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Duality, inverse problems and nonlinear problems in solid mechanics

Duality and symmetry lost in solid mechanics

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Dedicated to my teachers Paul Germain, Jean Mandel and Laurent Schwartz

Abstract

Some conservation laws in Solids and Fracture Mechanics present a lack of symmetry between kinematic and dynamic variables. It is shown that Duality is the right tool to re-establish the symmetry between equations and variables and to provide conservation laws of the pure divergence type which provide true path independent integrals. The loss of symmetry of some energetic expressions is exploited to derive a new method for solving some inverse problems. In particular, the earthquake inverse problem is solved analytically. ***To cite this article: H.D. Bui, C. R. Mecanique 336 (2008).***

Keywords: Conservation laws; Duality; Symmetry loss; Inverse problem

The notion of duality and symmetry is closely linked to the concept of 'Virtual Power'. This was introduced in the Mechanics of continuous media by my teacher Paul Germain for an adequate representation of the action (forces, stress, ...) on a body, [1]. In one of his papers, he wrote "*This concept is very seldom considered in the English scientific community, which directly made use of equations, for example the classical Newton law* ($\mathbf{f} = m\mathbf{a}$) *or Cauchy law* ($\operatorname{div}\sigma = \rho\mathbf{a}$) ". It originates from the mathematical concept of *spaces* and *dual spaces* of functions. As an example, to introduce the generalized functions, including the Dirac Delta function, my other teacher Laurent Schwartz invented distribution theory (for which he was awarded the Fields medal, 1951), [2]. A distribution is a continuous linear form in some space of function F, equipped with some topology. It belongs to the *dual* space F'. In the mechanics of continuous media, solids or fluids, duality is always present in the formulation of mechanical problems. As Paul Germain liked to tell us "*force is the dual of the mobility*", we kept in our mind that 'force' is indeed a dual vector, i.e. an element of V', the dual space of the space V of velocity fields, [3]. Stress is an element of the dual space D', which is dual to the strain rate space D etc., so that the stress space S is identical to $D' \equiv S$. By an extension, stress and strain are often considered as dual variables. The interpretation of stress as the dual of an element in the strain rate space D leads to the abstract definition of stress as a linear form on D, called a virtual power. The duality becomes the bilinear form denoted by $\langle d', d\rangle$, i.e. the map $D' \times D \to \mathbb{R} : (d', d) \to \langle d', d\rangle$.

Let us consider an elementary example. We wish to find a force F equal to the prescribed one F^d. It is thus equivalent to require the equality between scalars $\langle F, v^*\rangle = \langle F^d, v^*\rangle$ for any $v^* \in V$. From the linearity of the form $\langle .\,, .\rangle$ we

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Table 1
Duality in solid mechanics

Variables & functions	Action or results	Dual variables & conjugate functions	Remarks
Displacement $\mathbf{u}$	Work	Force $\mathbf{f}$, traction vector $\mathbf{T}$	
Virtual velocity $\mathbf{v}^*$	Virtual power	Force $\mathbf{f}$, traction vector $\mathbf{T}$	
Deformation ε, strain rate $\dot{\varepsilon}$		Stress σ	
Potential energy P		Complementary potential Q	
Thermoelastic fields $\mathbf{u}, \theta$	Conserv. law	Adjoint fields $\mathbf{u}^*, w^*$	Symmetry between $(\mathbf{u}, \theta)$ and $(\mathbf{u}^*, w^*)$
Internal variables α	Dissipation $\mathbf{A}.\dot{\alpha}$	Generalized forces $\mathbf{A}$	
$\Phi(\dot{\alpha})$ (pseudo-dissipation)		$\Psi(A)$ (conjugate function)	$\Phi(\dot{\alpha}) = \sup\{\mathbf{A}.\dot{\alpha} - \Psi(\mathbf{A})\}$
J-integral	Derivative of the energy $-\mathrm{d}W/\mathrm{d}a$	Dual I-integral	
$J(u, u^*)$	Virtual power		J-integral $= \frac{1}{2} J(\mathbf{u}, \mathbf{u})$
$(\mathbf{v}, \varepsilon)^\tau = C\mathbf{u}$	Tonti's diagram	$C^*(\mathbf{p}, \sigma)^\tau = (\mathbf{m}, \mathbf{e})$	
	Eq. of motion	$\mathbf{m} = \mathbf{0}$	
	Equilibrium Eq.	$\mathbf{e} = \mathbf{0}$	
$S[\mathbf{v}, \varepsilon]^\tau = [\eta, \zeta]^\tau$	Constitutive law	$S^*[Z, B] = [\mathbf{p}, \sigma]$	
$\eta = 0$	Compatibility $v \leftrightarrow \dot{\varepsilon}$		
$\zeta = 0$	Compatibility of ε		
Projection P	Tomography	Back projection P^*	The inverse Radon transform makes use of P and P^*
Propagation	Scattering of waves	Back propagation	
Forward equations	Reciprocity gap	Time reversal mirror	
	Functional (RG)	Adjoint equations	
Forward diffusion	RG functional	Backward diffusion	
State equation	Control theory	Adjoint state equation	
Primal problem	Convex analysis	Dual problem	
posi-functional spaces: D, S	Mathematical analysis	Dual spaces: D', S' (Schwartz's spaces)	D (space of C^∞ compactly supported function) D' (space of distributions) S (rapidly decreasing functions f at infinity, $f/\lvert x\rvert^n \to 0, \forall n$, n positive integer) S' (tempered distributions) $(1/p) + (1/p') = 1$
$H^{m,p}$		$H^{-m,p'}$ (Sobolev's spaces)	

get $\langle F - F^d, v^* \rangle = 0, \forall v^* \in V$. This concept was known in analytical mechanics since Lagrange, many centuries ago. In modern computational mechanics, virtual motion and virtual displacement are known as *test functions*. Therefore there are no new topics unfamiliar to everybody, but only new interpretations and new applications allowed by the concepts of duality, virtual power and symmetry lost, as illustrated by many papers devoted to inverse problems.

I mentioned my two teachers Paul Germain and Laurent Schwartz, to whom I owed the basic notions of duality in Mechanics and Mathematics. It appears that, as remarked by my colleague Xanthippi Markenscoff of UCSD, most of my works have been motivated by duality as a common theme, which may be simply explained by the teaching I received from them. Indeed, most of my works were published in the French Comptes rendus. Since she reads French very well, and perhaps the French Comptes rendus too, she remarkably noticed that *Duality, symmetry and symmetry lost* are the backbone of my works. I shall try to explain her remarks through some applications.

Applications of duality can be found in various fields of Mechanics: computational mechanics, mechanics of materials, fracture mechanics and inverse problems (Table 1). For example, in classical boundary integral equation methods, *dual* vectors $\mathbf{u}$ and $\mathbf{T} = \sigma.n$ are both considered in the boundary of the solid. Moreover *duality* has been used to derive the equations, via conservation laws and symmetry of the elastic tensor.

In the monograph of Tonti [4] one can find *duality* in various domains of Physics (electromagnetism, gravitation, thermodynamics, electrostatics, quasi-static elasticity, rod, strings, etc.). In his diagram for elastostatics, Fig. 1, there are two main charts, kinematics and statics, which are linked by the elastic constitutive law.

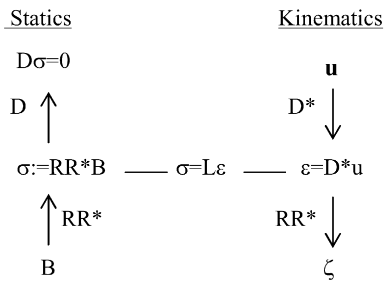

Fig. 1. Tonti's diagram in elastostatics.

In Fig. 1, $D := -\mathrm{div}$ and its adjoint $D^* = \frac{1}{2}(\nabla + \nabla^i)$, R is the curl (of second order symmetric tensor) and R^* its adjoint defined respectively by $RB := -(\partial_k B_{ij})\mathbf{e}^i \otimes (\mathbf{e}^j \wedge \mathbf{e}^k)$ and $R^*B := (\partial_k B_{ij})(\mathbf{e}^k \wedge \mathbf{e}^i) \otimes \mathbf{e}^j$. The operator RR^* is self-adjoint and B is the Beltrami tensor. In two dimensions, one has $B = \psi(x_1, x_2)\mathbf{e}^3 \otimes \mathbf{e}^3$ with the Airy function $\psi(x_1, x_2)$.

Traditionally, finite element methods do not consider the symmetry between stress and strain since one considers the chart beginning with the displacement field:

$$\mathbf{u} \to \varepsilon \to \sigma \to \ -\mathrm{div}\,\sigma = 0$$

A stress method based on the chart:

$$B \to \sigma \to \varepsilon \to RR^*\varepsilon = 0$$

does not consider the symmetry either. An hybrid method which takes account of the symmetry between the kinematic and static charts:

$$\mathbf{u} \to \varepsilon \cdots \sigma \leftarrow B$$

consists in satisfying the constitutive law, in the sense of minimum norm $\|\varepsilon - \sigma\|$. This is essentially the 'error in constitutive law' method proposed by Ladeveze [5].

When I generalized Tonti's diagram of elastostatics to dynamic elasticity (Comptes Rendus Acad. Sciences, Paris, 311, II, p. 7, 1990, [6]), I realized the beautiful structure of the equations using dual variables, Fig. 2. I discovered that the links between dual variables are always governed by operators and their adjoint ones (see the generalisation of Tonti's diagram to dynamics). The two charts are linked by two constitutive laws: the momentum/velocity relation $\mathbf{p} = \rho\mathbf{v}$ and the elastic law $\sigma = L\varepsilon$.

In elasticity or plasticity theory, most works presented the field equations of equilibrium and constitutive laws in *stress* space, ignoring another dual presentation of equations in *strain* space. It is not simply an academic point of view, but sometimes it is a necessity. An example is given by the softening of elastic–plastic materials which convinces us that, to describe the decrease of the load and to avoid the ambiguity between elastic unloading (a) and plastic loading with softening (b), Fig. 3, it is necessary to consider the strain space [Nguyen and Bui, Journal de Mécanique, vol. 13, No. 2, pp. 321–343, 1974, [7]].

Let me show now how duality is useful for solving some ill-posed problems with applications to the Non-Destructive Testing method. When I was still a research student, I wondered how the Dirichlet boundary value problem in elasticity, with the prescribed datum $u_i^d(x)$, $x \in \partial\Omega$, on the boundary can be replaced by the corresponding Neumann boundary value problem with the prescribed datum $T_i^d(x)$, $x \in \partial\Omega$. I found the solution to my problem for a half plane but ignored at this time that the map $u_i^d(x) \to T_i^d(x)$, called the *Dirichlet-to-Neumann* map (DN) or its inverse, the *Neumann-to-Dirichlet* map (ND), will have very important applications in the solution of some inverse problems in elasticity investigated in the nineties by mathematicians. This paper published in the sixties ("Transformation of boundary values on an elastic half-space (in French)", H.D. Bui, Comptes Rendus Acad. Sciences, t. 265, pp. 862–865, 1967, [9]) indicates that the concept of duality has been unconsciously the driving concept of my works for many decades. The Dirichlet-to-Neumann maps are today the key tools to solve the "Crack detection problem by a geometry approach" (H.D. Bui, in A propos des grands Systèmes des Sciences et de la Technologie, in the honor of Robert Dautray, by J. Horowitch & J.L. Lions (Eds.), Masson, Paris, 1993, [10]). To solve such a problem, one needs

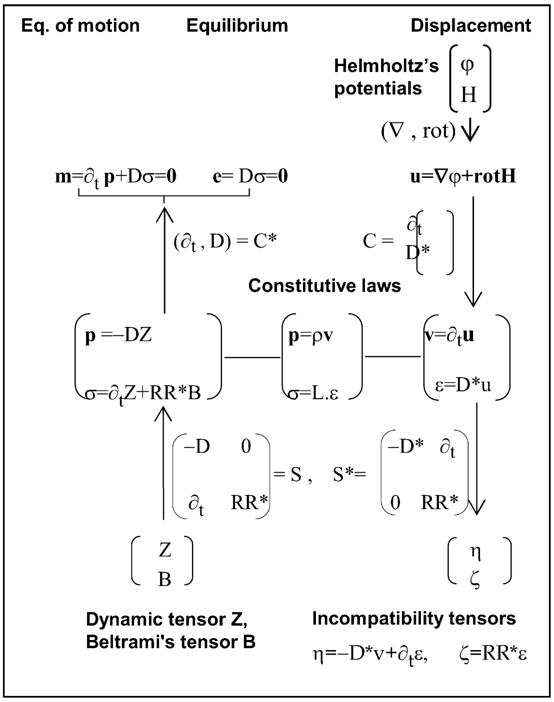

Fig. 2. Generalization of Tonti's diagram to elastodynamics: The potentials (ϕ, H) have no duals. Functions (ϕ, H) and (Z, B) are called conjugate pairs of functions, which are the key tools for a boundary integral equation method using symmetrical and regular kernels (Bui, [8]). They are like parents and parents-in-laws of children $(\mathbf{v}, \varepsilon)$ and $(\mathbf{p}, \sigma)$ who are related by a marriage (constitutive laws).

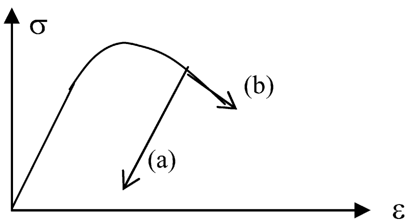

Fig. 3. Elastic–plastic behavior with softening.

a family of *dual vectors* $u_i^d(x:\lambda)$ and $T_i^d(x;\lambda) := DN(u_i^d(x:\lambda))$, depending on some parameter λ defining a family of surfaces S_λ (the boundary of the solid corresponds to $\lambda = 0$) and then to solve a Cauchy problem.

There are many methods for solving the Cauchy problem. One method is based on duality as explained in the following. Let the domain between S_0 and S_λ be denoted by Ω_λ, which is assumed to be free of defects. The variational form of elasticity in Ω_λ can be written as

$$a_\lambda(\mathbf{u}, \mathbf{v}) = b_\lambda(\mathbf{v}), \quad \forall \mathbf{v}$$

with test functions $\mathbf{v}$ assumed to be independent of λ. The notations a_λ and b_λ mean that the bilinear and linear forms depend on λ which parameterizes the domain Ω_λ and its boundary. A convected differentiation of the variational form on a moving domain Ω_λ yields the differential equation determining the evolution of the dual pair $(\mathbf{u}, \mathbf{T})$

$$\frac{d}{d\lambda} a_\lambda(\mathbf{u}, \mathbf{v}) = \frac{d}{d\lambda} b_\lambda(\mathbf{v}), \ \forall \mathbf{v} \quad \Rightarrow \quad \frac{d}{d\lambda}\begin{pmatrix}\mathbf{u}\\ \mathbf{T}\end{pmatrix} = A.\begin{pmatrix}\mathbf{u}\\ \mathbf{T}\end{pmatrix}$$

where A is a transfer matrix of tangential operators along S_λ which is an unbounded operator (ill-posedness of the Cauchy problem), Bui [10]. Therefore, an explicit integration of the differential equation is not possible. We need a regularization technique which consists in replacing operator A by AS, where S is a 'smoothing' operator (see Lorentz and Andrieux, [11]; Bui et al., [12]). Since AS is a bounded operator, the continuation of the pair $(\mathbf{u}, \mathbf{T})$ through S_λ

can be obtained by the integration of the differential equation with small integration steps. This geometrical method determines the greatest solid domain not containing defects, using data $(\mathbf{u}, \mathbf{T})$ on the outer boundary S_0.

Duality has also many applications in Fracture Mechanics. In 1973, I found that Rice's J-integral, [13], is only one description of the energy release rate G, by a path-independent integral and that the dual I-integral is another possible one. This offers a great advantage in considering both descriptions with dual variables and spaces, conservation laws and dual laws since the minimum theorems for the potential energy $W(\varepsilon)$ and the complementary potential $U(\sigma)$, under certain conditions, provide us exact bounds of the J-integral ("Dual path independent integrals in the boundary-value problems of cracks", H.D. Bui, Engineering Fracture Mechanics, 1974, 6, pp. 287–296, [14])

$$J = -\frac{\mathrm{d}W}{\mathrm{d}a}, \qquad I = -\frac{\mathrm{d}U}{\mathrm{d}a}$$

where a is the crack length, and $J = I = G$ only for exact solutions. Another important applications of the virtual power principle in Fracture Mechanics is provided by the notion of the *virtual crack propagation*. Classically, one deals with the energy release rate G as the *derivative* of the energy with respect to the crack length. Therefore G in mixed modes I + II is well known as the quadratic form $G = (1-\nu^2)(K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2)/E$. The question has been arisen on how to separately extract the stress-intensity factors. Many methods were proposed consisting in calculating the derivatives of the energy in the Ox_1 direction $J_1 = (1-\nu^2)(K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2)/E$ (crack propagation along Ox_1) and in the Ox_2 direction $J_2 = -(1-\nu^2)K_{\mathrm{I}}K_{\mathrm{II}}/E$ (crack translation out of its plane). Such an unphysical method (for an actual derivative) was criticized by many authors. I tried to look at the virtual power method, with arbitrary adjoint fields u^* and discovered that the virtual power of the energy of a cracked body, in two-dimensions, is equal to the bilinear form

$$G(u, u^*) = \frac{2(1-\nu^2)}{E}\{K_{\mathrm{I}}K_{\mathrm{I}}^* + K_{\mathrm{II}}K_{\mathrm{II}}^*\}$$

By choosing a symmetric adjoint field ($K_{\mathrm{I}}^* = 1$, $K_{\mathrm{II}}^* = 0$) we extracted the SIF in mode I by $G(u, u^*) = 2(1-\nu^2)K_{\mathrm{I}}/E$ and similarly by considering an anti-symmetric adjoint field ($K_{\mathrm{I}}^* = 0$, $K_{\mathrm{II}}^* = 1$) I obtained the stress-intensity factor in mode II. The virtual crack propagation is richer than the derivative of energy since it contains the classical result $\frac{1}{2}G(u, u) \equiv J_1$-integral. There is a profound difference between the virtual crack propagation method and the J_2- method which involves a crack translation out of its plane with the *same* loading. The virtual method $G(u, u^*)$ is based on a crack propagation in its direction, but under a *virtual load* giving rise to u^*, K_{I}^* and K_{II}^*.

In the seventies, I found some intriguing results for the energy release rate in elastodynamics. As a student, I always learned that a formula describing physical phenomena must be independent of the motion of the frame reference in which measurements are made. This is the objectivity principle in Physics. The energy release rate formula in elastodynamics for a moving crack with the velocity V does not satisfy this principle since the velocity is *explicitly* present in its expression, in plane strain mode I

$$G = \frac{1-\nu^2}{E}K_{\mathrm{I}}^2 f_{\mathrm{I}}(V)$$

where $f_{\mathrm{I}}(V) = \beta_1(1-\beta_2^2)/\{(1-\nu)(4\beta_1\beta_2 - (1+\beta_2^2)^2)\}$, $\beta_i = \sqrt{1-(V^2/c_i^2)}$, c_1 for P-wave, c_2 for S-wave. How to restore the objective formula for the dynamic G? My response to this question was *duality*.

Let me introduce for the mode I the same local definitions of *stress-intensity factor* and *crack displacement intensity factor* as known in quasi-statics, respectively

$$K_{\mathrm{I}}^{\sigma} = \lim_{r\to 0}\sigma_{22}\sqrt{2\pi r}, \qquad K_{\mathrm{I}}^{u} = \lim_{r\to 0}\frac{\mu}{4(1-\nu)}[\![u_2]\!]\sqrt{\frac{2\pi}{r}}$$

In quasi-statics, both definitions provide the same SIF. In elastodynamics, I found a symmetrical formula for the energy release rate which is nothing but the duality between stress and strain rate near the moving crack tip

$$G = \frac{1-\nu^2}{E}K_{\mathrm{I}}^{\sigma}K_{\mathrm{I}}^{u}$$

This objective formula agrees with the traditional one, Achenbach and Bazant [30], since it can be proved that $K_{\mathrm{I}}^{u} = K_{\mathrm{I}}^{\sigma} f_{\mathrm{I}}(V)$ (see H.D. Bui, Fracture, ICF4, Waterloo, June 19–24, 1977, pp. 91–95, [15]).

Another beautiful application of duality is about a conservation law in linear thermo-elasticity, with applications to fracture mechanics. It is well known that classical conservation law in thermo-elasticity includes a source term, namely in the form $\operatorname{div} A(\mathbf{u},\theta) = B(\mathbf{u},\theta)$, where θ is the temperature field. Precisely, because of the source term $B = (\partial W/\partial\theta)\operatorname{grad}\theta$, the thermo-elastic J_{th}-integral is not a purely path-independent integral, since it involves an area integral too

$$J_{\text{th}} = \int_{\Gamma} \left\{ W(\varepsilon,\theta) - \sigma_{ij} n_j u_{i,1} \right\} \mathrm{d}s - \int_{A(\Gamma)} w_{,\theta}\theta_{,1}\, \mathrm{d}A$$

We lose the main interest of path-independent integrals which is to avoid the calculation of singular fields near the crack tip. Does a path-independent integral exist in thermoelasticity? This was a question addressed to me by George Herrmann in the 1980s. The symmetry is lost when we consider the pair $(\mathbf{u},\theta)$ alone. We restored the symmetry by considering the dual pair $\{(\mathbf{u},\theta), (\mathbf{u}^*, w^*)\}$ and obtained a conservation law in the form $\operatorname{div} A(\mathbf{u},\theta;\mathbf{u}^*, w^*) = 0$, without a source term, using dual variables $(\mathbf{u}^*, w^*)$. The conservation law in linear thermoelasticity of the pure divergence form, for a line crack problem (along negative Ox_1) is given by

$$\frac{\partial}{\partial x_j}\left\{ u_i\sigma^*_{ij,1} - u^*_{i,1}\sigma_{ij} - \gamma\theta(u^*_{1,j} - w^*_{,j}) + \gamma\theta_{,j}(u^*_1 - w^*) \right\} = 0$$

where $\gamma = -\alpha\mu(3\lambda + 2\mu)/(\lambda+\mu)$, α is the thermal coefficient, λ and μ are Lamé's coefficients. The actual temperature field θ as well as the scalar adjoint field w^* are harmonic. The stress free is assumed on the crack $\sigma.\mathbf{n} = 0$ as well as the normal heat flux $\partial\theta/\partial n = 0$. The adjoint fields $(\mathbf{u}^*, w^*)$ are *not* the thermoelastic ones. There is a coupling between adjoint fields, by imposing the following condition on the crack faces $\partial(w^* - u^*_1)/\partial n = 0$. Under these conditions, we get a path-independent integral in linear thermoelasticity

$$T = \int_{\Gamma} \frac{1}{2}\left\{ u_i\sigma^*_{ij,1}n_j - u^*_{i,1}\sigma_{ij}n_j - \gamma\theta(u^*_{1,n} - w^*_{,n}) + \gamma(u^*_1 - w^*)\theta_{,n} \right\} \mathrm{d}s$$

$$T = \frac{1-\nu^2}{E}(K_{\mathrm{I}}K^*_{\mathrm{I}} + K_{\mathrm{II}}K^*_{\mathrm{II}})$$

The results were presented at the French Academy by Paul Germain himself and also at the Eshelby Symposium (*Fundamentals of deformation and Fracture*, April, 1984, pp. 2–5, [16]), in honour of a great scientist who impinged on many works in Fracture Mechanics, Bui [32].

Does symmetry exist in conservation laws in elastodynamics? It is clear that the conservation law $\operatorname{div}\sigma[\mathbf{u}] = \rho\ddot{\mathbf{u}}$ or those derived by Fletcher [17]

$$\operatorname{div}\left\{ W - \frac{1}{2}\rho\mathbf{v}.\mathbf{v} - \sigma.\nabla\mathbf{u} \right\} + \frac{\partial}{\partial t}(\rho\mathbf{v}.\nabla\mathbf{u}) = 0$$

are not symmetric. To restore the symmetry, it is necessary to introduce adjoint fields $\mathbf{v}(\mathbf{x},t;\tau)$ satisfying the elastodynamic wave equations $\operatorname{div}\sigma[\mathbf{v}] = \rho\ddot{\mathbf{v}}$ such that $\mathbf{v}(\mathbf{x},t;\tau) \equiv 0$ for $t > \tau$ where τ is an arbitrary constant. We obtained symmetric conservation laws in elastodynamics given in ("Facteur d'intensité des contraintes mécaniques globales", H.D. Bui and H. Maigre, C. R. Acad. Sci. Paris, II, 306, p. 1213, 1988, [18]) by

$$\operatorname{div}\left\{ \int_0^{\tau} \left(\mathbf{n}.\sigma[\mathbf{u}].\mathbf{v} - \mathbf{n}.\sigma[\mathbf{v}].\mathbf{u}\right)\mathrm{d}t \right\} = 0, \quad \text{for any } \mathbf{v}$$

These conservation laws have been exploited by Bui et al. [33] to extract the stress intensity factors in dynamic modes I and II by choosing appropriately adjoint dynamical fields.

Dual variables are crucial in the thermodynamics of irreversible processes. The contributions of J.J. Moreau, Q.S. Nguyen, P. Germain, P. Suquet, A. Ehrlacher, C. Stolz and others in France, during the period 1960–1990 are very important in clarifying the nature of dissipation in Plasticity and Fracture. Internal rate variables $\dot{\alpha}$, including the plastic strain rate $\dot{\varepsilon}^p$, $\dot{\alpha} = (\dot{\varepsilon}^p, \dot{\beta})$ describe the evolution of materials. The variable $\dot{\alpha}$ is the dual to the generalized force $\mathbf{A}$, so that $\mathbf{A}.\dot{\alpha} \geqslant 0$ represents the *dissipation rate*. If one introduces the free energy per unit volume $W(\varepsilon, \alpha)$ so that

$$\sigma = \frac{\partial W}{\partial \varepsilon}, \qquad \mathbf{A} = -\frac{\partial W}{\partial \alpha}$$

then one obtains the state equation. One needs to introduce a complementary law by introducing a pseudo-potential $\Phi(\dot{\alpha})$ so that (B. Halphen and Q.S. Nguyen, [19])

$$\mathbf{A} = \frac{\partial \Phi}{\partial \dot{\alpha}}$$

The dual presentation consists in introducing the *conjugate* function $\Psi(\mathbf{A})$ in the sense that conjugate functions $\Phi(\dot{\alpha})$ and $\Psi(\mathbf{A})$ are linked by Legendre transform

$$\Phi(\dot{\alpha}) = \sup_{A \in V} \{\mathbf{A}.\dot{\alpha} - \Psi(\mathbf{A})\}$$

where V is some convex of generalized forces. In the smooth convex case one has $\dot{\alpha} = \partial\Psi(\mathbf{A})/\partial\mathbf{A}$, while in the case of a non-differentiable convex, one may use Moreau's notion of sub-differential $\dot{\alpha} \in \partial\Psi(\mathbf{A})$.

If the crack length is considered as a state variable, then $\dot{a} J_{\text{tip}}$ can be identified as the *dissipation rate* due to fracture at the crack tip! Here we have defined J_{tip} as the J-integral for a vanishing contour around the crack tip. In plasticity J_{tip} is equal to zero, which is the paradoxical result revealed by Rice (1966), so that the dissipation rate in a cracked body is essentially distributed by plastic heating inside the solid domain rather than concentrated at the crack tip. In elasticity, J_{tip} is not equal to zero and $\dot{a} J_{\text{tip}}$ represents the dissipation rate the crack tip even in an elastic body (which is a non-dissipative medium in its volume!). The dissipative nature of crack propagation in an elastic body resulted in a new interpretation of the energy release rate and to the discovery of the positive *logarithmic singularity* of the temperature field T for a moving crack tip, which behaves like a moving point heat source (Bui et al., Comptes Rendus Acad. Sci. Paris, t. 289, pp. 211–214, 1979, [20])

$$T = -\frac{\dot{a} J_{\text{tip}}}{2k\pi} \log r$$

These are new aspects of Fracture Mechanics based on thermodynamical considerations. Such considerations have been introduced in Plasticity by my teacher, Professor Jean Mandel, with whom I wrote my first research paper on experimental Plasticity (1962). Later, in 1965, I published another paper [21] on the experimental verification of his plastic dissipation formula [22]

$$D_p = \int (\mathbf{T}.\dot{\mathbf{u}} - \dot{\mathbf{T}}.\mathbf{u})\, \mathrm{d}t$$

which is nothing but a *symmetry lost* in Plasticity (Cahier du Groupe Français de Rheologie, t. 1, no. 1, 1965, pp. 15–19, [21]).

Virtual power is more general than the time derivative of the energy. Dual variables in continuum Mechanics are more general than the variables considered in the formulation of equations. Consider the expression of the energy release rate in Linear Fracture Mechanics as the derivative of the energy with respect to the crack length

$$G = \frac{1}{2} \int_{\partial\Omega} \left(\mathbf{T}.\frac{\partial \mathbf{u}}{\partial a} - \mathbf{u}.\frac{\partial \mathbf{T}}{\partial a} \right) \mathrm{d}s$$

I remember a discussion with Paul Germain in which he questioned me about the anti-symmetry found in the above formula, in the sense that the interchanges $u \leftrightarrow \mathrm{d}u/\mathrm{d}a$ and $T \leftrightarrow \mathrm{d}T/\mathrm{d}a$ change the sign of G. It seems that G looks like a Poisson's bracket! like the Mandel formula of plastic dissipation. At this time I had no correct answer to his question on the anti-symmetry. Today, I can say that this is simply a *symmetry lost*. In recent works with my colleagues in two groups of research at École polytechnique and Electricité de France, and also with the University of Tunis, I discover that *symmetry lost* is a fundamental notion in crack detection problems. The reciprocity theorem in elasticity expresses the symmetry between two states $(\mathbf{u}^1, \mathbf{u}^2)$. Consider a homogeneous elastic solid with the two states and the integral R defined as

$$R = \int_{\partial\Omega} \big(\mathbf{u}^1.T(\mathbf{u}^2) - \mathbf{u}^2.T(\mathbf{u}^1)\big)\, \mathrm{d}s$$

which expresses the Betti reciprocity theorem by $R = 0$, revealing the *symmetry* between the two states. The symmetry is lost when $R \neq 0$, for instance in the case where the solid is not homogeneous or contains cracks. One field corresponds to the non-homogeneous or cracked body, another field for the homogeneous one.

Therefore R is a *defect indicator* (also called a *reciprocity gap*):

$$R = 0 \quad \Leftrightarrow \quad \text{no defect inside } \partial\Omega$$

$$R \neq 0 \quad \Leftrightarrow \quad \text{existence of a defect}$$

Finally, crack detection problems reduce to the search of the *zeros* of a functional. A series of recent papers of my groups, in EDF and École polytechnique, showed that the reciprocity gap functional method provided a closed form solution to some inverse crack detection problems, for electrostatics (Andrieux and Ben Abda, [23,31]), static elasticity (Andrieux et al., Inverse Problems, 15, pp. 59–65, 1999, [24]), diffusion equation (Ben Abda and Bui, Inverse and Ill-posed Problems, 11, no. 1, pp. 27–31, 2003, [25]), transient acoustics (Bui et al., Comptes Rendus Acad. Sci. Paris, 327, II, pp. 971–976, 1999, [26]), elasto-dynamics with the exact solution to an earthquake inverse problem (Bui et al., Inverse and Ill-posed Problems, vol. 13, no. 6, pp. 553–565, 2005, [27]). As shown above, the reciprocity gap R is the external boundary functional which is known from the data $\mathbf{u}^1 = \mathbf{u}^d$ and $T(\mathbf{u}^1) = \mathbf{T}^d$ and from the chosen adjoint functions $\{\mathbf{u}^2, T(\mathbf{u}^2)\}$. In planar crack detection problems (quasistatic elasticity), we can prove the following variational equation (R is an integral over S_{ext}, with known data and known adjoint functions)

$$\int_{\Sigma(\mathbf{u})} [\![\mathbf{u}]\!].T(\mathbf{u}^*)\,\mathrm{d}s = R\big(\mathbf{u}^d, \mathbf{T}^d; \mathbf{u}^*, T(\mathbf{u}^*)\big), \quad \forall \mathbf{u}^*$$

In the homogeneous body case (no crack), the left-hand side of the above equation equals zero. By $R = 0$, we recover the symmetry between fields $\mathbf{u}$ and $\mathbf{u}^*$. In the *symmetry lost* case, the above equation provides a *non-linear* equation for determining the crack plane (containing Σ) as well as the displacement discontinuity $[\![\mathbf{u}]\!]$. It is impossible to solve the non-linear inverse problem with classical methods based on the field equations, since the crack support $\Sigma(\mathbf{u})$ depends on the unknown $\mathbf{u}$. Now, the variational form makes it possible to solve the inverse problem in closed form, by first finding suitable adjoint fields to determine the crack plane and then considering a *linear* inverse problem, which is incomparably simpler than the original one. For more details on the solutions of the above variational equations, for different physics, the readers can refer to Bui [28]. An adequate choice of the adjoint function allows the invertibility of the above equation. We exploit here the arbitrariness of the choice of functions $\mathbf{u}^*$ to obtain the desired results. Classical methods deal with the fields equations (elastic equilibrium equation, boundary conditions, with an *unknown* geometry). Therefore the only possible method consists in finding the best fitting of true measurements with predicted data corresponding to some guessed geometry $\mathbf{S}$

$$\big\{\Sigma \text{ and } [\![\mathbf{u}]\!]\big\} = \arg\min_{S.[\![\mathbf{v}]\!]} \big\{|\mathbf{v} - \mathbf{u}^d|^2 + |T(\mathbf{v}) - T^d|^2\big\}$$

where $\mathbf{v}$ is the solution of the boundary value problem with the geometry $\Omega \backslash S$ and with *one* of the boundary condition, either $\mathbf{u}^d$ or $\mathbf{T}^d$ (two possible numerical solutions!). This classical method of solution is essentially a numerical one. It is well known that the above optimization procedure is mathematically ill-posed. One interesting statement of the work by Das and Suhadolc [29] on the earthquake inverse problem is the following one: "*even if the fitting of data seems to be quite good, the faulting process is poorly reproduced, so that in the real case, it would be difficult to know when one has obtained the correct solution*". Undoubtedly, the reciprocity gap functional method based on the symmetry lost and on duality (through the variational form) is the right tool to solve these inverse problems in closed form.

To illustrate the method of closed form solutions using the symmetry lost, let us consider first a parabolic inverse problem of crack detection using the heat diffusion equation (Ben Abda and Bui, [25]) and then a hyperbolic problem of elastodynamic inverse scattering by a crack (Bui et al., [12]). The first problem is particularly interesting to be investigated because measurements of surface temperature and heat flux are today technically possible with infra-red cameras.

The field equations of (forward) heat diffusion are:

$$\partial_t u - \Delta u = 0, \quad \text{in } (\Omega \backslash \Sigma) \times [0, T]$$

$$u(x, 0) = 0, \quad \text{in } (\Omega \backslash \Sigma)$$

$$u = u^d, \quad \text{on } S_{\text{ext}} \text{ (measured data)}$$
$$\partial_n u = 0, \quad \text{on } \Sigma \text{ (unknown crack)}$$
$$\partial_n u(x,0) = \Phi, \quad \text{on } S_{\text{ext}} \text{ (measured data)}$$

The adjoint (backwards) heat equations for an homogeneous solid are:

$$\partial_i w + \Delta w = 0, \quad \text{in } \Omega \times [0,T] \text{ (no crack)}$$
$$w(x,T) = 0, \quad \text{in } \Omega$$

Here, no boundary conditions are needed for w. The reciprocity gap functional is non-linear in u, because of the unknown integration domain $\Sigma(u)$

$$\int_0^{\pi} \int_{\Sigma(u)} [\![u]\!] \partial_n w \,\mathrm{d}S\,\mathrm{d}t = \int_0^{T} \int_{S_{\text{ext}}} (\Phi w - u^d \partial_n w) \,\mathrm{d}S\,\mathrm{d}t, \quad \forall w$$

Three steps are needed to solve the inverse problem to determine the crack $\Sigma(u)$:

- *First step*: Determination of the normal $\mathbf{N}$ to the crack plane. By using the following adjoint field, it can be proved that $\mathbf{N}$ is given by

$$\mathbf{N} = \arg\left\{ \min_{|n|=1} \max_{|m|=1,\ n.m=0} F(\mathbf{n} \wedge \mathbf{m}) \right\}$$
$$F(\mathbf{p}) = RG(w^{(\mathbf{p})})$$
$$w^{(\mathbf{p})} = \begin{cases} 1 - \operatorname{erf}\left\{ \dfrac{\mathbf{x}.\mathbf{n}}{2\sqrt{T-t}} \right\} & (t \leqslant T) \\ 0 & (t > T) \end{cases}$$

- *Second step*: Find the crack plane position. By taking Ox_3 along $\mathbf{N}$ and defining the crack plane by $x_3 - C = 0$, we determine the constant C by the zero of the reciprocity gap function $c \to F(c)$

$$F(c) = RG(w^{(c)})$$
$$w^{(c)}(x,t;T) = \frac{1}{\sqrt{4\pi(T-t)}} \exp\left\{ \frac{-(x_3 - c)^2}{4(T-t)} \right\}$$

- *Third step*: Find the crack geometry, defined as the *support* of the discontinuity $[\![u]\!]$. For this purpose, we introduce an adjoint function $w^{(s_1,s_2,q)}$ parametrized by (s_1, s_2, q), which is

$$w^{(s_1,s_2,q)}(\mathbf{x},t) = \exp(\mathrm{i}qt)\exp\{-\mathrm{i}(s_1x_1 + s_2x_2)\}\exp\{x_3(s_1^2 + s_2^2 - \mathrm{i}q)^{1/2}\}$$

with $q > 0$ and $(s_1, s_2) \in \mathbb{R}$. From the reciprocity gap functional we get:

$$\int_{\mathbb{R}^2} H(\mathbf{x},q)\exp\{-\mathrm{i}(s_1x_1 + s_2x_2)\}\,\mathrm{d}x_1\,\mathrm{d}x_2 = -(s_1^2 + s_2^2 - \mathrm{i}q)^{1/2} \int_0^{\infty} \mathrm{d}t \int_{S_{\text{ext}}} (\Phi w - u^d \partial_n w)\,\mathrm{d}S$$

with $H(\mathbf{x}, q)$ being the time Fourier transform of the crack discontinuity

$$H(\mathbf{x},q) = \int_0^{\infty} [\![u]\!] \exp(\mathrm{i}qt)\,\mathrm{d}t$$

We notice that both functions of $\mathbf{x}$, $[\![u(\mathbf{x},t)]\!]$ and $H(\mathbf{x},q)$, have the same spatial support! Therefore, the crack geometry Σ is nothing but the support of an *inverse* spatial Fourier transform

$$\Sigma = \operatorname{supp}\left\{ \mathbf{F}_x^{-1} \left\{ -(s_1^2 + s_2^2 - \mathrm{i}q)^{1/2} \int_0^{\infty} \mathrm{d}t \int_{S_{\text{ext}}} (\Phi w - u^d \partial_n w)\,\mathrm{d}S \right\} \right\}$$

All quantities in the right hand side are known. This formula solves the inverse heat diffusion problem.

The second problem on crack detection by elastic waves is an important one in engineering. This is generally solved for unbounded solids and under some restrictive assumptions: approximate ray theory or limitations on the frequency range (Born's approximation in low frequency, Kirchhoff's approximation in high frequency). Let us show how, without entering into the complete details of the derivation, the symmetry lost principle can provide us the exact solution to the inverse elastodynamic scattering problem in *time* domain. We restrict ourselves to the determination of the crack geometry, using data $(\mathbf{u}^d, \mathbf{T}^d)$ on the external boundary S_{ext} of a cracked solid (see Bui et al., [12]), by assuming that the crack plane has been previously determined by appropriate adjoint fields. The key tool is again the reciprocity gap functional R over the outer boundary and the full time of experiments

$$R := \int_0^\infty \int_{S_{\text{ext}}} \left\{ \mathbf{u}.\sigma[\mathbf{v}].\mathbf{n} - \mathbf{v}.\sigma[\mathbf{u}].\mathbf{n} \right\} \mathrm{d}S\, \mathrm{d}t$$

where $\mathbf{v}$ is the adjoint field. The internal boundary of the cracked solid Σ is yet unknown. From the elastodynamic conservation law, we get the fundamental variational equation

$$\int_0^\infty \int_\Sigma [\![\mathbf{u}]\!].\sigma[\mathbf{v}].\mathbf{n}\, \mathrm{d}S\, \mathrm{d}t = \int_0^\infty \int_{S_{\text{ext}}} \left\{ \mathbf{u}.\sigma[\mathbf{v}].\mathbf{n} - \mathbf{v}.\sigma[\mathbf{u}].\mathbf{n} \right\} \mathrm{d}S\, \mathrm{d}t := R^d(\mathbf{v}), \quad \forall \mathbf{v}$$

It is remarkable that this equation is valid either for a stress free crack as considered in Non-Destructive Testing method (NDT), or for a sliding crack (or fault) with friction as considered in the earthquake inverse problem. In the first case, fields $\mathbf{u}$ and $\sigma[\mathbf{u}].\mathbf{n} = \mathbf{T}^d$, are known as data on S_{ext}. In the second case, the solid is modelled by a half sphere of large radius, compare to the fault size. The stress is free on the ground while velocity and stress vectors can be estimated by geophysical considerations on the remaining half-spherical surface of S_{ext} embedded in the underground. If the seismic source is near the ground, spherical waves on the part of S_{ext} in the underground can be well estimated by a point source model. Other available data are the acceleration measured on the ground, due to the release of stresses on the fault. The above non-linear variational equation can be solved analytically, step by step, by determining the normal to the crack plane, the plane position and then the crack geometry. In what follows, we only determine the crack geometry, after finding the crack plane by suitable adjoint fields (for more details, see Bui et al., [27,12], Bui [28,34]).

The field equations for the NDT problem are:

$$\begin{aligned}
&\operatorname{div}\sigma[\mathbf{u}] - \rho\partial^2\mathbf{u}/\partial t^2 + \eta\partial\mathbf{u}/\partial t = 0, \quad \text{in } (\Omega - \Sigma) \times [0, \infty] \\
&\sigma[\mathbf{u}] = L.\varepsilon[\mathbf{u}], \qquad \sigma[\mathbf{u}].\mathbf{n} = \mathbf{T}^d \text{ on } S_{\text{ext}}, \qquad \sigma[\mathbf{u}].\mathbf{n} = 0 \text{ on } \Sigma \\
&\mathbf{u} = 0, \quad \partial\mathbf{u}/\partial t = 0 \quad \text{for } t \leqslant 0 \\
&t^2|\mathbf{u}| \to 0, \quad t^2|\partial\mathbf{u}/\partial t| \to 0 \quad \text{for } t \to \infty
\end{aligned}$$

In the first field equation, for mathematical reasons a damping term is introduced, assuming that η is a small positive and vanishing number. The wave equation is recovered in the limit $\eta \to 0^+$.

The adjoint equations are:

$$\begin{aligned}
&\operatorname{div}\sigma[\mathbf{v}] - \rho\partial^2\mathbf{v}/\partial t^2 - \eta\partial\mathbf{v}/\partial t = 0, \quad \text{in } (\Omega - \Sigma) \times [0, \infty] \\
&\sigma[\mathbf{v}] = L.\varepsilon[\mathbf{v}], \quad [\![\mathbf{v}]\!] = 0 \quad \text{on the crack}
\end{aligned}$$

Neither time conditions at infinity, nor boundary conditions on S_{ext} are demanded for the adjoint field. This makes it easier to find the solution for Σ. In the NDT problem, the adjoint field is given by ($\mathbf{s} = (s_1, s_2)$, the crack plane is $x_3 = 0$):

$$\begin{aligned}
&\mathbf{v}^{(s,q)}(\mathbf{x}, t) = \operatorname{grad}\phi(\mathbf{x}, t; \mathbf{s}, q) \\
&\phi(\mathbf{x}, t; \mathbf{s}, q) = \exp(\mathrm{i}qt - \eta t)\exp(\mathrm{i}\mathbf{s}.\mathbf{x})\exp\left[x_3\left\{|\mathbf{s}|^2 + (\mathrm{i}q - \eta)^2/c_p^2\right\}^{1/2}\right], \quad \eta \to 0^+
\end{aligned}$$

As in the inverse heat diffusion case, the geometry of the crack is identical to the support of $[\![\mathbf{u}]\!]$ in the crack plane, more precisely the support of $\operatorname{div}[\![\mathbf{u}]\!]$. From the fundamental variational equation, we can see that the time Fourier transform F_t combined with the spatial Fourier transform F_x of div$[\![\mathbf{u}]\!]$ is precisely related to the reciprocity gap functional (to within a factor depending on **s** and q). Therefore, by inversion we get div$[\![\mathbf{u}]\!]$ in explicit form

$$\mathrm{div}[\![\mathbf{u}]\!] = \frac{1}{2\mu}(F_t)^{-1}(F_x)^{-1}R^d\big(\mathbf{v}(\mathbf{s},q)\big)\big\{|\mathbf{s}|^2 - (q + \mathrm{i}0^+)^2/c_p^2\big\}^{-1/2}$$

This gives the exact solution to the inverse problem, $\Sigma \equiv$ Support div $[\![\mathbf{u}]\!]$. Due to the presence of the imaginary term $\mathrm{i}0^+$, the bracketed term {.} in the right-hand side of the above equation does not vanish in the **s**-plane. Thus, it can be rigorously proved that div$[\![\mathbf{u}]\!]$ is a compactly supported function of **x**.

Remark. In the earthquake inverse problem, the adjoint field to be considered is the solenoidal field $\mathbf{v} = \mathrm{curl}\{\phi(\mathbf{x},t;\mathbf{s},q)\mathbf{e}^3\}$ with shear wave velocity c_s instead of c_p for ϕ. The solution is $\Sigma = \mathrm{Support}\{\mathrm{div}[\![\mathbf{u}^\perp]\!]\}$, with $[\![\mathbf{u}^\perp]\!] = ([\![\mathbf{u_2}]\!], -[\![\mathbf{u_1}]\!], 0)$ and div$[\![\mathbf{u}^\perp]\!]$ given by similar formulae.

Conclusion

As a concluding remark, I would like to mention that *duality* which is found all along this paper is a very old philosophical principle in Asia. Duality is synonym of parallelism, or complementary things, sometimes an opposition between things tied together in an integral whole: Yin and Yang in China, Âm and Duong in Vietnam (the Vietnamese words for Female and Male respectively), Positive and Negative, the Sky and the Earth, Water and Fire, etc.

References

[1] P. Germain, The method of virtual power in continuum mechanics. Part II, in: J.J.D. Domingos, et al. (Eds.), Applications to Continuum Thermodynamics, J. Wiley, New York, 1973, pp. 317–333.
[2] L. Schwartz, Théorie des distributions, Hermann, Paris, 1978.
[3] P. Germain, Duality and convection in continuum mechanics, in: G. Fichera (Ed.), Trends in Applications to Mechanics, Pitman, London, 1978, pp. 107–128.
[4] E. Tonti, On the formal structure of physical theories, Cooperative Library Instituto di Polytechnico di Milano, Milano, 1975.
[5] Ladeveze, Comparison of models of continuous media (in French), PhD Thesis, Univ. of Paris VI, 1975.
[6] H.D. Bui, Comptes Rendus Acad. Sciences, Paris II 311 (1990) 7.
[7] Q.S. Nguyen, H.D. Bui, Sur les matéraux à écrouissage positif ou négatif, J. Mécanique 13 (2) (1974) 321–342.
[8] H.D. Bui, On the variational boundary integral equations in elastodynamics with the use of conjugate functions, J. Elasticity 28 (1992) 247.
[9] H.D. Bui, Comptes Rendus Acad. Sciences 265 (1967) 862–865.
[10] H.D. Bui, Detection de fissure par une méthode géométrique, in: J. Horowitch, J.L. Lions (Eds.), A propos des grands Systèmes des Sciences et de la Technologie, Masson, Paris, 1993.
[11] E. Lorentz, S. Andrieux, Inter. J. Solids Struct. 40 (12) (2003) 2905–2936.
[12] H.D. Bui, A. Constantinescu, H. Maigre, The reciprocity gap functional for identifying defects and cracks, in: Z. Mroz, G.E. Stavroulakis (Eds.), Parameter Identification of Materials and Structures, in: CISM Course and Lecture, vol. 469, Springer, Wien, New York, 2005.
[13] J.R. Rice, Mathematical analysis in the mechanics of fracture, in: H. Liebowitz (Ed.), Fracture, Academic Press, 1968, p. 191.
[14] H.D. Bui, Dual path-independent integrals in the boundary-value problems of cracks, Eng. Fracture Mech. 6 (1974) 287–296.
[15] H.D. Bui, Stress and crack displacement intensity factors in elastodynamics, in: Proc. 4th Int. Conf. Fract., vol. 3, Waterloo, 1977, p. 91.
[16] B.A. Bilby, K.J. Miller, J.R. Willis (Eds.), Fundamentals of Deformation and Fracture, Cambridge University Press, 1984.
[17] D.C. Fletcher, Conservation laws in linear elastodynamics, Arch. Rat. Mech. Anal. 60 (1976) 329.
[18] H.D. Bui, H. Maigre, Extraction of stress intensity factors from global mechanical quantities, C. R. Acad. Sci. Paris (II) 306 (1988) 1213.
[19] B. Halphen, Q.S. Nguyen, Sur les matériaux standards généralisés, J. Mécanique 14 (1975) 39.
[20] H.D. Bui, et al., Comptes Rendus Acad. Sci. Paris 289 (1979) 211–214.
[21] H.D. Bui, Dissipation of energy in plasticity. Cahier Groupe Français de Rhéologie, 1, 1965, p. 15.
[22] J. Mandel, Energie élastique et travail dissipé dans les modèles rhéologiques. Cahier Groupe Français de Rhéologie, 1, 1965, p. 1.
[23] S. Andrieux, A. Ben Abda, Identification de fissures planes par une donnée de bord unique: un procédé direct de localisation et d'identification, C. R. Acad. Sci. Paris (I) 315 (1992) 1323–1328.
[24] S. Andrieux, H.D. Bui, A. Ben Abda, Reciprocity and crack identification, Inverse Problems 15 (1999) 59–65.
[25] A. Ben Abda, H.D. Bui, Planar cracks identification for the transient heat equation, Inverse and Ill-Posed Problems 11 (1) (2003) 67–86.
[26] H.D. Bui, A. Constantinescu, H. Maigre, Inverse scattering of a planar crack in 3D acoustics: closed form solution for a bounded solid, C. R. Acad. Sci. Paris 327 (II) (1999) 971–976.
[27] H.D. Bui, A. Constantinescu, H. Maigre, An exact inversion formula for determining a planar fault from boundary measurements, Inverse Ill-Posed Problems 13 (6) (2005) 553–565.
[28] H.D. Bui, Fracture Mechanics: Inverse Problems and Solutions, Springer, 2006.
[29] S. Das, P. Suhadolc, On the inverse problem for earthquake rupture: The Hasskel-type source model, J. Geophys. Res. 101 (B3) (1996) 5725–5738.

[30] J.D. Achenbach, Z.P. Bazant, Elastodynamic near-tip stress and displacement fields for rapidly propagating cracks in orthotropic materials, J. Appl. Mech. 97 (1972) 183.
[31] S. Andrieux, A. Ben Abda, Identification of planar cracks by complete overdetermined data inversion formulae, Inverse Problems 12 (1996) 553–563.
[32] H.D. Bui, A path-independent integral for mixed modes of fracture in linear thermo-elasticity, in: IUTAM Symposium on Fundamental of Deformation and Fracture, Sheffield, p. 597, April 1984.
[33] H.D. Bui, H. Maigre, D. Rittel, A new approach to the experimental determination of the dynamic stress intensity factors, Int. J. Solids & Struct. 29 (1992) 2881–2895.
[34] H.D. Bui, Inverse Problems in the Mechanics of Materials: An Introduction, CRC Press, Boca Raton, 1994.