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ScienceDirect
C. R. Mecanique 336 (2008) 238–244
COMPTES RENDUS MECANIQUE
http://france.elsevier.com/direct/CRAS2B/

Duality, inverse problems and nonlinear problems in solid mechanics

Optimal control approach in nonlinear mechanics

Claude Stolz

Laboratoire de mécanique des solides – CNRS UMR7649, École polytechnique, 91128 Palaiseau cedex, France

Available online 16 January 2008

Abstract

The purpose of this article is to present some applications of optimal control theory for nonlinear mechanical problems. In particular we consider some inverse problems dealing with the determination of unknown boundary conditions, the determination of internal state resulting of unknown loading history. Extensions to asymptotic behavior due to cyclic loading are also presented. ***To cite this article: C. Stolz, C. R. Mecanique 336 (2008).***

Résumé

Applications du contrôle optimal en mécanique non linéaire. La théorie du contrôle optimal est utilisée pour résoudre divers problèmes de mécanique non linéaire. Par exemple, le contrôle de conditions aux limites est utilisé pour déterminer à l'aide de données surabondantes sur une partie de la frontière du solide, les conditions aux limites manquantes sur la partie complémentaire de la frontière. À partir de données initiales et des déplacements résiduels dus à un chargement inconnu on détermine un état interne et une histoire du chargement compatibles avec l'état final. Enfin dans le cas de chargements cycliques, on caractérise les états asymptotiques (adaptation, accommodation, ...) atteints pour des lois à comportement standard. ***Pour citer cet article : C. Stolz, C. R. Mecanique 336 (2008).***

Keywords: Optimal control; Inverse problems; Cyclic loading

Mots-clés : Contrôle optimal ; Problème inverse ; Chargements cycliques

1. Introduction

The purpose of this article is to present some applications of the optimal control theory in order to solve inverse problems in nonlinear mechanics. We consider a body of volume Ω. The constitutive equations of the material are known. We are dealing with the determination of unknown boundary conditions, with the determination of internal state resulting of unknown loading history or with the asymptotic behavior due to cyclic loading.

In many situations we are not able to know exactly the boundary conditions imposed on the body over one part Γ_i of the boundary $\partial\Omega$. The first class of the considered problem is to determine these conditions knowing the two fields (temperature and flux for thermal problem, displacement and traction for mechanical problems) on the complementary part Γ_o of the boundary. All these problems are not well-posed in the sense of Hadamard [1].

E-mail address: stolz@lms.polytechnique.fr.

1631-0721/$ – see front matter
doi:10.1016/j.crme.2007.11.015

In the second class of inverse problems, knowing the initial state and the final state of the body, we propose a method to estimate the history of loading and the internal state knowing the mechanical quantities on the boundary at the final state.

Finally, using of theoretical results on cyclic response of an anelastic system to cyclic loading, an optimal control approach is proposed to determine the asymptotic behavior. The classical theorems on elastic or plastic shakedown give conditions for the existence of a periodic answer but they don't provide an explicit manner to determine the cyclic answer. The proposed new approach gives the limit cyclic behavior depending on the initial conditions.

2. Determination of unknown boundary conditions

Many methods are dedicated to this problem, here we present methods based on the optimal control theory using of an adjoin problem. As introduction to more complex situation we consider a linear thermal problem on the body Ω. The boundary of the body $\delta\Omega = \Gamma$, is decomposed in two parts Γ_i and Γ_o. The conduction law is an isotropic Fourier law with constant modulus κ. We search an estimation of unknown conditions on Γ_i by knowing the temperature field T_o and the flux field Q_o on the complementary part Γ_o.

So, we must determine the temperature field which satisfies:

Conduction law	$\underline{q} = -\kappa \nabla T,$	over Ω
Equilibrium	$0 = \operatorname{div} \underline{q},$	over Ω
Boundary conditions	$T = T_o,\ \underline{q}.\underline{n} = Q_o,$	along Γ_o

This problem is not well posed in the sense of Hadamard. Many methods of resolution can be used to determine the unknown quantities: integration of the Cauchy problem [1], making use of the quasi-reversibility of J.-L. Lions [2,3] or an optimal control approach as proposed in [3,4]. The last formulation is defined in two steps. The first step is the definition of the dynamical equations; they are defined as a well posed problem. The solution $T(v)$ of this problem is governed by the control variables v. The state equations are those of a classical thermal problem: find the field T satisfying:

Conduction law	$\underline{q} = -\kappa \nabla T,$	over Ω
Equilibrium	$0 = \operatorname{div} \underline{q},$	over Ω
Boundary conditions	$T = v,$	along Γ_i, $\underline{q}.\underline{n} = Q_o$, along Γ_o

The field $T(v)$ is then a functional of the control variable v. The second step is to find an optimal control v. We control the temperature v over Γ_i such that the discrepancy between the resulting temperature $T(v)$ over Γ_o is closed to the given temperature T_o. This optimality is obtained by minimizing an accurate cost function.

The cost function is a functional $J(v)$ of the control variables

$$J(v) = \int_{\Gamma_o} \frac{1}{2} \left\| T(v, Q_o) - T_o \right\|^2 \mathrm{d}S + r \int_{\Gamma_i} \frac{1}{2} \|v\|^2 \mathrm{d}S \tag{1}$$

The constant r must be strictly positive to avoid local minima [2]. Introducing an adjoin field T^* such that $T^* = 0$ over Γ_i, the problem of optimal control can be rewritten as a minimization on T, T^*

$$\mathcal{J}(T, T^*) = \int_{\Omega} \nabla T.\kappa.\nabla T^* \,\mathrm{d}\Omega + \int_{\Gamma_o} T^* Q_o \,\mathrm{d}S + h \int_{\Gamma_o} \frac{1}{2} \|T - T_o\|^2 \mathrm{d}S + r \int_{\Gamma_i} \frac{1}{2} \|T\|^2 \mathrm{d}S \tag{2}$$

The minimization of $\mathcal{J}$ gives the set of equations:

Adjoin problem $\quad 0 = \operatorname{div}(\kappa.\nabla T^*),\ \underline{n}.\nabla T^* = h(T - T_o)$ over $\Gamma_o,\ 0 = T^*$ over Γ_i

Primal problem $\quad 0 = \operatorname{div}(\kappa.\nabla T),\ \underline{n}.\nabla T = Q_o$ over Γ_o

and the optimality condition gives the corresponding boundary condition for the primal problem:

$$rT = \underline{n}.\nabla T^*, \quad \text{over } \Gamma_i \tag{3}$$

The primal problem in T and the adjoin problem in T^* are both linear, with the same constitutive law. They differ only by the prescribed boundary conditions. This formulation has been applied to determine heat source in structure [4]. In this article, comparisons between solutions of the inverse problem with some reference conditions in different configuration are shown. Therefore the presented mathematical and experimental results provide us a way of evaluating the reliability and accuracy of the proposed method. Extension of the method to the determination of the transient solution has also been proposed in [5].

The same method can be applied in linear elasticity. On Γ_o, we know the displacement and the traction $\underline{T}_o$, we search an estimation of the displacement u on the complementary part Γ_i. In this case, the inverse problem is to find a cinematically admissible displacement u satisfying the system of equations:

Compatibility	$\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla^t u)$,	over Ω
BC0	$u = u_o$,	on Γ_o
Equilibrium	$\operatorname{div} \sigma = 0$,	over Ω, $\underline{n}.\sigma = \underline{T}_o$ on Γ_o
Constitutive law	$\sigma = \mathbb{C} : \varepsilon$,	over Ω

As previously, this problem is not well posed. Replacing the boundary condition (BC0) by (BC1): $u = v$ on Γ_i, the primal problem becomes well-posed and we add an optimality process to find the best control v. The optimal displacement v on Γ_i is controlled such that the functional J defined by:

$$J(v) = \int_{\Gamma_o} \frac{1}{2} \left\| u(v, T_o) - u_o \right\|^2 \mathrm{d}S + r \int_{\Gamma_i} \frac{1}{2} \|v\|^2 \mathrm{d}S, \tag{4}$$

is minimum. A solution of the inverse problem is governed by the optimization of the functional $\mathcal{J}$

$$\mathcal{J}(u, u^*) = -\int_{\Omega} \varepsilon(u).\mathbb{C}.\varepsilon(u^*) \,\mathrm{d}\Omega + \int_{\Gamma_o} u^*.T_o \,\mathrm{d}S + h \int_{\Gamma_o} \frac{1}{2} \|u - u_o\|^2 \mathrm{d}S + r \int_{\Gamma_i} \frac{1}{2} \|u\|^2 \mathrm{d}S \tag{5}$$

on the set of cinematically admissible adjoin fields u^* such that $u^* = 0$ over Γ_o.

3. Boundary control and extension to visco-plasticity

Let Ω a domain with external boundary $\partial\Omega = \Gamma = \Gamma_u \cup \Gamma_T$. The body has an elastoviscoplastic behavior. The internal variables are denoted α. The local behavior is defined by a free energy $w(\varepsilon, \alpha)$ and we assume that the internal state evolution is determined by a normality rule $\dot{\alpha} = \frac{\partial \Phi}{\partial A}$ associated with the convex potential $\Phi(A)$ of dissipation, where $A = -\frac{\partial w}{\partial \alpha}$.

We search an estimation of a loading history along a part of the boundary Γ_T knowing both the initial state of the body and the final position at final time t_f of Γ_T. In elasticity, this problem is easy to solve. Assuming that we know the displacement over the boundary, it is clear that we can determine the resulting traction on the boundary. We present here an approach based on optimal control theory to the more complex case of elastoviscoplasticity.

We consider that the initial state $u(x, t_o) = 0$, $\alpha(x, t_o) = 0$ is given over Ω. On Γ_u the displacement is prescribed $u(x, t) = 0$. The final state $u(x, t_f) = u_m(x)$ is known on the complementary part Γ_T. We search the best history of the loading $T(x, t)$ applied on the boundary Γ_T and an internal state $\alpha(x, t)$ such that the resulting displacement along Γ_T at time t_f is closed to $u_m(x)$.

For a given history $T(x, t)$ along Γ_T, we determine the displacement $u(x, t)$, the stress $\sigma(x, t)$, the internal state $\alpha(x, t)$ such that all these quantities satisfy the primal problem corresponding to the set of equations:

Compatibility	$2\varepsilon(u) = \nabla u + \nabla^t u$,	over Ω, $u = 0$, along Γ_u
Equilibrium	$\operatorname{div} \sigma = 0$,	$\underline{n}.\dot{\sigma} = \dot{T}$, along Γ_T
Constitutive laws	$\sigma = \frac{\partial w}{\partial \varepsilon}$,	$A = -\frac{\partial w}{\partial \alpha}$, $\dot{\alpha} = \frac{\partial \Phi}{\partial A}$

In this case, the displacements u, the strains ε, the stresses σ are functions of position and time. In particular, we know the displacement $u(x,t_f)$ along Γ_T. We must determine now the best history $T(x,t)$, such that the displacement at time t_f is closed to the given displacement $u_m(x)$ along Γ_T. The function to minimize is obviously

$$J(u,T)=\int_{\Gamma_T}\frac{1}{2}k\left\|u(x,t_f)-u_m(x)\right\|^2\,\mathrm{d}S+\int_0^{t_f}\int_{\Gamma_T}\frac{1}{2}\dot{T}.H.\dot{T}\;\mathrm{d}S\,\mathrm{d}t \tag{6}$$

Denoting by $\mathcal{W}$ the second derivative of w at state ε,α. We introduce an adjoin state and a functional $\mathcal{L}$ to take into account of all local fields equations and prescribed boundary conditions. Then, we control the history $T(t)$, $t\in[0,t_f]$, in order to ensure the stationarity of the functional $\mathcal{J}(T,u,u^*,A,A^*,\alpha,\alpha^*)=\mathcal{L}+J$ given by:

$$\mathcal{J}=\mathcal{L}+\frac{1}{2}\int_{\Gamma_T}k\left\|u(t_f)-u_o\right\|^2\mathrm{d}S+\int_o^{t_f}\int_{\Gamma_T}\frac{1}{2}\dot{T}.H.\dot{T}\,\mathrm{d}S\,\mathrm{d}t$$

$$\mathcal{L}=-\int_o^{t_f}\int_{\Omega}(\dot{\varepsilon},\dot{\alpha})^t.\mathcal{W}.(\varepsilon^*,\alpha^*)\,\mathrm{d}\Omega\,\mathrm{d}t+\int_o^{t_f}\int_{\Gamma_T}\dot{T}.u^*\,\mathrm{d}S\,\mathrm{d}t+\int_o^{t_f}\int_{\Omega}\left(A^*.\left(-\dot{\alpha}+\frac{\partial\Phi}{\partial A}\right)-\alpha^*\dot{A}\right)\mathrm{d}\Omega\,\mathrm{d}t.$$

Let us introduce the notations $(\dot{\sigma},\dot{B})^t=\mathcal{W}:(\dot{\varepsilon},\dot{\alpha})$, $(\sigma^*,B^*)^t=\mathcal{W}:(\varepsilon^*,\alpha^*)$. The variations of $\mathcal{L}$ and the variations of J are given by:

$$\delta\mathcal{L}=-\int_o^{t_f}\int_{\Omega}\delta\dot{\varepsilon}:\sigma^*+\delta\dot{\alpha}B^*+\delta\varepsilon^*:\dot{\sigma}+\delta\alpha^*\dot{B}\,\mathrm{d}\Omega\,\mathrm{d}t+\int_o^{t_f}\int_{\Gamma_T}(\delta\dot{T}.u^*+\dot{T}.\delta u^*)\,\mathrm{d}S\,\mathrm{d}t$$
$$+\int_o^{t_f}\int_{\Omega}\delta A^*.\left(-\dot{\alpha}+\frac{\partial\phi}{\partial A}\right)+A^*\left(-\delta\dot{\alpha}+\frac{\partial^2\phi}{\partial A\partial A}\delta A\right)-\delta\alpha^*\dot{A}-\alpha^*\delta\dot{A}\,\mathrm{d}\Omega\,\mathrm{d}t$$

$$\delta J=\int_{\Gamma_T}k\big(u(t_f)-u_o\big).\delta u(t_f)\,\mathrm{d}S+\int_o^{t_f}\int_{\Gamma_T}\dot{T}.H.\delta\dot{T}\,\mathrm{d}S\,\mathrm{d}t$$

Then the stationarity of $\mathcal{J}$ is equivalent to the conditions of:

Equilibrium	$\operatorname{div}\dot{\sigma}=0,\ \operatorname{div}\sigma^*=0,\ \underline{n}.\dot{\sigma}=\dot{T},\ \text{over }\Gamma_T,\underline{n}.\dot{\sigma}^*=0$
Constitutive laws	$A^*+B^*=0,\ \dot{A}+\dot{B}=0,\ \dot{\alpha}=\frac{\partial\phi}{\partial A},\quad \dot{\alpha}^*=-\frac{\partial^2\phi}{\partial A\partial A}.A^*,\ \alpha^*(t_f)=0$
Optimality	$\int_{\Gamma_T}\big(\underline{n}.\sigma^*(t_f)+k\big(u(t_f)-u_o\big)\big).\delta u_{t_f}\,\mathrm{d}S=0$

To ensure the existence of a solution, the two potential w,ϕ must have a regular second derivative. Discussions and examples can be found in [3]. Particular cases of viscoplastic potential are regularization of plastic potential, and the previous results should be generalized to elastoplasticity. Some cases in elastoplasticity have analytical solutions. The case of an elastoplastic beam is presented in [6].

4. Application for cyclic loadings

It is well known that an elastoplastic structure under cyclic loading can exhibit different kinds of asymptotic behavior depending on the magnitude of the load: shakedown, alternate plasticity, ratcheting. In structural design, knowing the asymptotic behavior for a given loading is very important to estimate the durability of the structure. Ratcheting must always be avoided, it implies rapidly the collapse of the structure due to the accumulation of plastic strain. Elastic shakedown is often considered as safe, because the structure behaves elastically after a finite number of cycles. Plastic shakedown can be accepted or not depending on the structure. However, knowing the asymptotic behavior is generally not sufficient to ensure the structure safety because of fatigue phenomenon and in these cases an assessment on the asymptotic residual stress field is needed to check some fatigue criterion.

Estimating the asymptotic behavior by a step by step incremental analysis is a tedious task: the computation of the structural response requires often a great number of cycles, and the accuracy of the results can be prohibitive in terms of calculation cost. So a direct method for elastoviscoplastic structures is proposed to provide information about the asymptotic behavior without computing the total evolution of the internal state of the structure. By using both known theoretical results and the optimal control approach, the asymptotic field is found to minimize a functional J defined in a adapted manner [7].

As previously, the local behavior is defined by a free energy w which is quadratic in elastic strain, and depending on internal variable $\alpha = (\varepsilon_p, \beta)$

$$w(\varepsilon, \alpha) = \frac{1}{2}(\varepsilon - \varepsilon_p) : \mathbb{C} : (\varepsilon - \varepsilon_p) + \frac{1}{2}\beta.Z.\beta \tag{7}$$

The thermodynamical forces associated with the internal parameters $\alpha = (\varepsilon_p, \beta)$ are denoted by $A = (\sigma, Y)$ where

$$\sigma = \frac{\partial w}{\partial \varepsilon} = -\frac{\partial w}{\partial \varepsilon_p} = \mathbb{C} : (\varepsilon - \varepsilon_p); \quad Y = -\frac{\partial \beta}{\partial \beta} = Z.\beta \tag{8}$$

On the space of stresses A we define a scalar product

$$\mathcal{Q}(A, A') = \int_\Omega \left(\sigma.E^{-1} : \sigma + Y.Z^{-1}.Y'\right) \mathrm{d}\Omega = \int_\Omega A.H.A' \, \mathrm{d}\Omega \tag{9}$$

It can be noticed that $A.H.\delta A = \sigma.\delta\varepsilon - A.\delta\alpha$. Given a convex viscoplastic potential, $\Phi(A) = \phi(\sigma, Y)$ the internal parameters follow the normality rule:

$$\dot{\alpha} = (\dot{\varepsilon}_p, \dot{\beta}) = \left(\frac{\partial \phi}{\partial \sigma}, \frac{\partial \phi}{\partial Y}\right) = \frac{\partial \Phi}{\partial A} \tag{10}$$

Shakedown theorems have been stated for a wider class of non-associated material [8], but the convergence of the stress towards a T-periodic solution has been proved so far for C-class standard generalized materials [9,10]. In this case, under boundary conditions periodic in time with period T, for any solution σ, α of the problem of evolution, a periodic solution $\sigma_\infty, \alpha_\infty$ exists; the rate $\dot{\alpha}$ are periodic, and the functions $\sigma(t), \alpha(t)$ tend toward $\sigma_\infty(t), \alpha_\infty(t)$ as time tends to ∞. The last property means that the asymptotic rates $\dot{\sigma}_\infty(t), \dot{\varepsilon}_{p\infty}(t), \dot{\beta}_\infty(t)$ defined on the period T are unique, and that the asymptotic fields depend on the initial state. The determination of the asymptotic state is now turned into a minimization problem.

The characterization of the asymptotic behavior is the periodicity of the asymptotic fields, we introduce the measure of deviation to the periodicity using the functional J based on the quadratic energy $\mathcal{Q}$

$$J = \mathcal{Q}\big(A(T) - A(o), A(T) - A(o)\big) \tag{11}$$

Introducing as previously the functional $\mathcal{L}$ (changing t_f in T) and the new functional $\mathcal{J} = J + \mathcal{L}$, the solution is given by the stationarity of $\mathcal{J}$. The variations of J satisfy

$$\delta J = \int_\Omega \sigma(T)\delta\varepsilon(T) - \sigma(o).\delta\varepsilon(o) - A(T)\delta\alpha(T) + A(o)\delta\alpha(o) \, \mathrm{d}\Omega \tag{12}$$

As previously, the stationarity of $\mathcal{J}$ gives the conditions of equilibrium, the local constitutive laws and the prescribed boundary conditions. The condition of optimality is then deduced.

$$0 = \delta J - \int_\Omega \big(\sigma^*(T)\delta\varepsilon(T) - \sigma^*(o)\delta\varepsilon(o)\big) \, \mathrm{d}\Omega - \int_\Omega \big(\alpha^*(T)\delta A(T) - \alpha^*(o)\delta A(o)\big) \, \mathrm{d}\Omega \tag{13}$$

Then the variations with respect to $\varepsilon(T)$ and to $\alpha(T)$ give the conditions

$$-\sigma^*(T) + \sigma(T) - \sigma(o) = 0, \qquad -A^*(T) + A(T) - A(o) = 0 \tag{14}$$

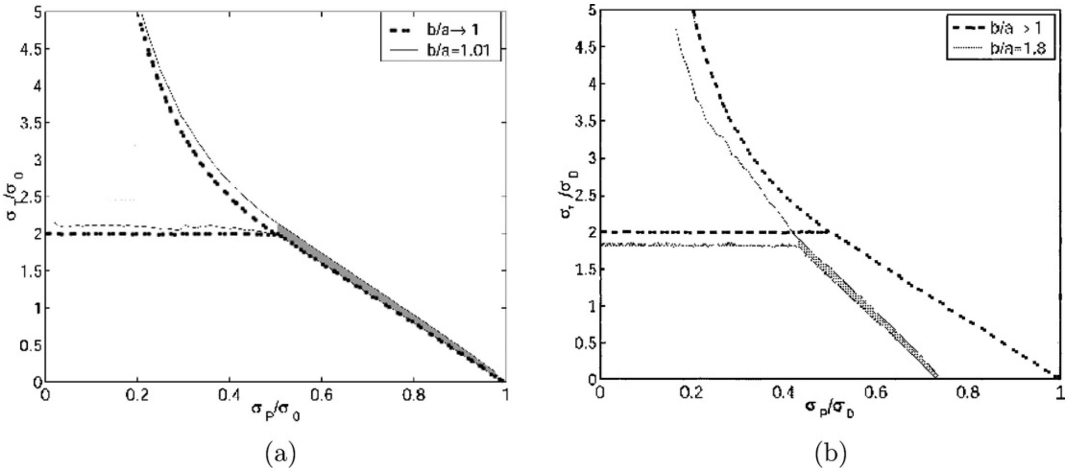

Fig. 1. Thin shell: (a) $b/a \simeq 1$, (b) $b/a > 1$.

the last terms gives

$$\int_{\Omega} \big(-\sigma^*(T) + \sigma^*(o)\big)\delta\varepsilon(o) + \big(A^*(T) - A^*(o)\big)\delta\alpha(o)\,\mathrm{d}\Omega = 0 \tag{15}$$

The first terms is null, due to boundary conditions on $\sigma^*(t)$, and the last one is exactly the condition of optimality. Further results and discussion are founded in [7,11]. Applications, results and commentaries are found in these papers.

4.1. Bree diagram

For elastic perfectly plastic material analytical results have been obtained for a spherical cavity of radius a surrounded by a thin shell [12]. The elastic limit in tension is denoted σ_o. The external radius is b. The asymptotic behaviour is given in terms of two parameters

$$\sigma_p = \frac{p}{2(\frac{b}{a} - 1)}, \qquad \sigma_T = \frac{E\alpha T_o}{2(1-\nu)} \tag{16}$$

The internal pressure is a constant and the periodic prescribed temperature is $\tau(t) = T(b,t) - T(a,t) = T_o\delta(t)$ where $\delta(t)$ has saw shape. For homogeneous conduction the local temperature is given by

$$T(r,t) = T_o\delta(t)\frac{b}{b-a}\left(\frac{a}{r} - \frac{2a}{a+b}\right) \tag{17}$$

For testing the ability of the proposed method, we consider that the potential of dissipation of the shell is

$$\phi(\sigma) = \frac{c}{m+1}\left\langle \frac{\sqrt{J_2}}{k} - 1\right\rangle_+^{m+1}, \quad J_2 = \frac{1}{2}s_{ij}s_{ij}, \quad s_{ij} = \sigma_{ij} - \frac{1}{3.}\operatorname{tr}\sigma\delta_{ij} \tag{18}$$

where $k = \sigma_o/\sqrt{3}$, $m = 1.8$, the Young modulus is $E = 2 \times 10^5$ MPa, the Poisson ratio is $\nu = 0.3$, the thermal coefficient is $\alpha = 16.3 \times 10^{-6}C^{-1}$, and the ratio $cTp/k = 10$. Tp is the period of loading. The last ratio ensures that locally the stresses and the strains obtained with the elastoviscoplastic law have values closed to those of elastoplastic behavior.

On Fig. 1, we compare the Bree Diagram (dashed line) with the computed diagram for a thin shell. The two curves are closed. But for increasing value of b/a the analytical value of Bree are not available, however, the diagram can be computed with the proposed method. We see that the diagram has the same shape but the domain of elastic shakedown is reduced.

5. Conclusion

We have presented some applications of optimal control theory to solve inverse problems in nonlinear mechanics. An adjoin state is introduced in order to find the condition of optimality. This formulation preserves particular properties which are useful for an implementation in classical finite element scheme.

These results can be extended to other problems of nonlinear mechanics such as wear, as identification of plastic zones or of damaged zones. Other methods can be applied, but it is important to notice that the optimal control theory is a general framework to be use for many applications. The main results are founded on the fact that the behavior is described by two potentials, the free energy and the potential of dissipation. All the preceding functional can be generalized to fracture and damage in classical continuum mechanics as well as for generalized continua such beams or plates.

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