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Duality, inverse problems and nonlinear problems in solid mechanics

Travelling interface waves in a brake-like system under unilateral contact and Coulomb friction

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Abstract

This article considers the frictional interface waves generated by the flutter instability of the sliding steady state for an elastic tube in frictional contact with a rigid and rotating shaft. According to the values of the contact pressure, the rotation velocity and the friction coefficient, several periodic dynamical responses can be found under the form of travelling surface waves. Such a periodic solution may be interesting in the study of a possible dynamic transition from the sliding steady state in the spirit of Andronov–Hopf bifurcation. Examples of stick-slip, stick-slip-separation and stick-slip-separation-reverse-slip waves propagating along the contact surface, obtained by various semi-analytical and numerical approaches, are reported here. Some results on the stability of these travelling waves are also indicated. ***To cite this article: Q.S. Nguyen et al., C. R. Mecanique 336 (2008).***
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Résumé

Ondes d'interface sous contact unilatéral et frottement de Coulomb. Cet article est consacrée aux ondes d'interface induites par l'instabilité dynamique de glissement stationnaire à l'interface d'un tube élastique en contact frottant avec un cylindre rigide en rotation uniforme. Selon les valeurs de la pression de contact, de la vitesse de rotation et du coefficient de frottement, plusieurs réponses dynamiques périodiques peuvent être observées. Nous présentons des exemples d'ondes stick-slip, stick-slip-separation et stick-slip-separation-slip obtenues par des approches numériques et semi-analytiques. Quelques résultats concernant l'analyse de stabilité d'une telle onde sont aussi rapportés. ***Pour citer cet article : Q.S. Nguyen et al., C. R. Mecanique 336 (2008).***
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Keywords: Friction; Stick-slip-separation-slip wave; Stability of waves

Mots-clés : Frottement ; Onde glissement-adhérence-décollement-glissement ; Étude de stabilité

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1. Introduction

This paper is dedicated to Professor Huy Duong Bui whose scientific contributions are exceptionally rich and widely recognized in several fields, especially in fracture mechanics, inverse problems and dual methods in mechanics. Since the problem of frictional contact of solids is also one of his centers of interest [1,2], we present here some results and some open problems on interface waves resulting from the flutter instability of a system of coaxial cylinders in contact with Coulomb friction.

During the last decade, several works have been devoted to the study of the instability of the sliding steady state between solids, cf. [3–5]. It is found that there is a strong connection between the dynamic instability and the existence of certain interfacial waves in frictionless contact [6]. Precisely, it has been shown that, when the generalized Rayleigh wave exists, steady sliding with Coulomb friction is dynamically unstable for arbitrarily small values of the friction coefficient. Moreover, this instability was discussed numerically in many works ([7–9], etc.).

Although the mechanism of instability is now well understood, only a few works have been carried out to investigate the friction-induced oscillations in continuum bodies. Comninou et al. [10] and Adams [4] showed that self-sustained waves are mathematically feasible between two debonded identical half-planes and two sliding different half-planes, respectively.

This Note presents some theoretical and numerical results for the flutter instability of the steady sliding response in view of the interpretation of the brake squeal phenomenon. In particular, the possibility of dynamic bifurcation to a periodic self-excited response, in the spirit of Andronov–Hopf bifurcation, is illustrated in a simple example of coaxial and rotating cylinders in frictional contact. In this example, the bifurcated solution consists of stick-slip-separation waves propagating at high speed on the contact surface.

2. The problem of coaxial cylinders

Consider a brake-like system composed of an elastic annular tube with internal radius R and external radius R^* in frictional contact with a rotating rigid shaft of radius $R+d$ ($d \geqslant 0$) and of angular velocity Ω, cf. Fig. 1. The elastic cylinder is fixed at its outer surface and the frictional model is Coulomb's law with a constant coefficient f. The mismatch d is considered as a load parameter controlling the normal contact pressure. Within the framework of linear elasticity, the dynamic equations of the motion with the corresponding boundary and unilateral frictional contact read:

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma} = \gamma \ddot{\mathbf{u}} \\ E\boldsymbol{\sigma} = \lambda \operatorname{tr}(\epsilon)\mathbf{I} + 2\mu\boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon} = \operatorname{grad}_s \mathbf{u} \\ u(\xi,\varphi,t) = v(\xi,\varphi,t) = 0 \\ \sigma_{rr}(1,\varphi,t) = -p(\varphi,t), \sigma_{r\varphi}(1,\varphi,t) = -q(\varphi,t) \\ u \geqslant \delta, \quad p \geqslant 0, \quad p(u-\delta) = 0 \\ |q| \leqslant fp, \quad q(1-\dot{v}) - fp|1-\dot{v}| = 0 \end{cases} \tag{1}$$

in terms of polar coordinates (r,φ) and nondimensional variables:

$$\mathbf{u} = \frac{\bar{\mathbf{u}}}{R}, \quad \boldsymbol{\sigma} = \frac{\bar{\boldsymbol{\sigma}}}{E}, \quad r = \frac{\bar{r}}{R}, \quad \gamma = \frac{\rho R^2 \Omega^2}{E} \quad \xi = \frac{R^*}{R}, \quad \delta = \frac{d}{R}, \quad t = \Omega \bar{t}, \quad \dot{\mathbf{u}} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}$$

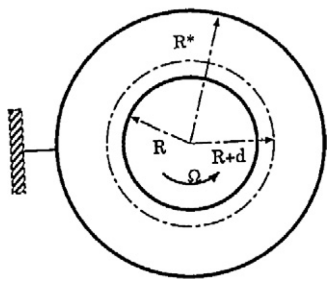

Fig. 1. The problem of the encased cylinders.

This boundary value problem (1) has the following steady state sliding solution:

$$\begin{cases} u_e = \delta \dfrac{1}{\xi^2 - 1}\left(\dfrac{\xi^2}{r} - r\right), \quad v_e = \delta f \dfrac{1}{\xi^2 - 1}\left(\dfrac{\xi^2}{r} - r\right)\left(1 + \dfrac{1}{\xi^2(1-2\mu)}\right) \\ p_e = \delta \dfrac{1}{1-\xi^2}\dfrac{1}{1+\mu}\left(\xi^2 + \dfrac{1}{1-2\mu}\right), \quad q_e = f p_e \end{cases} \tag{2}$$

It has been shown in [11] that the steady sliding equilibrium is unstable. The proof of this result can be shown under the assumption of sliding motions, in the same spirit as in the sliding of elastic layers, cf. [4] or [5].

The governing equations (1) are highly nonlinear because of the nonlinearity introduced by Coulomb's law and by unilateral conditions at the interface between the cylinder and the shaft. Since nontrivial closed form solutions cannot be achieved, an interesting simplification to the problem has been proposed and discussed in [11,12] where the displacement is sought in the form:

$$u(r,\varphi,t) = U(\varphi,t)\frac{u_e(r)}{\delta}, \qquad v(r,\varphi,t) = V(\varphi,t)\frac{u_e(r)}{\delta} \tag{3}$$

From the Principle of Virtual Work, the following system of partial differential equations is derived:

$$\begin{cases} \ddot{U} + bU'' + DV' - gU + P = 0 \\ \ddot{V} + aV'' - DU' - hV + Q = O \\ P \geqslant 0, \quad U \geqslant \delta, \quad P(U-\delta) = 0 \\ |Q| \leqslant fP, \quad Q(1-\dot{V}) - fP|1-\dot{V}| = 0 \end{cases} \tag{4}$$

The steady sliding solution, given by $U_e = \delta$, $V_e = \delta fg/h$, is unstable for the reduced system (4). When $f > 0$ and $D > 0$, it has been proved that a small perturbation of the steady sliding solution will lead to a growing wave propagating in the sense of the imposed rotation, and a decaying wave propagating in the opposite direction. If $f > 0$ and $D < 0$, the growing wave propagates in the opposite sense.

Due to the nonlinear conditions of contact and friction, several different contact regimes are possible:

- stick regime: the elastic tube rotates with the rigid cylinder,

$$U = \delta, \quad \dot{V} = 1, \quad P > 0, \quad |Q| < fP \tag{5}$$

- positive-slip regime: the elastic tube slides along the rigid cylinder,

$$U = \delta, \quad \dot{V} < 1, \quad P > 0, \quad Q = fP \tag{6}$$

- negative-slip (or reverse-slip) regime: the elastic tube rotates faster than the shaft,

$$U = \delta, \quad \dot{V} > 1, \quad P > 0, \quad Q = -fP \tag{7}$$

- separation: the contact between the bodies is lost

$$P = Q = 0, \quad U > \delta \tag{8}$$

On the other hand, since the steady response is unstable due to a flutter instability, the perturbed motion may eventually tend toward a periodic response. Therefore, the search for periodic solutions of stick-slip, or stick-separation, or slip-separation, or stick-slip-separation or stick-slip-separation-slip waves is a priori an interesting problem. Periodic solutions of (4) are sought under the form of travelling waves of non-dimensional celerity c:

$$U = U(\theta), \quad V = V(\theta), \quad \theta = \varphi - ct \tag{9}$$

where U and V are periodic functions of $\theta \in [0, \frac{2\pi}{k}]$ for a mode-k wave. It follows that a travelling wave is governed by the system of differential equations

$$\begin{cases} (c^2 - b)U'' - DV' + gU = P \\ (c^2 - a)V'' + DU' + hV = Q \\ P \geqslant 0, \quad U \geqslant \delta, \quad P(U-\delta) = 0 \\ |Q| \leqslant fP, \quad Q(1+cV') - fP|1+cV'| = 0 \end{cases} \tag{10}$$

3. Travelling interface waves

3.1. Stick-slip and stick-slip-separation waves

The semi-analytic approach developed in [11,12] enables us to find stick-slip and stick-slip-separation waves, cf. Fig. 2.

Moreover, in [11,12], with the finite element method a solution of the frictional contact between the two coaxial cylinders is computed using an explicit time-discretization scheme proposed by Carpentar et al. [13]. This trial-error type algorithm relies on the forward Lagrange multipliers method to enforce the non-interpenetrability constraint and the Coulomb's friction law along the contact surface. This algorithm has been applied for the problem of the coaxial cylinders. It is checked that one obtains different regimes of stick-slip or stick-slip-separation waves depending essentially on the values of the mismatch d, the angular velocity Ω and the coefficient of friction f. For example, Fig. 3 shows numerical results on the radial displacement U for stick-slip and stick-slip-separation travelling waves with a mesh size of 132 nodal points on the contact surface.

These two approaches (semi-analytical and numerical methods) permit a detailed analysis of the influence of the data (friction coefficient f, Poisson coefficient ν, ratio R/R^*, angular velocity Ω, mismatch d) on the nature of the dynamic response of the brake-like system. The reader can refer to [12] for more details.

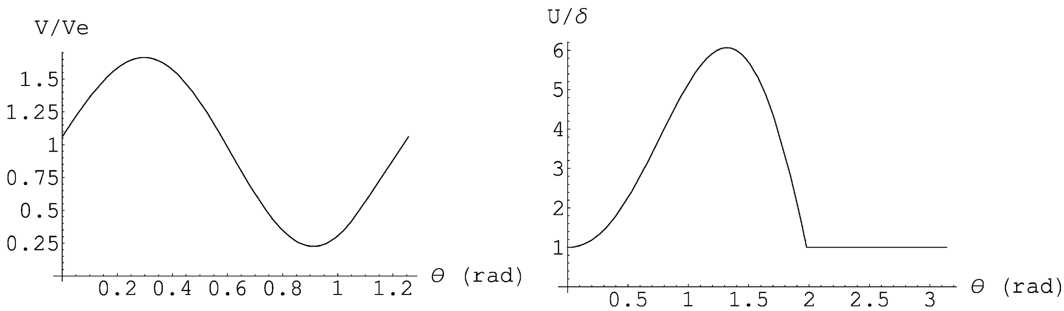

Fig. 2. Tangential and normal displacements for a stick-slip wave and for a stick-slip separation wave respectively.

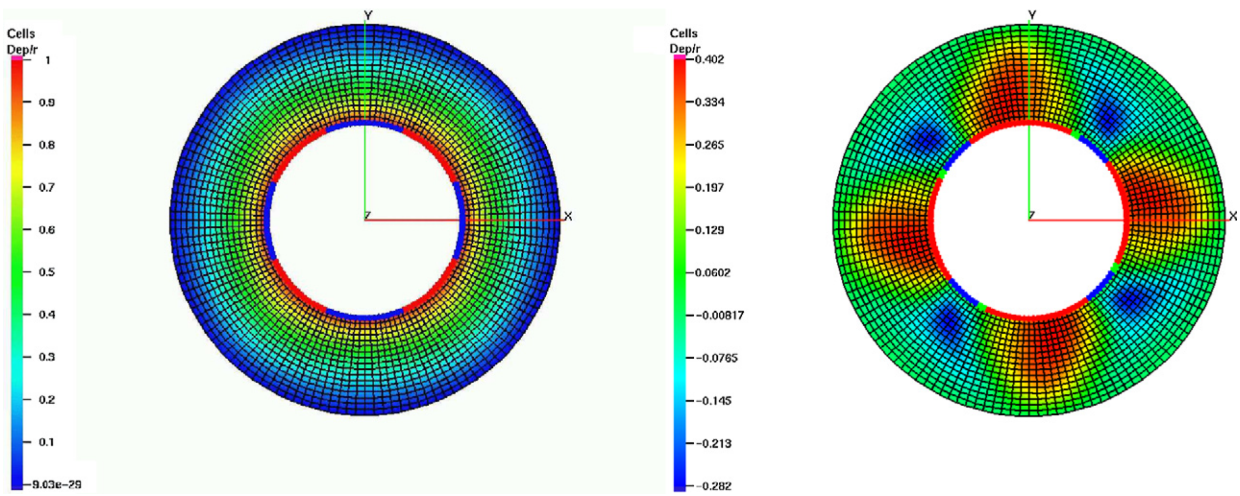

Fig. 3. Stick-slip and stick-slip-separation waves in mode 4, obtained by the F.E.M. [12].

3.2. *Stick-slip-separation-reverse slip*

This section presents a new family of solutions: the stick-slip-separation-reverse slip self-excited waves. This type of regime was also found for a simple friction oscillator with a more complicated dry friction law in [15].

The nonlinear BVP (10) is solved with the program *Boundsco* based on the multiple shooting method [14], which is able to compute the switching points between different regions automatically. In fact, at the borders between the different regions appropriate switching conditions must be fulfilled, e.g. $P = 0$ at the start of the separation and $U = \delta$ and U' jumps back to 0 at the end. For the sticking region we have $V' = 1/c$ at the start and $Q = fP$ at the end.

In Fig. 4, the pressure is drawn as function of the circumferential angle θ for a mode-4 solution for various values of the mismatch d as parameter. While for larger values of d just a slip-stick solution occurs, the contact pressure becomes zero, if d is decreased below a certain value. If we decrease the mismatch further, a separation interval occurs, as it can be seen in Fig. 5, which shows the radial displacement for different values of d and the loci of the switching points τ_1. If the mismatch d becomes very small, the switching points τ_2 and τ_3 coalesce and the slip region right of the separation zone vanishes.

In Fig. 6 a phase plane plot for the travelling wave in mode-8 with reverse slip (overshooting) is depicted. Fig. 7 shows the relative velocity and the friction force. The short segment between $\theta = 0.62$ and $\theta = 0.73$, where the relative velocity is negative, is clearly visible.

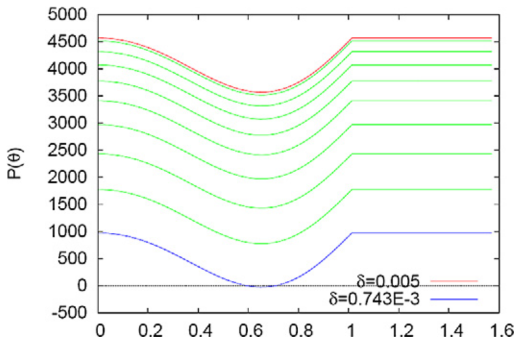

Fig. 4. Pressure for a stick-slip solution depending on the mismatch d for a mode-4 solution.

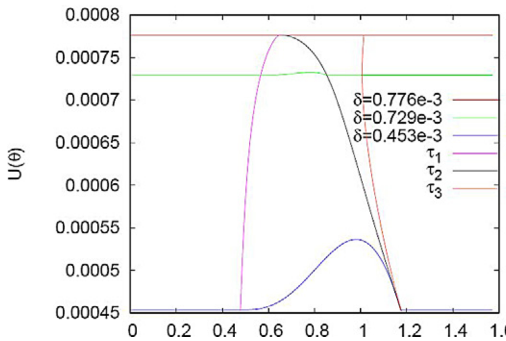

Fig. 5. Radial displacement u showing clear separation for small mismatch d.

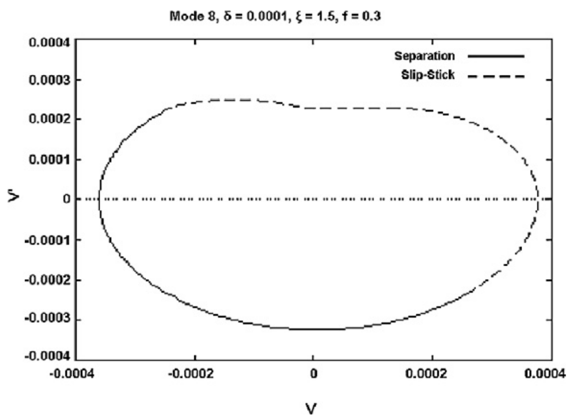

Fig. 6. Phase plane plot of the tangential displacement of a travelling wave with reverse slip (overshooting).

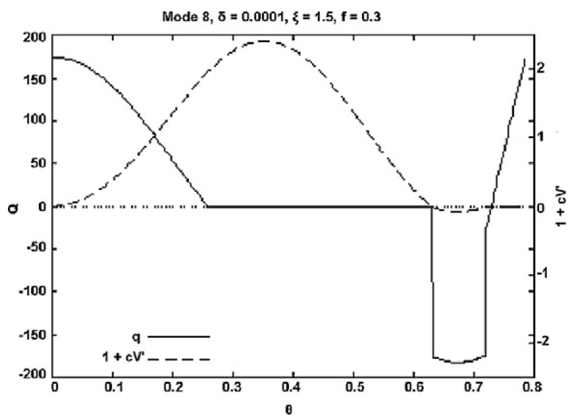

Fig. 7. Friction force and relative velocity for a mode-8 travelling wave with reverse slip.

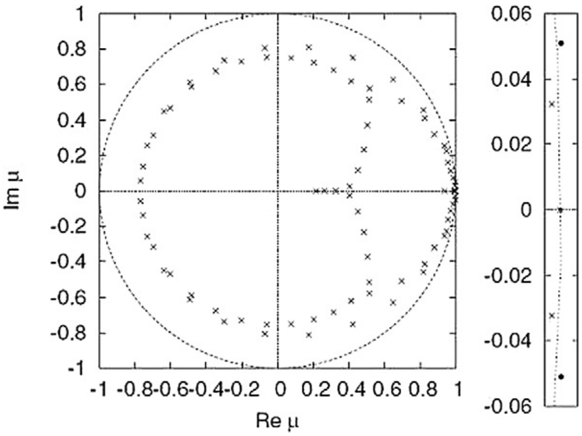

Fig. 8. Floquet multipliers of the discretized system. The enlarged view of the eigenvalues close to 1 shows that there is a pair of eigenvalues outside the unit circle. Therefore, the travelling wave is unstable and one has to expect an oscillating behaviour.

4. Stability analysis of the waves

In order to calculate the stability of the computed solutions, we have to investigate the linearized PDE. For simplicity we consider only the simple slip-stick solution. We used two methods to estimate the stability of the system, cf. [16].

By replacing the spatial derivatives by finite differences, we obtain a large system of ODEs. In order to avoid problems with the discontinuities, we approximate the sign function by $\tanh(x/\epsilon)$, where ϵ is a moderately small number in the range $[10^{-5}, 0.2]$. Travelling waves are obtained by enforcing the boundary conditions

$$\mathbf{y}_i(T) = \mathbf{y}_{i+1}(0), \qquad \mathbf{y}_{N+1} := \mathbf{y}_1 \tag{11}$$

Every component $\mathbf{y}_i(t)$ represents a small part of the travelling wave, connecting a grid point to the next one. The matrix $\mathbf{A} = \mathbf{S}^{-1}Y(0,T)$, where the permutation matrix $\mathbf{S}$ acts by the cyclic shift $\mathbf{y}_i \to \mathbf{y}_{i+1}$ and $Y(0,T)$ is the fundamental solution matrix of the linearized system. It governs the stability of the discretized wave, which is a fixed point of the map $\mathbf{y}(0) = \mathbf{S}^{-1}\mathbf{y}(T)$. Due to the autonomy of the original system one eigenvalue of A will be 1. If all remaining eigenvalues μ lie within the unit circle, the travelling wave is asymptotically stable. With this method one obtains many eigenvalues at once, but these eigenvalues are usually not very accurate, cf. Fig. 8.

In order to improve the accuracy of calculated eigenvalues, we derive a BVP for the eigenfunctions of the system. Since we have to linearize along a stick-slip solution, we make the ansatz $v_L = \exp(\lambda t)\psi(\varphi - ct)$ for the linearized component v_L. From (4), the following second order equation is obtained:

$$c^2\psi'' - 2\lambda\psi' + \lambda^2\psi = \begin{cases} a\psi'' - h\psi - fD\psi', & \text{slip zone} \\ 0, & \text{stick zone} \end{cases} \tag{12}$$

At the boundary between slip and stick regimes, $\dot{v}_L$ jumps to 0. Furthermore $\psi(\theta)$ and $\psi'(\theta)$ are periodic in θ with period 2π, and we also add a scaling condition $\psi(0)^2 + \psi'(0)^2 = 1$. It is necessary to consider the eigenvalue problem on the whole interval $\theta \in [0, 2\pi]$, because for many parameters some k-mode solutions are asymptotically stable, as long as only k-mode perturbations are considered. But these solutions become unstable if the whole interval is considered, unless material damping is introduced. Solving this system together with the wave problem, we obtain one eigenvalue and eigenvector. It should also be noted, that one needs already a rather good initial guess for the BVP solver. However, contrary to the first method the eigenvalues are very accurate and it is quite simple to study the behavior of eigenvalues under parameter variations.

5. Conclusion

The existence of stick-slip and stick-slip-separation-slip waves, travelling either in the same direction as the rigid shaft or rotating in the opposite direction depending on the characteristics of the data has been obtained. Phenomena

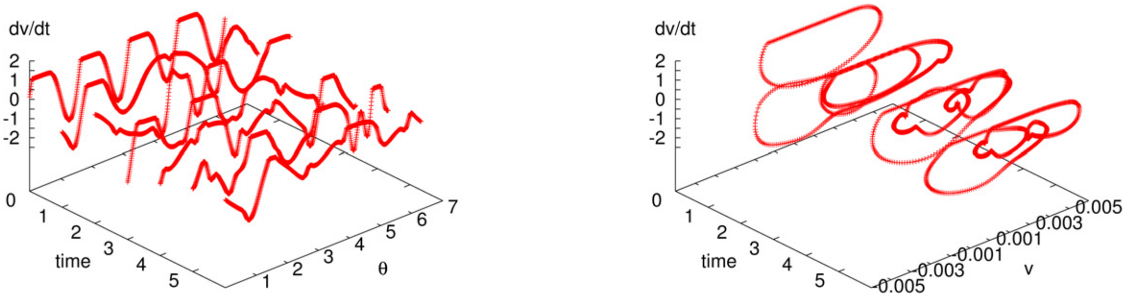

Fig. 9. The left figure displays the evolution of $\dot{v}(\varphi)$ over time. From a mode-4 solution, a quite irregular function develops. The right figure shows the same solution in the $(v, \dot{v})$ phase plane.

like reverse slip or loss of contact have been discussed. Some numerical simulations have been performed for the search of a limit cyclic response and for the stability analysis. However, a global stability analysis of these travelling waves and the general question of transition to a limit cyclic response still remain open problems.

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