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Duality, inverse problems and nonlinear problems in solid mechanics

Dual integrals in small strain elasticity with body forces

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Abstract

Dual integrals of small strain elasticity are derived and related to the energy release rates associated with a defect motion in the presence of body forces. A modified energy momentum tensor is used, which includes a work term due to body forces, and which yields simple expressions for the configurational forces in terms of the J_k, L_k, and M integrals. Since the complementary potential energy does not include body forces explicitly, the complementary energy momentum tensor has the same structure as in the elasticity without body forces. The expressions for the nonconserved J_k, L_k, and M integrals, and their duals, are related to the volume integrals whose integrands depend on body forces and their gradients. If body forces are spatially uniform, the conservation laws $J_k = \hat{J}_k = 0$ hold for both 2D and 3D problems, and $L_3 = \hat{L}_3 = 0$ for the antiplane strain problems. The conservation law $M = \hat{M} = 0$ holds if body forces are absent, or if they are homogeneous functions of particular degree in spatial coordinates. ***To cite this article: V.A. Lubarda, X. Markenscoff, C. R. Mecanique 336 (2008).***

Résumé

Intégrales duales en élasticité infinitésimale avec forces de masse. Des intégrales duales en élasticité infinitésimale sont obtenues et reliées aux taux de restitution d'énergie associés au mouvement d'un défaut en présence de forces de masse. On définit un tenseur d'énergie–impulsion qui inclut un terme de travail des forces de masse, et qui fournit des expressions simples des forces configurationnelles en fonction des intégrales J_k, L_k et M. Du fait que l'énergie potentielle complémentaire n'inclut pas explicitement les forces de masse, le tenseur d'énergie-impulsion complémentaire a la même structure qu'en élasticité sans forces de masse. Les expressions des intégrales non-conservées J_k, L_k et M et de leurs duales sont reliées à des intégrales de volume dont les intégrandes dépendent des forces de masse et de leurs gradients. Si les forces de masse sont spatialement uniformes, les lois de conservation $J_k = \hat{J}_k = 0$ s'appliquent aux problèmes tant 2D que 3D, de même que la loi $L_3 = \hat{L}_3 = 0$ aux problèmes antiplans. La loi de conservation $M = \hat{M} = 0$ s'applique en l'absence de forces de masse ou si ce sont des fonctions homogènes de degré particulier des coordonnées. ***Pour citer cet article : V.A. Lubarda, X. Markenscoff, C. R. Mecanique 336 (2008).***

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1. Introduction

In absence of body forces, the conservation laws $J_k = 0$, $L_k = 0$, and $M = 0$ hold for any closed surface that does not embrace a singularity or defect [1–4]. The energy momentum tensor (or Eshelby stress) used to construct the J_k, L_k, and M integrals depends on spatial gradients of displacements. If the surface used to evaluate these integrals surrounds a defect, the integrals do not vanish, but represent the configurational forces associated with particular defect motions and the corresponding potential energy release rates [5].

In a dual analysis, the complementary or dual energy momentum tensor, expressed in terms of spatial gradients of stresses, is used to construct the dual $\hat{J}_k$, $\hat{L}_k$, and $\hat{M}$ integrals, which are related to the release rates of the complementary potential energy. The study of dual integrals was initiated by Bui's [6,7] introduction of the $\hat{J}$ integral as a dual to Rice's [8] J integral of fracture mechanics. In the context of elastodynamics, the dual integrals were introduced in [9]. In the subsequent work, the dual integrals were studied in [10,11], although they were there incorrectly related to the release rates of the complementary potential energy. This was corrected in [12] by an extension of the analysis from [5], which involves the complementary energy considerations and an appropriate incorporation of the rates of stress and the change of the surface orientation of the moving defect. Other work on dual conservation integrals, in both nonpolar or micropolar elasticity, includes Refs. [13–22].

The evaluation of the configurational force on a defect in the presence of body forces, thermal strains, or in nonhomogeneous elastic media, was considered in [23–32]. In the presence of body forces, the stress tensor and the energy momentum tensor are not divergence-free tensors, which precludes the existence of the J_k, L_k, and M conservation laws. In most of the previous work, the energy momentum tensor was defined by the same expression as in the case of elasticity without body forces, which leads to less appealing relationships between the integrals, the energy release rates and the corresponding configurational forces on moving defects. In the present article, we use a modified energy momentum tensor, which includes a work term due to body forces, and which yields simple expressions for the configurational forces on defects, in terms of the J_k, L_k, and M integrals evaluated over the unloaded surface of the defect. Since the complementary potential energy does not include a body force term, the complementary energy momentum tensor has the same structure as in the elasticity without body forces. The expressions for the nonconserved J_k, L_k, and M integrals, and their dual $\hat{J}_k$, $\hat{L}_k$, and $\hat{M}$ integrals, are derived and related to the volume integrals whose integrands depend on the body forces and their gradients. In particular case, when the body forces are spatially uniform, we show that the conservation laws $J_k = \hat{J}_k = 0$ hold for both 2D and 3D problems, and $L_3 = \hat{L}_3 = 0$ for the antiplane strain problems. The conservation law $M = \hat{M} = 0$ holds if the body forces are absent, or if they are homogeneous functions of particular degree in spatial coordinates.

The considerations in this paper are restricted to small deformations of an elastic material, which are geometrically described by the displacement vector with rectangular components u_i, and the corresponding infinitesimal strain components $\epsilon_{ij} = (u_{i,j} + u_{j,i})/2$. The surface tractions T_i are in equilibrium with the symmetric Cauchy stress σ_{ij}, such that $T_i = n_j \sigma_{ji}$, where n_j are the components of the unit vector orthogonal to the surface element under consideration. If the components of body forces (per unit volume) are b_i, the differential equations of equilibrium are

$$\sigma_{ji,j} + b_i = 0 \tag{1}$$

The elastic strain energy, $W = W(\epsilon_{ij})$, and the complementary strain energy, $\Phi = \Phi(\sigma_{ij})$, are related by

$$\Phi(\sigma_{ij}) = \sigma_{ij}\epsilon_{ij} - W(\epsilon_{ij}) \tag{2}$$

The corresponding constitutive relations are

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}, \qquad \epsilon_{ij} = \frac{\partial \Phi}{\partial \sigma_{ij}} \tag{3}$$

2. Dual J_k integrals

A spatial gradient of the strain energy function $W = W(\epsilon_{ij})$ is

$$W_{,k} = \frac{\partial W}{\partial \epsilon_{ij}} \epsilon_{ij,k} = \sigma_{ji} u_{i,jk} \tag{4}$$

In view of equilibrium equations (1), this can be rewritten as

$$(W\delta_{jk} - \sigma_{ji}u_{i,k})_{,j} = b_i u_{i,k} \tag{5}$$

This can be rewritten as

$$\big[(W - b_i u_i)\delta_{jk} - \sigma_{ji}u_{i,k}\big]_{,j} = -b_{i,k}u_i \tag{6}$$

which specifies the energy momentum tensor in the presence of body forces as [33]

$$P_{jk} = (W - b_i u_i)\delta_{jk} - \sigma_{ji}u_{i,k}, \quad P_{jk,j} = -b_{i,k}u_i \tag{7}$$

As shown in Section 5, this definition of the energy momentum tensor is directly related to the release rates of the potential energy due to defect motion in the presence of body forces. The J_k integrals, corresponding to (7), are

$$J_k = \int_S P_{jk}n_j\,\mathrm{d}S = -\int_V b_{i,k}u_i\,\mathrm{d}V \tag{8}$$

where S is the bounding surface of the volume V which does not include any singularity of defect.[1]

If the body forces are spatially uniform ($b_{i,k} = 0$), we have

$$J_k = \int_S P_{jk}n_j\,\mathrm{d}S = 0, \quad P_{jk} = (W - b_i u_i)\delta_{jk} - \sigma_{ji}u_{i,k} \tag{9}$$

i.e.,

$$J_k = \int_S \big[(W - b_i u_i)n_k - T_i u_{i,k}\big]\,\mathrm{d}S = 0 \tag{10}$$

In the absence of body forces ($b_i = 0$), this result is originally due to Eshelby [1,2].

In a dual analysis, a spatial gradient of the complementary strain energy function $\Phi = \Phi(\sigma_{ij})$ is

$$\Phi_{,k} = \frac{\partial \Phi}{\partial \sigma_{ij}}\sigma_{ij,k} = u_{i,j}\sigma_{ji,k} \tag{11}$$

In view of equilibrium equations (1), the above can be recast as

$$(\Phi\delta_{jk} - u_i\sigma_{ji,k})_{,j} = u_i b_{i,k} \tag{12}$$

which defines a dual energy momentum tensor, such that

$$\hat{P}_{jk} = \Phi\delta_{jk} - u_i\sigma_{ji,k}, \quad \hat{P}_{jk,j} = u_i b_{i,k} \tag{13}$$

This definition of the dual energy momentum tensor will be later shown to be directly related to the release rates of the complementary potential energy associated with a defect motion, in the presence of body forces. The corresponding dual $\hat{J}_k$ integrals are

$$\hat{J}_k = \int_S \hat{P}_{jk}n_j\,\mathrm{d}S = \int_V u_i b_{i,k}\,\mathrm{d}V \tag{14}$$

for any closed surface S that does not embrace a singularity or a defect. While J_k in (8) is expressed in terms of spatial gradients of displacement, $\hat{J}_k$ in (14) is expressed in terms of the stress gradients.[2] If the body forces are spatially uniform, $\hat{J}_k = 0$ for any closed surface which does not surround a singularity. In the absence of body forces, and in two-dimensional context, this result is originally due to Bui [6,7].

Since the right-hand sides in (8) and (14) are opposite, we conclude that

$$J_k + \hat{J}_k = 0 \tag{15}$$

[1] The body force term is also included in the structure of the J_P integral used in the study of the progressive failure of over-consolidated clay [34].

[2] Computational aspects of the evaluation of dual integrals via the displacement-based and hybrid finite element calculations have been discussed in [27].

It is also noted that

$$P_{jk} + \hat{P}_{jk} = (W + \Phi - b_i u_i)\delta_{jk} - (\sigma_{ji} u_i)_{,k}$$
$$P_{kk} = 3(W - b_k u_k) - \sigma_{jk} u_{k,j}, \qquad \hat{P}_{kk} = 3\Phi + b_k u_k \tag{16}$$

If the strain energy W is a homogeneous function of degree r in strain components ($1 < r \leqslant 2$), the complementary strain energy Φ is a homogeneous function of degree $s = r/(r-1)$ in stress components ($s \geqslant 2$), and $\Phi = rW/s$. In this case it readily follows that

$$rJ_k - s\hat{J}_k = \int_S (su_i\sigma_{ij,k} - r\sigma_{ij}u_{i,k} - rb_i u_i \delta_{jk}) n_j \, \mathrm{d}S = -rs \int_V b_{i,k} u_i \, \mathrm{d}V \tag{17}$$

Combining (15) and (17), it follows that

$$J_k = \int_S \left(\frac{1}{r} u_i \sigma_{ij,k} - \frac{1}{s} \sigma_{ij} u_{i,k} - \frac{1}{s} b_i u_i \delta_{jk} \right) n_j \, \mathrm{d}S \tag{18}$$

In absence of body forces and for homogeneous materials of degree two ($r = s = 2$), the last expression reduces to the reciprocal representation of the J_k integral [14],

$$J_k = \frac{1}{2} \int_S (u_i \sigma_{ij,k} - \sigma_{ij} u_{i,k}) n_j \, \mathrm{d}S \tag{19}$$

3. Dual M integrals

Let the strain energy $W = W(\epsilon_{ij})$ be a homogeneous function of degree r in strain components, so that

$$W = \frac{1}{r} \sigma_{jk} \epsilon_{jk} \tag{20}$$

The energy momentum tensor (7) satisfies the equation

$$(P_{jk} x_k)_{,j} - P_{kk} = -u_i b_{i,k} x_k \tag{21}$$

In view of (16) and (20), we have

$$P_{kk} = \frac{3-r}{r} (\sigma_{jk} u_k)_{,j} + \frac{3-4r}{r} b_k u_k \tag{22}$$

and the substitution into (21) gives

$$\left(P_{jk} x_k - \frac{3-r}{r} \sigma_{jk} u_k \right)_{,j} = u_i \left(\frac{3-4r}{r} b_i - b_{i,k} x_k \right) \tag{23}$$

Upon the application of the Gauss divergence theorem, this yields

$$M = \int_S \left(P_{jk} x_k - \frac{3-r}{r} \sigma_{jk} u_k \right) n_j \, \mathrm{d}S = \int_V u_i \left(\frac{3-4r}{r} b_i - b_{i,k} x_k \right) \mathrm{d}V \tag{24}$$

In the absence of body forces, the M integral vanishes for any closed surface that does not embrace a singularity or defect [3,4].[3]

A dual energy momentum tensor (13) satisfies the equation

$$(\hat{P}_{jk} x_k)_{,j} - \hat{P}_{kk} = u_i b_{i,k} x_k \tag{25}$$

[3] In contrast to nonpolar elasticity, there is no conservation law $M = 0$ in couple stress and micropolar elasticity, due to an inherent material length scale present in these material models; e.g., [20–22].

The complementary strain energy, corresponding to (20), is

$$\Phi = \frac{1}{s}\sigma_{jk}\epsilon_{jk}, \quad s = \frac{r}{r-1} \tag{26}$$

so that

$$\hat{P}_{kk} = \frac{3}{s}(u_k\sigma_{jk})_{,j} + \frac{3+s}{s}b_k u_k \tag{27}$$

The substitution into (25) gives

$$\left(\hat{P}_{jk}x_k - \frac{3}{s}u_k\sigma_{jk}\right)_{,j} = u_i\left(\frac{3+s}{s}b_i + b_{i,k}x_k\right) \tag{28}$$

Consequently, there is a dual M integral

$$\hat{M} = \int_S \left(\hat{P}_{jk}x_k - \frac{3}{s}u_k\sigma_{jk}\right)n_j\,\mathrm{d}S = \int_V u_j\left(\frac{3+s}{s}b_j + b_{j,k}x_k\right)\mathrm{d}V \tag{29}$$

The duality is such that M is expressed in terms of spatial gradients of displacements, while $\hat{M}$ is in terms of the stress gradients. In absence of body forces, $\hat{M} = 0$ for any surface which does not embrace a defect [10–12]. The conservation law $M = \hat{M} = 0$ also holds if the body forces are homogeneous functions of degree $-(1 + 3/s) = -(4 - 3/r)$ in spatial coordinates x_k, although this type of body forces is probably of little practical interest.

Since $rs = r + s$, the right-hand sides in (24) and (29) are opposite, and we conclude that

$$M + \hat{M} = 0 \tag{30}$$

It also follows that

$$\begin{aligned} rM - s\hat{M} &= \int_S \left[(su_i\sigma_{ij,k} - r\sigma_{ij}u_{i,k})x_k + r(\sigma_{jk}u_k - b_k u_k x_j)\right]n_j\,\mathrm{d}S \\ &= -rs\int_V u_i\left(\frac{3+s}{s}b_i + b_{i,k}x_k\right)\mathrm{d}V \end{aligned} \tag{31}$$

Combining (30) and (31), it follows that

$$\begin{aligned} M &= \int_S \left[\left(\frac{1}{r}u_i\sigma_{ij,k} - \frac{1}{s}\sigma_{ij}u_{i,k}\right)x_k + \frac{1}{s}(\sigma_{jk}u_k - b_k u_k x_j)\right]n_j\,\mathrm{d}S \\ &= -\int_V u_i\left(\frac{3+s}{s}b_i + b_{i,k}x_k\right)\mathrm{d}V \end{aligned} \tag{32}$$

In absence of body forces, this simplifies to

$$M = \int_S \left[\left(\frac{1}{r}u_i\sigma_{ij,k} - \frac{1}{s}\sigma_{ij}u_{i,k}\right)x_k + \frac{1}{s}\sigma_{jk}u_k\right]n_j\,\mathrm{d}S = 0 \tag{33}$$

which parallels the reciprocal representation (18) of the J_k integral.

4. Dual L_k integrals

To derive the L_k integrals of isotropic infinitesimal elasticity, we begin from an identity

$$c_k = e_{kij}(\sigma_{il}\epsilon_{jl} + \sigma_{li}\epsilon_{lj}) = e_{kij}(\sigma_{il}u_{l,j} + \sigma_{li}u_{j,l}) = 0 \tag{34}$$

The components of the permutation tensor are e_{ijk}. This identity holds because the tensor $(\sigma_{il}\epsilon_{jl} + \sigma_{li}\epsilon_{lj})$ is symmetric in ij (for isotropic elasticity), as can be verified by the substitution of the constitutive expression for stress. By using the definition of the energy momentum tensor (7), we can write $e_{kij}P_{ji} = e_{kij}\sigma_{il}u_{l,j}$, and (34) becomes

$$c_k = e_{kij}(P_{ji} + \sigma_{li}u_{j,l}) \tag{35}$$

In view of (1) and (7), this can be expressed as

$$c_k = d_{kl,l} + e_{kij}(b_i u_j + b_{l,i} u_l x_j), \quad d_{kl} = e_{kij}(P_{li} x_j + \sigma_{li} u_j) \tag{36}$$

Since $c_k = 0$, this yields the integrals

$$L_k = \int_S d_{kl} n_l \, \mathrm{d}S \tag{37}$$

which is, in the expanded form,

$$L_k = e_{kij} \int_S (P_{li} x_j + \sigma_{li} u_j) n_l \, \mathrm{d}S = -e_{kij} \int_V u_l (\delta_{lj} b_i + b_{l,i} x_j) \, \mathrm{d}V \tag{38}$$

In absence of body forces, $L_k = 0$ for any closed surface S that does not embrace a singularity or defect [3,4].

To derive dual $\hat{L}_k$ integrals, introduce the components of a dual vector $\hat{c}_k$, defined by $\hat{c}_k + c_k = 0$. From (34) it follows that

$$\hat{c}_k = e_{kij}(u_{i,l}\sigma_{lj} + u_{l,i}\sigma_{jl}) \tag{39}$$

Since, by Eq. (13),

$$e_{kij}(\hat{P}_{ji} + u_l \sigma_{jl,i}) = 0 \tag{40}$$

we rewrite (39) as

$$\hat{c}_k = e_{kij}(\hat{P}_{ji} + u_{i,l}\sigma_{lj} + u_{l,i}\sigma_{jl} + u_l \sigma_{jl,i}) \tag{41}$$

In view of (1) and (13), Eq. (41) can be expressed in the following form

$$\hat{c}_k = \hat{d}_{kl,l} - e_{kij} u_l (b_{l,i} x_j - \delta_{li} b_j), \quad \hat{d}_{kl} = e_{kij}(\hat{P}_{li} x_j + u_i \sigma_{lj} + \delta_{il} u_r \sigma_{jr}) \tag{42}$$

Consequently, the dual integrals are

$$\hat{L}_k = \int_S \hat{d}_{kl} n_l \, \mathrm{d}S \tag{43}$$

i.e., in the expanded form,

$$\hat{L}_k = e_{kij} \int_S (\hat{P}_{li} x_j + u_i \sigma_{lj} + \delta_{il} u_r \sigma_{jr}) n_l \, \mathrm{d}S = e_{kij} \int_V u_l (b_{l,i} x_j - \delta_{li} b_j) \, \mathrm{d}V \tag{44}$$

In absence of body forces, $\hat{L}_k = 0$ for any surface not surrounding a defect [11,12].

Since $(b_{l,i} x_j + b_{l,j} x_i)$ is symmetric and e_{kij} skew-symmetric in ij, the right-hand sides in (38) and (44) are opposite, and we conclude that

$$L_k + \hat{L}_k = 0 \tag{45}$$

It also follows that

$$\begin{aligned} L_k - \hat{L}_k &= e_{kij} \int_S \big[(u_r \sigma_{lr,i} - \sigma_{lr} u_{r,i}) x_j + 2\sigma_{li} u_j - \delta_{il} u_r \sigma_{jr}\big] n_l \, \mathrm{d}S \\ &= 2e_{kij} \int_V u_l (\delta_{li} b_j + b_{l,j} x_i) \, \mathrm{d}V \end{aligned} \tag{46}$$

Combining (45) and (46), we obtain

$$\begin{aligned} L_k &= \frac{1}{2} e_{kij} \int_S \big[(u_r \sigma_{lr,i} - \sigma_{lr} u_{r,i}) x_j + 2\sigma_{li} u_j - \delta_{il} u_r \sigma_{jr}\big] n_l \, \mathrm{d}S \\ &= e_{kij} \int_V u_l (\delta_{li} b_j + b_{l,j} x_i) \, \mathrm{d}V \end{aligned} \tag{47}$$

In absence of body forces, this simplifies to

$$L_k = \frac{1}{2} e_{kij} \int_S \left[(u_r \sigma_{lr,i} - \sigma_{lr} u_{r,i}) x_j + 2\sigma_{li} u_j - \delta_{il} u_r \sigma_{jr} \right] n_l \, \mathrm{d}S = 0 \tag{48}$$

which is the reciprocal representation of the type (18) and (33).

5. Dual integrals and energy release rates

The physical interpretation of the dual integrals is given in this section, based on the consideration of the energy release rates of the potential and complementary potential energies. The analysis is an extension of the analysis of the conservation integrals and the release rates of the potential energy, presented by Budiansky and Rice [5]. Consider the body of volume V loaded by the surface tractions $T_i = \bar{T}_i$ over the portion S_T of its external surface S. The displacements $u_i = \bar{u}_i$ are prescribed over the remaining part S_u. Suppose that within a body there is an unloaded cavity of the bounding surface S_0. The potential energy of such body is

$$\Pi = \int_V W \, \mathrm{d}V - \int_{S_T} \bar{T}_i u_i \, \mathrm{d}S - \int_V b_i u_i \, \mathrm{d}V \tag{49}$$

Without changing the boundary conditions on S, the rate of change of the potential energy associated with the spatial variation of the cavity surface S_0, described by its velocity field $\dot{u}_i^0$, is

$$\dot{\Pi} = \int_V \dot{W} \, \mathrm{d}V - \int_{S_0} W \dot{u}_i^0 n_i \, \mathrm{d}S - \int_{S_T} \bar{T}_i \dot{u}_i \, \mathrm{d}S - \int_V b_i \dot{u}_i \, \mathrm{d}V + \int_{S_0} b_j u_j \dot{u}_i^0 n_i \, \mathrm{d}S \tag{50}$$

where $\dot{u}_i$ is the associated velocity field within $V(t)$ due to imposed velocity $\dot{u}_i^0$. Body forces are assumed to be unaffected by the cavity motion (dead body forces). The surface integrals over S_0 on the right-hand side follow from the Reynolds transport theorem, where n_i is the unit normal to S_0 directed into the material. Assuming that $\dot{u}_i$ is a kinematically admissible field within $V(t)$, and by using the Gauss divergence theorem, it readily follows that [33]

$$\dot{\Pi} = -\int_{S_0} (W - b_j u_j) \dot{u}_i^0 n_i \, \mathrm{d}S \tag{51}$$

The rate of energy release due to spatial variation of S_0, specified by a prescribed velocity field $\dot{u}_i^0$, is $f = -\dot{\Pi}$. This represents an energetic or configurational force on the cavity (defect). Since $(W - b_j u_j) n_i = P_{ji} n_j$ over the unloaded S_0, we obtain

$$f = -\dot{\Pi} = \int_{S_0} P_{ji} \dot{u}_i^0 n_j \, \mathrm{d}S \tag{52}$$

If the cavity translates with a unit velocity in the k-direction, then $\dot{u}_i^0$ can be replaced by δ_{ik}, and (52) gives the rate of energy release per unit cavity translation in the k-direction,

$$f_k = \int_{S_0} P_{jk} n_j \, \mathrm{d}S = J_k(S_0) \tag{53}$$

Since the cavity is unloaded, this is equal to J_k evaluated over S_0. By applying the Gauss divergence theorem to the surface $S_0 + S$ bounding a region between S_0 and any closed surface S around the cavity, and by using (7), the configurational force f_k is also equal to

$$f_k = J_k(S) + \int_V b_{j,k} u_j \, \mathrm{d}V \tag{54}$$

where

$$J_k(S) = \int_S P_{jk} n_j \, \mathrm{d}S \tag{55}$$

If the body forces are spatially uniform, there is a conservation law $J_k = 0$ over the closed surface that does not enclose a cavity, so that $f_k = J_k(S_0) = J_k(S)$. In the absence of body forces, that result was originally derived in [5].

If the cavity transforms such that $\dot{u}_i^0 = x_i$,

$$f = \int_{S_0} P_{ji} x_i n_j \, \mathrm{d}S = M(S_0) \tag{56}$$

Alternatively, by using any other closed surface S around the cavity,

$$f = M(S) - \int_V u_i \left(\frac{3-4r}{r} b_i - b_{i,k} x_k \right) \mathrm{d}V \tag{57}$$

where

$$M(S) = \int_S \left(P_{jk} x_k - \frac{3-r}{r} \sigma_{jk} u_k \right) n_j \, \mathrm{d}S \tag{58}$$

If the absence of body forces, there is a conservation law $M = 0$ over the closed surface that does not enclose a cavity, so that $f = M(S_0) = M(S)$, as originally shown in [5].

If the cavity is given a unit angular velocity around the k-axis, then $\dot{u}_i^0$ in (52) can be replaced by $-e_{kil} x_l$, and

$$f_k = -e_{kil} \int_{S_0} P_{ji} x_l n_j \, \mathrm{d}S = -L_k(S_0) \tag{59}$$

When expressed in terms of the surface integral over S, this is

$$f_k = -L_k(S) - e_{kij} \int_V u_l (\delta_{lj} b_i + b_{l,i} x_j) \, \mathrm{d}V \tag{60}$$

where

$$L_k(S) = e_{kij} \int_S (P_{li} x_j + \sigma_{li} u_j) n_l \, \mathrm{d}S \tag{61}$$

If the absence of body forces, there is a conservation law $L_k = 0$ over the closed surface that does not enclose a cavity [5], so that $f_k = L_k(S_0) = L_k(S)$.

5.1. Complementary energy release rates

We now relate the release rates of the complementary potential energy to the previously derived dual integrals. The complementary potential energy is defined by

$$\Omega = \int_V \Phi \, \mathrm{d}V - \int_{S_u} \bar{u}_i T_i \, \mathrm{d}S \tag{62}$$

such that $\Pi + \Omega = 0$. The rate of the complementary potential energy associated with spatial variation of the cavity due to its velocity field $\dot{u}_i^0$ is

$$\dot{\Omega} = \int_V \dot{\Phi} \, \mathrm{d}V - \int_{S_0} \Phi \dot{u}_i^0 n_i \, \mathrm{d}S - \int_{S_u} \bar{u}_i \dot{T}_i \, \mathrm{d}S \tag{63}$$

where $\dot{T}_i$ is the induced loading rate on S_u due to infinitesimal motion of S_0. In geometrically linear theory, we ignore the change of S due to $\dot{u}_i^0$. Assuming the stress rate field within $V(t)$ is statically admissible, and that body forces are unaffected by the motion of the cavity, we can write

$$\dot{\Phi} = \epsilon_{ij} \dot{\sigma}_{ij} = (u_j \dot{\sigma}_{ij})_{,i} \tag{64}$$

The stress rate $\dot{\sigma}_{ij}$ is the stress rate at fixed points in space, i.e., a nonconvected stress rate. Thus,

$$\int_V \dot{\Phi}\,\mathrm{d}V = \int_S u_j \dot{\sigma}_{ij} n_i\,\mathrm{d}S - \int_{S_0} u_j \dot{\sigma}_{ij} n_i\,\mathrm{d}S \tag{65}$$

For a geometrically linear theory, $\dot{\sigma}_{ij} n_i = \dot{T}_j$ on S ($\dot{T}_j$ being equal to zero on S_T). Consequently, (65) can be rewritten as

$$\int_V \dot{\Phi}\,\mathrm{d}V = \int_{S_u} \bar{u}_j \dot{T}_j\,\mathrm{d}S - \int_{S_0} u_j \dot{\sigma}_{ij} n_i\,\mathrm{d}S \tag{66}$$

The substitution into (63) yields

$$\dot{\Omega} = -\int_{S_0} (\Phi \dot{u}_i^0 + u_j \dot{\sigma}_{ij}) n_i\,\mathrm{d}S \tag{67}$$

The surface of the cavity is unloaded, so that its traction $T_j = n_i \sigma_{ij}$ remains zero throughout the motion. Thus,

$$\frac{\mathrm{d}T_j}{\mathrm{d}t} = \frac{\mathrm{d}n_i}{\mathrm{d}t}\sigma_{ij} + n_i \frac{\mathrm{d}\sigma_{ij}}{\mathrm{d}t} = 0 \tag{68}$$

where $\mathrm{d}/\mathrm{d}t$ designates the material time derivative, following the particle. Expressing the material derivative of stress as the sum of its local ($\dot{\sigma}_{ij}$) and convected ($\sigma_{ij,l}\dot{u}_l^0$) part, (68) gives

$$n_i \dot{\sigma}_{ij} = -\frac{\mathrm{d}n_i}{\mathrm{d}t}\sigma_{ij} - n_i \sigma_{ij,l} \dot{u}_l^0 \tag{69}$$

If the cavity translates, or expands in a self-similar manner, then $\mathrm{d}n_i/\mathrm{d}t = 0$ and

$$n_i \dot{\sigma}_{ij} = -n_i \sigma_{ij,l} \dot{u}_l^0 \tag{70}$$

When (70) is introduced in (67), there follows

$$\dot{\Omega} = \int_{S_0} (-\Phi \delta_{il} + u_j \sigma_{ij,l}) n_i \dot{u}_l^0\,\mathrm{d}S = -\int_{S_0} \hat{P}_{il} n_i \dot{u}_l^0\,\mathrm{d}S \tag{71}$$

Since $\Pi + \Omega = 0$, the release rate of the complementary potential energy due to spatial variation of S_0 is

$$f = -\dot{\Pi} = \dot{\Omega} = -\int_{S_0} \hat{P}_{il} n_i \dot{u}_l^0\,\mathrm{d}S \tag{72}$$

If the cavity translates with a unit velocity in the k-direction, then $\dot{u}_l^0$ is replaced by δ_{kl}, and (72) gives the release rate of the complementary potential energy per unit cavity translation in the k-direction,

$$f_k = -\int_{S_0} \hat{P}_{ik} n_i\,\mathrm{d}S = -\hat{J}_k(S_0) \tag{73}$$

Furthermore, by (14) and the Gauss divergence theorem, it follows that

$$f_k = -\hat{J}_k(S) + \int_V u_i b_{i,k}\,\mathrm{d}V \tag{74}$$

where

$$\hat{J}_k(S) = \int_S \hat{P}_{jk} n_j\,\mathrm{d}S \tag{75}$$

By comparing with (54) we also conclude that

$$\hat{J}_k(S) = -J_k(S) \tag{76}$$

for any surface S surrounding the cavity.

If the cavity transforms such that $\dot{u}_l^0 = x_l$, the energy release rate is

$$f = -\int_{S_0} \hat{P}_{il} n_i x_l \, \mathrm{d}S = -\hat{M}(S_0) \tag{77}$$

In view of (29), the configurational force can be expressed as

$$f = -\hat{M}(S) + \int_V u_i \left(\frac{3+s}{s} b_i + b_{i,k} x_k \right) \mathrm{d}V \tag{78}$$

where

$$\hat{M}(S) = \int_S \left(\hat{P}_{jk} x_k - \frac{3}{s} u_k \sigma_{jk} \right) n_j \, \mathrm{d}S \tag{79}$$

By comparing with (57), we conclude that

$$\hat{M}(S) = -M(S) \tag{80}$$

for any surface S surrounding the cavity.

If the cavity rotates within the material, then

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} = -n_j Q_{ji} \tag{81}$$

where Q_{ji} are the components of antisymmetric spin matrix, and $\dot{u}_i^0 = Q_{ij} x_j$. When this is introduced into (69), there follows

$$n_i \dot{\sigma}_{ij} = (\delta_{ik} \sigma_{lj} - \sigma_{ij,k} x_l) n_i Q_{kl} \tag{82}$$

and (67) gives

$$f = \dot{\Omega} = -\int_{S_0} (\hat{P}_{ik} x_l + \delta_{ik} u_j \sigma_{lj}) n_i Q_{kl} \, \mathrm{d}S \tag{83}$$

If the spin is of unit magnitude and about the k-axis, then $Q_{ij} = -e_{ijk}$ and from (83) the corresponding configurational force is

$$f_k = e_{ijk} \int_{S_0} (\hat{P}_{li} x_j + \delta_{li} u_r \sigma_{jr}) n_l \, \mathrm{d}S = \hat{L}_k(S_0) \tag{84}$$

If an arbitrary surface S around the cavity is used, and in view of (44), we obtain

$$f_k = \hat{L}_k(S) - e_{kij} \int_V u_l (b_{l,i} x_j - \delta_{li} b_j) \, \mathrm{d}V \tag{85}$$

where

$$\hat{L}_k(S) = e_{kij} \int_S (\hat{P}_{li} x_j + u_i \sigma_{lj} + \delta_{il} u_r \sigma_{jr}) n_l \, \mathrm{d}S \tag{86}$$

By comparing with (60) we conclude that

$$\hat{L}_k(S) = -L_k(S) \tag{87}$$

for any surface S surrounding the cavity.

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Appendix A. Dual integrals for plane strain

In the case of plane strain, the energy momentum tensor and its dual are

$$P_{\alpha\beta} = (W - b_\gamma u_\gamma)\delta_{\alpha\beta} - \sigma_{\alpha\gamma} u_{\gamma,\beta}, \quad P_{\alpha\beta,\alpha} = -b_{\alpha,\beta} u_\alpha$$
$$\hat{P}_{\alpha\beta} = \Phi \delta_{\alpha\beta} - u_\gamma \sigma_{\alpha\gamma,\beta}, \quad \hat{P}_{\alpha\beta,\alpha} = b_{\alpha,\beta} u_\alpha$$

where the Greek subscripts range from 1 to 2. The dual J integrals are

$$J_\beta = \int\limits_C P_{\alpha\beta} n_\alpha \, \mathrm{d}C = -\int\limits_A b_{\alpha,\beta} u_\alpha \, \mathrm{d}A$$
$$\hat{J}_\beta = \int\limits_C \hat{P}_{\alpha\beta} n_\alpha \, \mathrm{d}C = \int\limits_A b_{\alpha,\beta} u_\alpha \, \mathrm{d}A$$

for any closed contour C which does not surround a singularity or defect. The area within C is denoted by A. If the body forces are spatially uniform, there is a conservation law $J_\beta = \hat{J}_\beta = 0$.

The energy momentum tensor satisfies the equation

$$(P_{\alpha\beta} x_\beta)_{,\alpha} - P_{\alpha\alpha} = -u_\alpha b_{\alpha,\beta} x_\beta$$

where

$$P_{\alpha\alpha} = \frac{2-r}{r} \sigma_{\alpha\beta} u_{\beta,\alpha} - 2 b_\alpha u_\alpha$$

Thus,

$$M = \int\limits_C \left(P_{\alpha\beta} x_\beta - \frac{2-r}{r} \sigma_{\alpha\beta} u_\beta \right) n_\alpha \, \mathrm{d}C = \int\limits_A u_\alpha \left(\frac{2-3r}{r} b_\alpha - b_{\alpha,\beta} x_\beta \right) \mathrm{d}A$$

Similarly, the complementary energy momentum tensor satisfies the equation

$$(\hat{P}_{\alpha\beta} x_\beta)_{,\alpha} - \hat{P}_{\alpha\alpha} = u_\alpha b_{\alpha,\beta} x_\beta$$

where

$$\hat{P}_{\alpha\alpha} = \frac{2}{s} \sigma_{\alpha\beta} u_{\beta,\alpha} + b_\alpha u_\alpha$$

Consequently,

$$\hat{M} = \int\limits_C \left(\hat{P}_{\alpha\beta} x_\beta - \frac{2}{s} \sigma_{\alpha\beta} u_\beta \right) n_\alpha \, \mathrm{d}C = \int\limits_A u_\alpha \left(\frac{2+s}{s} b_\alpha + b_{\alpha,\beta} x_\beta \right) \mathrm{d}A$$

If body forces are absent, or if they are homogeneous functions of degree $-(3 - 2/r) = -(1 + 2/s)$, there is a conservation law $M = \hat{M} = 0$ for any contour C that does not embrace a singularity or defect.

Finally, the dual L integrals of plane strain elasticity are

$$L_3 = e_{3\alpha\beta} \int\limits_C (P_{\gamma\alpha} x_\beta + \sigma_{\gamma\alpha} u_\beta) n_\gamma \, \mathrm{d}C = -e_{3\alpha\beta} \int\limits_A u_\gamma (\delta_{\gamma\beta} b_\alpha + b_{\gamma,\alpha} x_\beta) \, \mathrm{d}A$$
$$\hat{L}_3 = e_{3\alpha\beta} \int\limits_C (\hat{P}_{\gamma\alpha} x_\beta + u_\alpha \sigma_{\gamma\beta} + \delta_{\alpha\gamma} u_\delta \sigma_{\beta\delta}) n_\gamma \, \mathrm{d}C = e_{3\alpha\beta} \int\limits_A u_\gamma (b_{\gamma,\alpha} x_\beta - \delta_{\gamma\alpha} b_\beta) \, \mathrm{d}A$$

Appendix B. Dual integrals for anti-plane strain

In the case of anti-plane strain, the dual energy momentum tensors are

$$P_{\alpha\beta} = (W - b_3 u_3)\delta_{\alpha\beta} - \sigma_{\alpha 3} u_{3,\beta}, \quad P_{\alpha\beta,\alpha} = -b_{3,\beta} u_3$$
$$\hat{P}_{\alpha\beta} = \Phi \delta_{\alpha\beta} - u_3 \sigma_{\alpha 3,\beta}, \quad \hat{P}_{\alpha\beta,\alpha} = b_{3,\beta} u_3$$

The corresponding dual J_β integrals are given by

$$J_\beta = \int_C P_{\alpha\beta} n_\alpha \, \mathrm{d}C = -\int_A b_{3,\beta} u_3 \, \mathrm{d}A$$
$$\hat{J}_\beta = \int_C \hat{P}_{\alpha\beta} n_\alpha \, \mathrm{d}C = \int_A b_{3,\beta} u_3 \, \mathrm{d}A$$

The energy momentum tensor satisfies the equation

$$(P_{\alpha\beta} x_\beta)_{,\alpha} - P_{\alpha\alpha} = -u_3 b_{3,\alpha} x_\alpha$$

where

$$P_{\alpha\alpha} = \frac{2-r}{r} \sigma_{\alpha 3} u_{3,\alpha} - 2 b_3 u_3$$

Thus, since $\sigma_{\alpha 3,\alpha} + b_3 = 0$, we obtain

$$M = \int_C \left(P_{\alpha\beta} x_\beta - \frac{2-r}{r} \sigma_{\alpha 3} u_3 \right) n_\alpha \, \mathrm{d}C = \int_A u_3 \left(\frac{2-3r}{r} b_3 - b_{3,\alpha} x_\alpha \right) \mathrm{d}A$$

Similarly, the complementary energy momentum tensor satisfies the equation

$$(\hat{P}_{\alpha\beta} x_\beta)_{,\alpha} - \hat{P}_{\alpha\alpha} = u_3 b_{3,\alpha} x_\alpha$$

where

$$\hat{P}_{\alpha\alpha} = \frac{2}{s} \sigma_{\alpha 3} u_{3,\alpha} + b_3 u_3$$

Consequently,

$$\hat{M} = \int_C \left(\hat{P}_{\alpha\beta} x_\beta - \frac{2}{s} u_3 \sigma_{\alpha 3} \right) n_\alpha \, \mathrm{d}C = \int_A u_3 \left(\frac{2+s}{s} b_3 + b_{3,\alpha} x_\alpha \right) \mathrm{d}A$$

Finally, the dual L_3 integrals are

$$L_3 = e_{\alpha\beta 3} \int_C P_{\gamma\alpha} x_\beta n_\gamma \, \mathrm{d}C = -e_{3\alpha\beta} \int_A u_3 b_{3,\alpha} x_\beta \, \mathrm{d}A$$
$$\hat{L}_3 = e_{\alpha\beta 3} \int_C (\hat{P}_{\gamma\alpha} x_\beta + \delta_{\alpha\gamma} u_3 \sigma_{\beta 3}) n_\gamma \, \mathrm{d}C = e_{3\alpha\beta} \int_A u_3 b_{3,\alpha} x_\beta \, \mathrm{d}A$$

If the body force b_3 is uniform, or absent, the conservation laws $J_\beta = \hat{J}_\beta = 0$ and $L_3 = \hat{L}_3 = 0$ hold for any contour C which does not embrace a singularity or defect. There is also a conservation law $M = \hat{M} = 0$ if the body force b_3 is absent, or of it is a homogeneous function of degree $-(3 - 2/r) = -(1 + 2/s)$ in spatial coordinates x_1 and x_2.

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