plate. Also, the curvature of the Earth needs to be taken into consideration. However, making this assumption could be acceptable if the target of analysis is a thin surface layer of the Japanese Islands. This is because the surface of the Japanese Islands is traction free and the effects of gravity and past plate movement on the islands' *permanent* deformation are not observed by the GPS array; in the time scale of the GPS array operation, deformation due to gravity or deformation caused by the past plate movement is regarded as permanent.

In applying the stress inversion method to the GPS array data, we use an increment formulation to emphasize the fact that the displacement measured by the array is the displacement during the observation period; the array cannot measure permanent displacement. Since the array provides displacement increment at the GPS stations, a strain increment field can be computed by interpolating the measured displacement increment.

There are two tasks needed to apply the stress inversion method to the GPS array data. The first task is to set the function for the dilatant stress increment, f. For the inelastic deformation of the Japanese Islands, we may assume

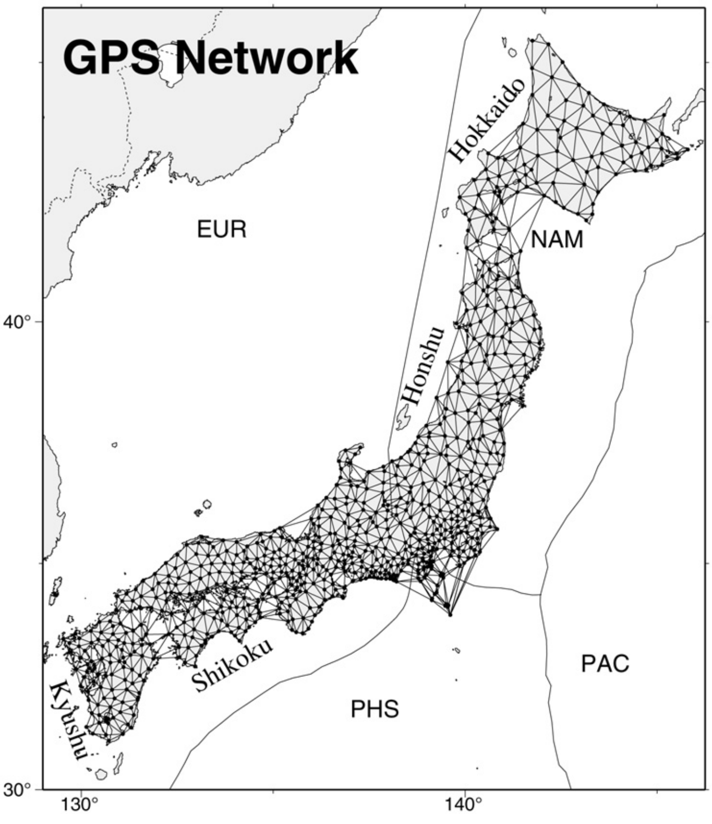


Fig. 2. GPS array of the Japanese Islands.

that it is mainly due to the fault sliding that produces shear deformation. Thus, this assumption implies that the dilatant stress increment is linearly related to the dilatant strain increment, i.e.,

$$f = \kappa(\mathrm{d}\epsilon_{11} + \mathrm{d}\epsilon_{22}) \tag{9}$$

Here, κ is the two-dimensional bulk modulus. This assumption is applicable to incompressible elasto-plastic materials; σ is elastically related to the two-dimensional volumetric strain when $\mathrm{d}\epsilon_{33}$ is neglected. It is certainly true that neglecting the vertical component of strain increment is an approximation. This component can be measured by using another space monitoring technique such as SAR; see [11].

The second task is to prescribe the boundary conditions. It is not possible to measure the boundary traction increment of the Japanese Islands. As an alternative, we make another assumption that the boundary traction rate is generated by a certain uniform stress increment. Actually, the movement of the four plates near the Japanese Islands produces a complicated distribution of stress increment, and hence this assumption is regarded as the first-order approximation which uses the average of the stress increment; as will be shown later, the uniform stress increment is calculated as the average of the stress increment. Expressing boundary traction and resultant force increments in terms of (unknown) uniform stress increment, we can derive equations which prescribe boundary conditions in terms of the uniform stress increment. However, it is easier to introduce Airy's stress function which generates a uniform stress increment field, in order to prescribe the boundary conditions. This Airy's stress function is given as

$$\mathrm{d}a^{\mathrm{o}}(\mathbf{x}) = \frac{1}{2}\left(x_2^2\,\mathrm{d}\sigma_{11}^{\mathrm{o}} - 2x_1x_2\,\mathrm{d}\sigma_{12}^{\mathrm{o}} + x_2^2\,\mathrm{d}\sigma_{22}^{\mathrm{o}}\right)$$

where $\mathrm{d}\sigma_{ij}^{\mathrm{o}}$ is the uniform stress increment. The function g that prescribes the boundary condition is thus given as

$$g = n_i(\mathbf{x})\,\mathrm{d}a_{,i}^{\mathrm{o}}(\mathbf{x}) \quad \text{for } \mathbf{x} \text{ on } \partial S$$

since g gives the Neumann boundary conditions of $\mathrm{d}a$. The uniform stress increment is determined by using the consistency condition, Eq. (7). Indeed, when f is given, Eq. (7) leads to $\int_S f\,\mathrm{d}s = \int_{\partial S} n_i\,\mathrm{d}a_{,i}^{\mathrm{o}}\,\mathrm{d}\ell$ or

$$\int_S f(\mathrm{d}\bar{\boldsymbol{\epsilon}})\,\mathrm{d}s = S(\mathrm{d}\sigma_{11}^{\mathrm{o}} + \mathrm{d}\sigma_{22}^{\mathrm{o}})$$

where S stands for the area of the surface S. If the form of $\mathrm{d}\sigma_{ij}^{\mathrm{o}}$ is assumed to be $\mathrm{d}\sigma_{ij}^{\mathrm{o}} = \mathrm{d}\sigma^{\mathrm{o}}\delta_{ij}$, this $\mathrm{d}\sigma^{\mathrm{o}}$ is determined as

$$\mathrm{d}\sigma^{\mathrm{o}} = \frac{1}{2S}\int_S F(\mathrm{d}\bar{\boldsymbol{\epsilon}})\,\mathrm{d}s \tag{10}$$

Hence, g is finally expressed in terms of $\mathrm{d}\sigma^{\mathrm{o}}$ as

$$g = \big(n_1(\mathbf{x})x_2 + n_2(\mathbf{x})x_1\big)\,\mathrm{d}\sigma^{\mathrm{o}}\,\mathrm{d}s \tag{11}$$

As mentioned, $\mathrm{d}\sigma^{\mathrm{o}}$ is the average of normal stress increment taken over S.

The boundary value problem for an increment of Airy's stress function is thus constructed by using f and g given by Eqs. (9) and (11), as follows:

$$\begin{cases} \mathrm{d}a_{,ii}(\mathbf{x}) = \kappa\,\mathrm{d}\bar{\epsilon}_{ii}(\mathbf{x}) & \text{for } \mathbf{x} \text{ in } S \\ n_i(\mathbf{x})\,\mathrm{d}a_{,i}(\mathbf{x}) = (n_1(\mathbf{x})x_2 + n_2(\mathbf{x})x_1)\,\mathrm{d}\sigma^{\mathrm{o}} & \text{for } \mathbf{x} \text{ on } \partial S \end{cases} \tag{12}$$

Here, $\mathrm{d}\sigma^{\mathrm{o}}$ is explicitly computed by using $\mathrm{d}\boldsymbol{\epsilon}$; see Eq. (10). When a suitable weight function, ϕ, is used, a weak form of this linear boundary value problem is $\int_S \phi(\mathrm{d}a_{,ii} - \kappa\,\mathrm{d}\bar{u}_{i,i})\,\mathrm{d}s = 0$, or

$$\int_S \phi_{,i}(\mathrm{d}a_{,i} - \kappa\,\mathrm{d}\bar{u}_i)\,\mathrm{d}s + \int_{\partial S} \phi\big((n_1x_2 + n_2x_1)\,\mathrm{d}\sigma^{\mathrm{o}} - \kappa n_i\,\mathrm{d}\bar{u}_i\big)\,\mathrm{d}\ell = 0 \tag{13}$$

This weak form is used when Eq. (12) is numerically solved. As is seen, the displacement increments that are measured by the GPS array are used in Eq. (13), and it is not necessary to take the derivative of $\mathrm{d}\bar{\mathbf{u}}$ to compute $\mathrm{d}\bar{\boldsymbol{\epsilon}}$.

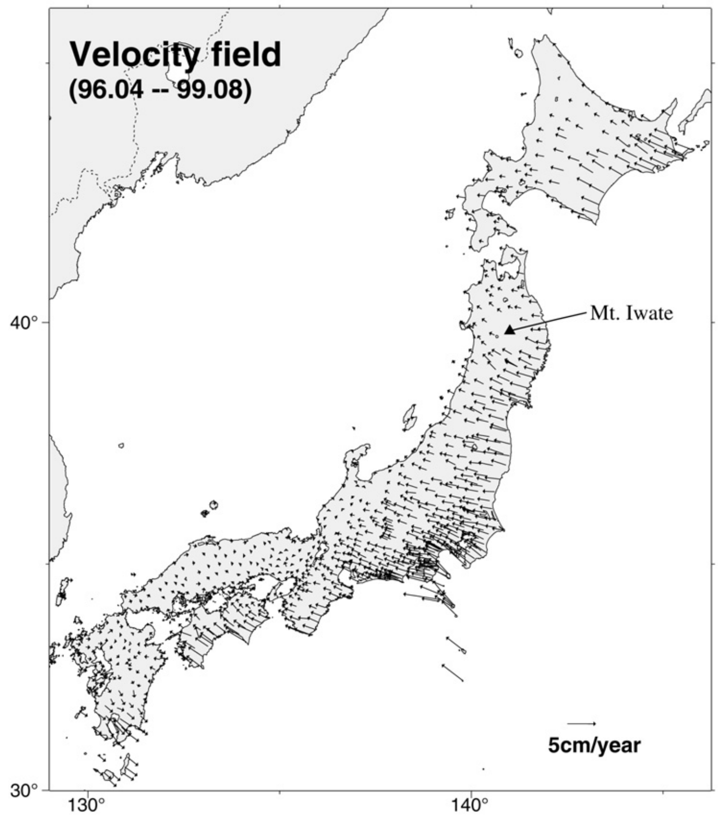


Fig. 3. Displacement increment measured by the GPS array.

Once da is obtained from Eq. (13), it is not difficult to compute the regional strain and stress increment. For a triangle region Ω which is formed by connecting three neighboring GPS stations, the average of regional strain and stress increment are

$$\begin{bmatrix} \langle \mathrm{d}\epsilon_{11} \rangle \\ \langle \mathrm{d}\epsilon_{22} \rangle \\ \langle \mathrm{d}\epsilon_{12} \rangle \end{bmatrix} = \frac{1}{\Omega} \int_{\partial\Omega} \begin{bmatrix} n_1\,\mathrm{d}u_B 1 \\ n_2\,\mathrm{d}u_B 2 \\ \frac{1}{2}(n_1\,\mathrm{d}\bar{u}_2 + n_2\,\mathrm{d}\bar{u}_1) \end{bmatrix} \mathrm{d}\ell$$

and

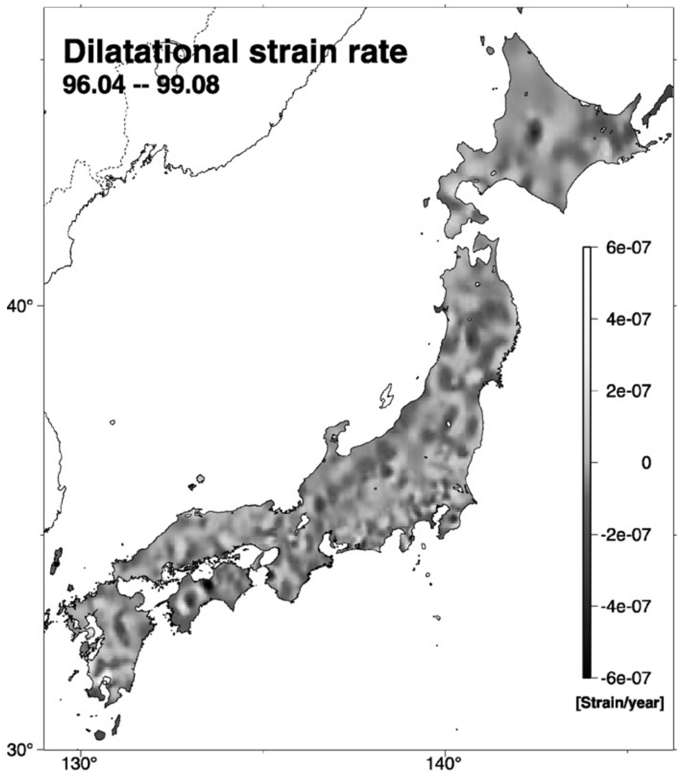
$$\begin{bmatrix} \langle \mathrm{d}\sigma_{11} \rangle \\ \langle \mathrm{d}\sigma_{22} \rangle \\ \langle \mathrm{d}\sigma_{12} \rangle \end{bmatrix} = \frac{1}{\Omega} \int_{\partial\Omega} \begin{bmatrix} n_2\,\mathrm{d}a_{,2} \\ n_1\,\mathrm{d}a_{,1} \\ -n_2\mathrm{d}a_{,1} \end{bmatrix} \mathrm{d}\ell$$

As is seen, these average quantities are computed by using the displacement increment and the gradient of Airy's stress increment, without taking higher derivatives.

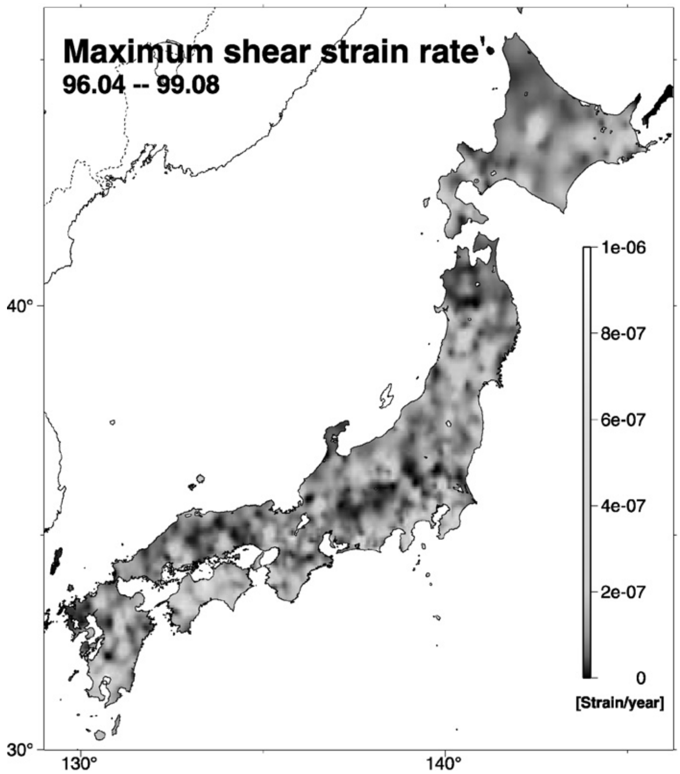
The stress inversion method is now applied to the GPS array data, by using a finite element method with a triangle element of 15 nodes to solve Eq. (13); see [11]. A triangle element is formed by connecting three neighboring GPS network stations, and twelve other nodes are equally spaced in the element. Meshing is made so that triangles become equilateral. The displacement increment measured at each station is assigned to the corner points, and other nodes have linearly interpolated values; see Fig. 3 for the displacement increment distribution which is used in the present analysis.

First, we plot the distribution of regional average strain increment in Fig. 4; Fig. 4(a) and (b) shows the dilate strain increment and for the maximum shear strain increment,

$$\mathrm{d}\gamma = \sqrt{\frac{1}{4}(\mathrm{d}\epsilon_{11} - \mathrm{d}\epsilon_{22})^2 + \mathrm{d}\epsilon_{12}^2}$$



(a)



(b)

Fig. 4. Distribution of the strain increment: (a) dilate strain increment; (b) maximum shear strain increment.

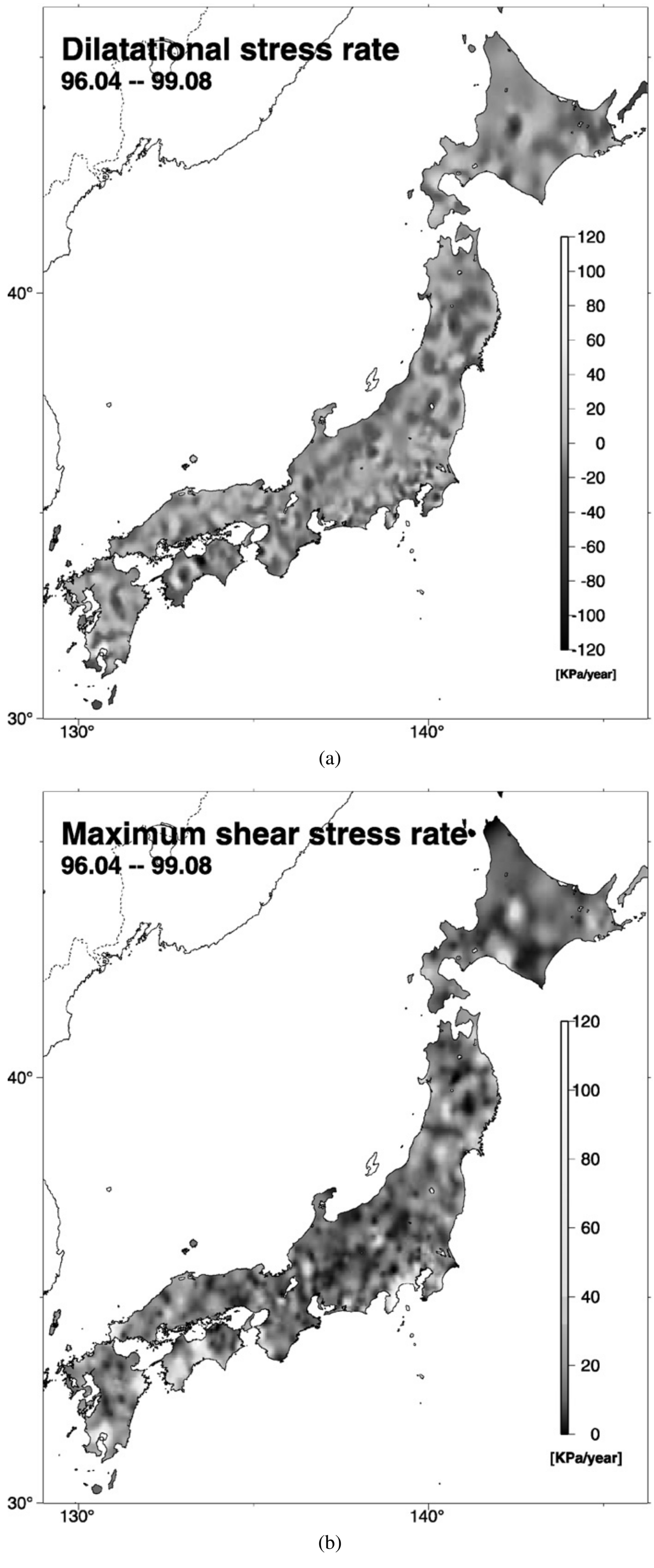


Fig. 5. Distribution of the stress increment: (a) dilate stress increment; (b) maximum shear stress increment.

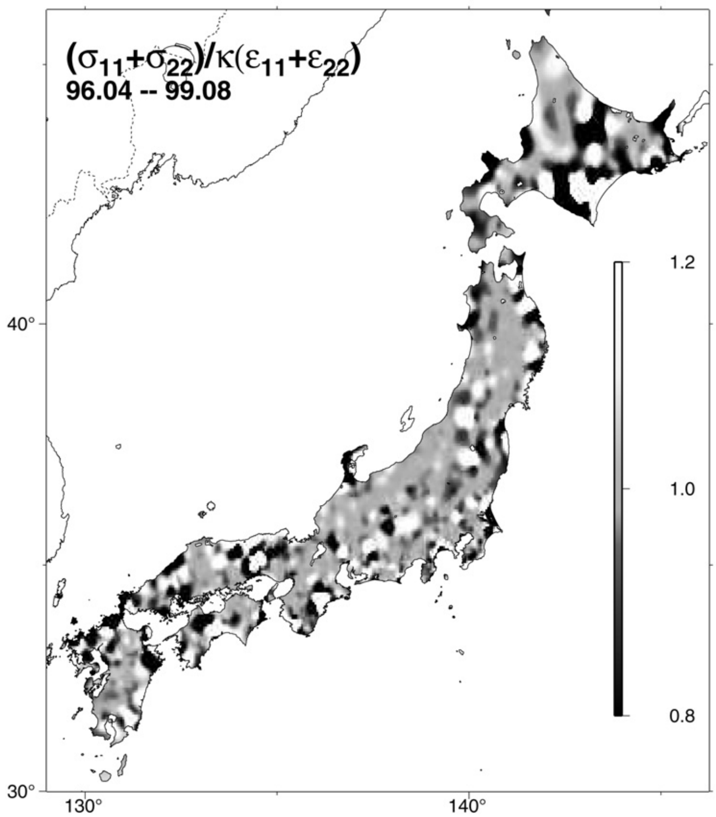


Fig. 6. Distribution of computed κ.

respectively. As is seen, the distribution of strain increment is not uniform. In particular, it is shown that there are some regions where the strain increment takes much larger values than in the surrounding regions. Such large spatial change in strain increment has not been observed. The validity of the strain increment distribution shown in Fig. 4 should be verified by comparing it with other geodesic data, such as the measurement of strain meters. To this end, the quality of the GPS data needs to be improved. Most domain measurement noise which correspond to rigid body motion of neighboring GPS stations is removed by computing the average strain increment, and hence we need a further study to get rid of other noise.

Next, we plot the distribution of regional average stress increment in Fig. 5, which is associated with the distribution of the regional average strain increment; Fig. 5(a) and (b) shows for the volumetric stress increment and for the maximum shear stress increment,

$$\mathrm{d}\tau = \sqrt{\frac{1}{4}(\mathrm{d}\sigma_{11} - \mathrm{d}\sigma_{22})^2 + \mathrm{d}\sigma_{12}^2}$$

respectively. The value of κ is set as $\kappa = 200$ GPa. Comparing Fig. 5 with Fig. 4, it is seen that the stress increment distribution is slightly different from the strain increment distribution, even though overall patterns are similar to each other. This is because the apparent stress increment, which is computed by multiplying the measured strain increment with a certain elasticity tensor, does not satisfy the equation of equilibrium, while the stress increment obtained by the stress inversion method does. Therefore, it is worth monitoring the stress increment that is computed by applying the stress inversion method to the GPS array data, in order to observe the crustal state. The crustal deformation monitor that is developed by the authors continuously checks the change in the average strain and stress increment.

As mentioned, a wild change in the strain increment distribution is first observed by computing the average strain that is obtained as the linear integration of the measured displacement along the triangle element edges. The corresponding stress increment also shows similar regional changes although the distribution is different from place to place. In the present analysis, we assume that $\mathrm{d}\sigma = \kappa\, \mathrm{d}\epsilon$ is uniform. However, it is possible to set a more complicated (and more realistic) condition for f, as more GPS array data are accumulated. Such data analysis will contribute to constructing a more rational model of the Japanese Islands. As a trial, we determine the distribution of κ assuming that it is not uniform. Instead, we assume isotropy, i.e., the coaxiality of the computed stress increment tensor with the measured strain increment tensor. The distribution of the computed κ is plotted in Fig. 6. While the validity of assuming the isotropy should be discussed, the stress inversion method yields a non-uniform distribution of κ. Since κ is related to the P-wave velocity, this distribution of κ should be compared with the velocity structure of the Japanese Islands which is constructed by analyzing earthquake waves.

4. Alternative to the stress inversion method

The stress inversion method is aimed at finding stress components that satisfy the equation of equilibrium. It requires partial information of stress–strain relations and boundary tractions, which are used to prescribe the boundary conditions, in addition to the distribution of strain in a body. We have to make two assumptions for the stress–strain relation and the boundary tractions in applying the stress inversion to the GPS array data. The validity of these assumptions is not verified by observation, and making such unverified assumptions is the limitation of the stress inversion method.

As an alternative to the stress inversion method, the authors are proposing another method of analyzing the GPS array data. This method is called an elasticity inversion method; see [12]. While it is applicable only to a body consisting of linearly elastic material, the method does not need to make any further assumptions for stress–strain relations or additional data except for strain distribution. Indeed, the elasticity inversion method is aimed at finding linear elastic constants by measuring a displacement field. If out-of-plane components are neglected, there are six independent components for linear elasticity in a state of plane stress. The elasticity inversion method seeks these six components. It should be pointed out that the deformation of the Japanese Islands is regarded as being linearly elastic during a short period in the geological time scale. If an anomaly is found in analyzing the GPS array data with the assumption of linear elasticity, it indicates inelastic deformation of the Japanese Islands.

To explain the elasticity inversion, we call the GPS station a node, and a triangle which is formed by connecting three neighboring nodes an element, and pick up one block which consists of several elements and has one node in it, with other nodes being located on the boundary, see Fig. 7. A displacement increment field in this block is computed by assuming a suitable elasticity and solving the resulting boundary value problem. The most suitable elasticity is thus found so that the solution of the inner node displacement coincides with the measured value; the displacement increment measured at the boundary nodes serve as boundary conditions. In solving this boundary value problem, there is no need to consider the stress state outside of the block. In this sense, we can *isolate* this block and analyze only the displacement increment of the inner and boundary nodes.

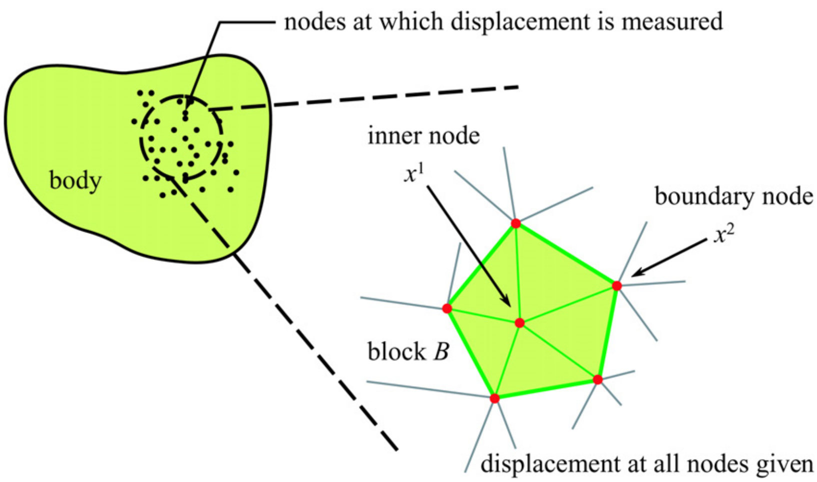


Fig. 7. A block consisting of nodes.

We formulate the elasticity inversion in a discrete manner which is used for a finite element method. Let the block B contain n nodes and $n-1$ triangular elements, denoted by N^P's and Ω^α ($P=1,2,\dots,n$, $\alpha=1,2,\dots,n-1$); the location of N^P is x_i^P and the inner node is the first node, N^1. For the Pth node, the nodal displacement increment and shape function are denoted by $\mathrm{d}u^P$ and ϕ^P, respectively. When the elasticity tensor is given as c_{ijkl}, the weak form of the governing equation of the displacement increment, $(c_{ijkl}\,\mathrm{d}u_{k,l})_{,i}=0$, leads to

$$\sum_{Q=1}^{n}\left(\int_B \phi^P_{,i}(\mathbf{x})c_{ijkl}(\mathbf{x})\phi^Q_{,l}(\mathbf{x})\,\mathrm{d}s\right)\mathrm{d}u_k^Q=0 \quad \text{for } P=1,2,\dots,n \text{ and } j=1,2 \tag{14}$$

We assume that c_{ijkl} is uniform, i.e., all the elements share the same c_{ijkl}. When the boundary node displacement increments, $\mathrm{d}\bar{u}_i^Q$ ($Q=2,3,\dots,n$), are used as the boundary conditions, Eq. (14) for $P=1$ determines the inner node displacement increment, i.e.,

$$\sum_{\alpha=1}^{n-1}\left(B_i^{1\alpha}B_l^{1\alpha}c_{ijkl}\right)\mathrm{d}u_k^1+\sum_{Q=2}^{n}\sum_{\alpha=1}^{n-1}\left(B_i^{1\alpha}B_l^{Q\alpha}c_{ijkl}\right)\mathrm{d}\bar{u}_k^Q=0 \tag{15}$$

where

$$B_i^{P\alpha}=\int_{\Omega^\alpha}\phi^P_{,i}(\mathbf{x})\,\mathrm{d}s \tag{16}$$

As is seen, solving this linear equation, we can obtain $\mathrm{d}u_i^1$, and find suitable c_{ijkl}'s or $\mathbf{c}$ so that the computed $\mathrm{d}u_i^1$ coincides with the measured value, $\mathrm{d}\bar{u}_i^1$. This is the basic idea of the elasticity inversion.

To find $\mathbf{c}$, it is better to substitute all the measured displacement increments into $\mathrm{d}\mathbf{u}^Q$'s and to minimize the resulting error with respect to unknown $\mathbf{c}$. When m sets of the measured data of displacement increment are available, there are $2m$ conditions for 6 unknown components of $\mathbf{c}$. Thus, $\mathbf{c}$ is determined by accumulating sufficiently many sets of the displacement increment which is measured by the GPS array. It should be noted that if isotropy is assumed for each element, the number of unknown components of $\mathbf{c}$ is reduced to 2, and the most suitable unknown is found by minimizing the error of Eq. (15).

Minimizing the error of Eq. (15) with respect to $\mathbf{c}$ seems trivial if sufficiently many data sets are provided. However, some attention must be paid to the fact that Eq. (15) is derived from the equation of equilibrium. First, we have to recall that rigid body motion, translation and rotation, is a trivial solution for the equation of equilibrium which does not generate stress; the rigid body motion is given as

$$\mathrm{d}u_i^P=\mathrm{d}u_i \quad \text{or} \quad \mathrm{d}u_i^P=\mathrm{d}\omega\,\varepsilon_{3ij}\left(x_j^P-x_j^1\right)$$

where $\mathrm{d}u_i$ and $\mathrm{d}\omega$ are constant and ε_{ijk} is permutation symbol. Displacement increment which corresponds to the rigid body motion vanishes when it is multiplied by $B_i^{P\alpha}$ of Eq. (16), and hence no information about $\mathbf{c}$ can be extracted from the rigid body motion.

Next, due to the absence of body force, the inner node displacement increment is formally expressed in terms of Green's function of the block, as

$$\mathrm{d}u_i(\mathbf{x})=\int_{\partial B}G_{ij}(\mathbf{x},\mathbf{y})\,\mathrm{d}u_j(\mathbf{y})\,\mathrm{d}\ell_{\mathbf{y}}$$

where G_{ij} is Green's function which gives a displacement in the block for given boundary displacement. This is a merely formal expression since G_{ij} cannot be obtained. Discretizing this formal expression, we can obtain an expression of the inner node displacement increment as

$$\mathrm{d}u_i^1=\sum_{Q=2}^{n}G_{ij}^Q\,\mathrm{d}u_j^Q \tag{17}$$

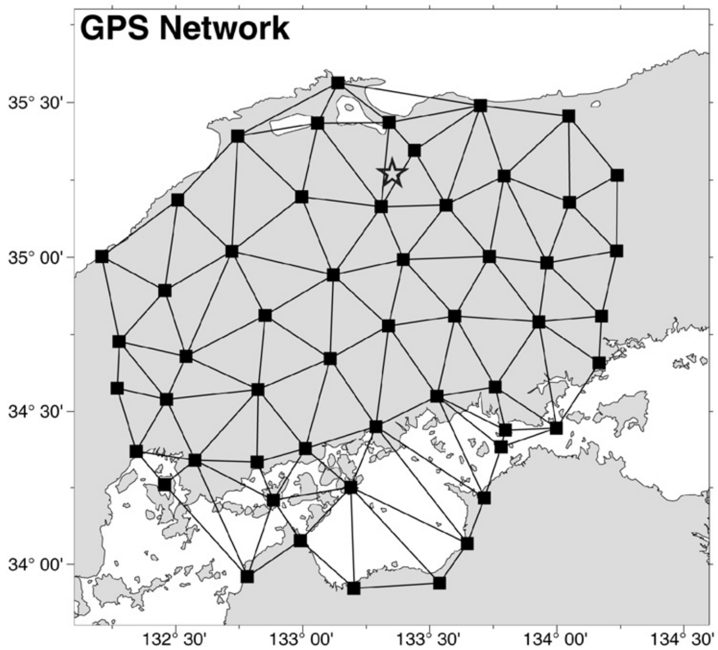


Fig. 8. GPS array of Tottori Prefecture.

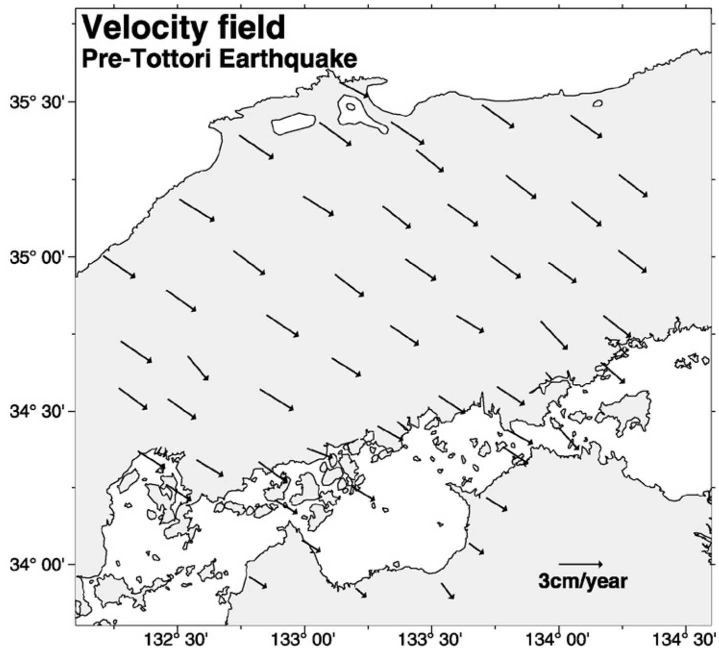
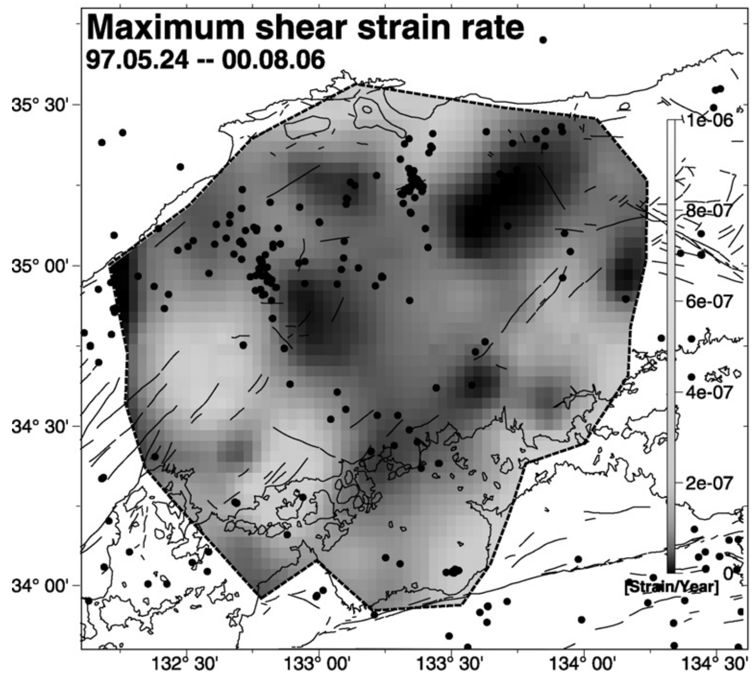


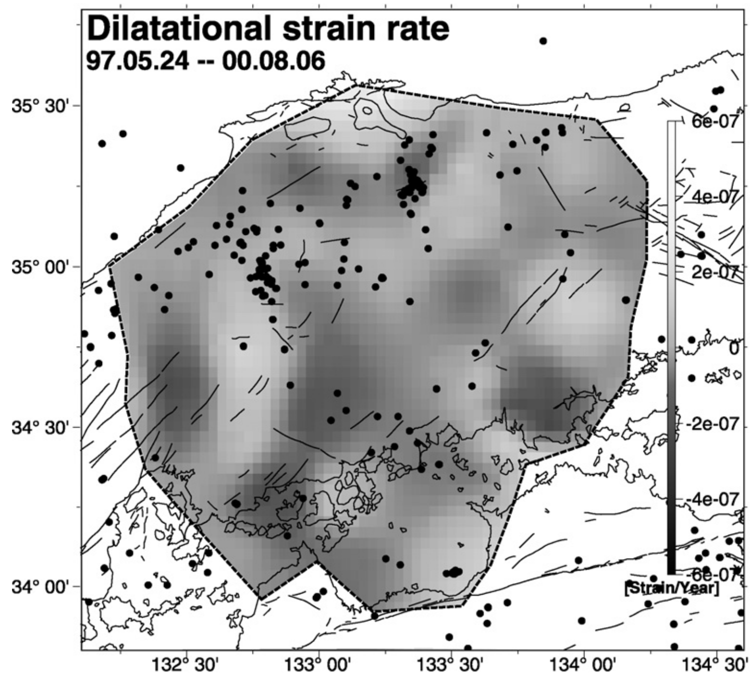
Fig. 9. Displacement increment of Tottori Prefecture.

This means that du_i^1 is given as linear combination of $2(n-1)$ modes that correspond to G_{ij}^Q. Even if many data sets are available, the number of the modes cannot be increased. These data should be used to determine the modes. In determining the modes, that fact that Eq. (17) is satisfied by the rigid body motion should be used, i.e.,

$$1=\sum_{Q=2}^{n} G_{ij}^{Q} \quad \text{or} \quad 0=\sum_{Q=2}^{n} G_{ij}^{Q}\varepsilon_{3jk}\left(x_k^Q-x_k^1\right)$$



(a)

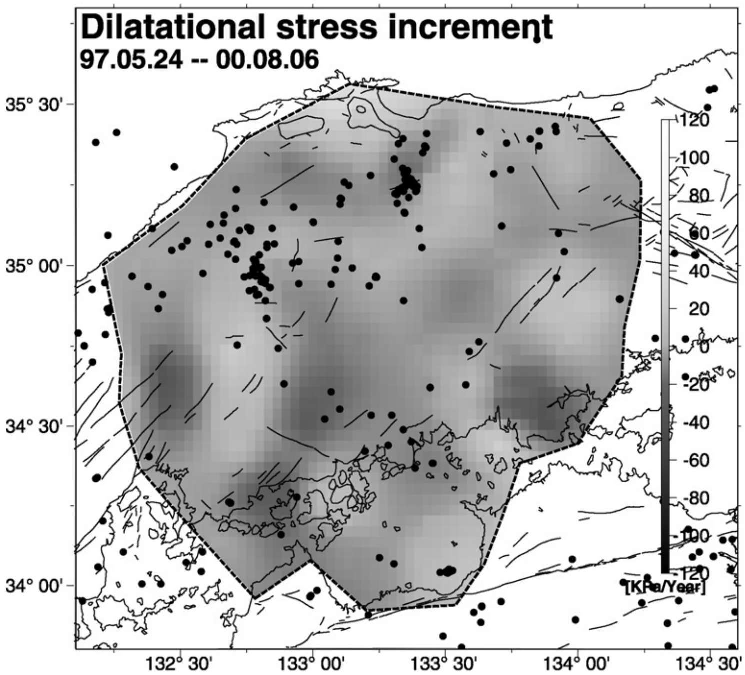


(b)

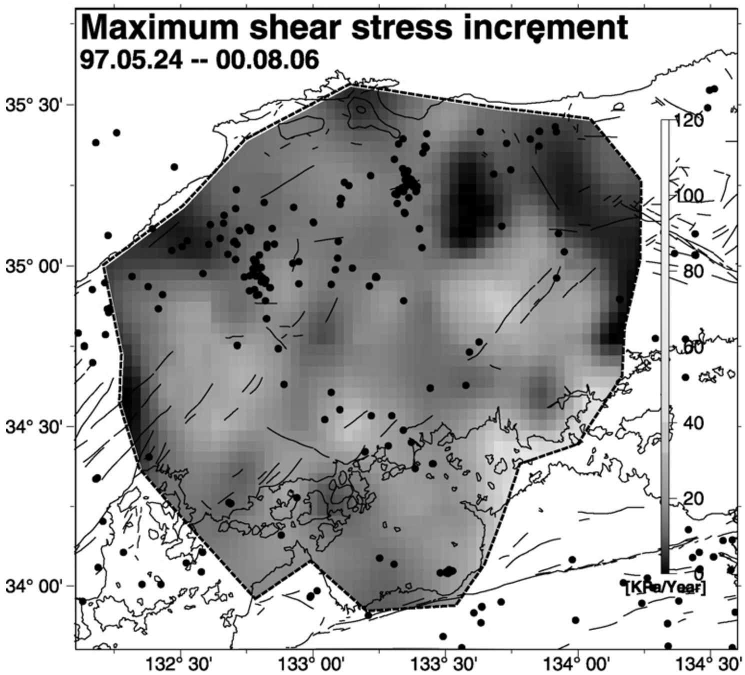
Fig. 10. Distribution of the strain increment of Tottori Prefecture: (a) dilatant strain increment; (b) maximum shear strain increment.

When the modes are determined by using the measured data, it is straightforward to determine unknown **c**, just by minimizing the error for each mode. That is,

$$\sum_{i,k,l.\alpha} \left(B_i^{1\alpha} B_l^{1\alpha} c_{ijkl}\right) G_k^Q + \sum_{i,l,\alpha} \left(B_i^{1\alpha} B_l^{Q\alpha} c_{ijKl}\right) = 0 \tag{18}$$



(a)



(b)

Fig. 11. Distribution of the stress increment of Tottori Prefecture: (a) dilatant stress increment; (b) maximum shear stress increment.

for $Q = 2, 3, \ldots, n$ and $j = 1, 2$; note that K in the second summation is not summed. A more complicated setting can be made for the elasticity inversion method, such as non-uniformity of the block (i.e., each element Ω^α has its own elasticity $\mathbf{c}^\alpha$); see [12].

The elasticity inversion method is applied to a region around Tottori Prefecture, Japan; the 2000 Western Tottori Earthquake, of magnitude 7.3 in the Japan Meteorological Agency scale, has occurred in this region; see [11]. There

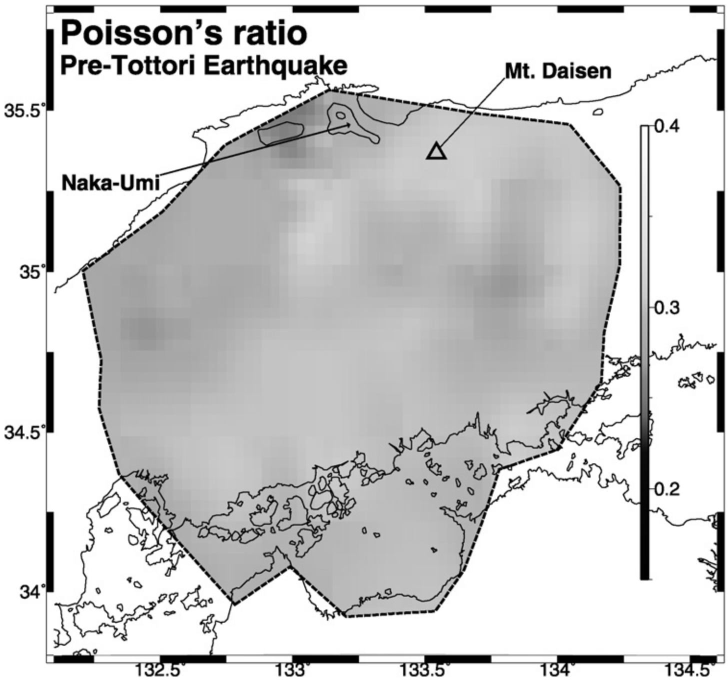


Fig. 12. Distribution of the computed Poisson's ratio of Tottori Prefecture.

are 53 GPS stations, as shown in Fig. 8. Annual and biannual sinusoidal variations are excluded from the displacement increment which has been measured from 1997 to 2002, and the displacement increment used for the present elasticity inversion method is shown in Fig. 9. The isotropy is assumed for each block, and Poisson's ratio is treated as an unknown. The distributions of the computed strain and stress increment are plotted in Figs. 10 and 11, respectively; in each figure, (a) and (b) are for the dilatant component and the maximum shear component, respectively, and $E =$ 100 GPa is used. The distribution of the computed Poisson's ratio is shown in Fig. 12. As is seen, a non-uniform distribution is observed for strain, stress and Poisson's ratio. It is shown that the computed stress increment is smoothly changing as the strain increment is changes, and that the distribution of Poisson's ratio is not similar to that of the strain or stress increment. Unlike the stress inversion method, the elasticity inversion method is able to produce these results without making any assumptions for the boundary traction of the block. It should be pointed out that the GPS array data after the 2000 Western Tottori Earthquake are analyzed by applying the elasticity inversion method, and the distribution of strain and stress increment and the distribution of Poisson's ratio are computed; see [11]. Geophysical discussions have been made by analyzing the change in the strain and strain increment before and after the earthquake, as well as the change in the Poisson ratio distribution.

5. Concluding remarks

The stress inversion method is developed for finding a self-equilibrating stress field. When one stress–strain relation is known and strain and traction are measured, the mathematical problem which the stress inversion method solves is well posed in the sense that the solution is unique. The method is applicable to the GPS array data, if assumptions are made for a relation between dilatant stress and strain increment and for traction acting on the boundary. While the validity of these assumptions and hence the results of the stress inversion method are not verified at this moment, the method serves as a data analysis method to monitor the change in crustal deformation of the Japanese Islands.

The elasticity inversion method is an alternative to the stress inversion method; it is based on the assumption of a linear elastic response, and does not require any data for traction. This method contributes to the analysis of a small region where it is impossible to measure the boundary traction acting.

We should emphasize that these two methods solve a mathematically well-posed problem. There is no ambiguity in solving the problem when sufficient sets of data are provided. From the mechanical view-point, however, these methods have drawbacks; assumptions have to be made for the deformation state or stress–strain relation in order to pose the problems. Thus, these methods are applicable to a limited class of mechanical problems.

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