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Yield criterion for a rigid-ideally plastic material with randomly oriented cracks

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Abstract

This Note presents an estimate of the yield criterion for a material with a von Mises matrix and randomly oriented ellipsoidal oblate cavities. The particular case where the cavities are penny-shaped cracks is detailed. The study is based on the problem of the unit cell made of an ellipsoidal oblate volume containing a confocal ellipsoidal oblate cavity. Bounds are looked for in order to verify the validity of the proposed estimate. ***To cite this article: P.-G. Vincent, Y. Monerie, C. R. Mecanique 336 (2008).***

Résumé

Surface de charge pour un matériau rigide plastique parfait avec fissures orientées aléatoirement. Cette Note présente une estimation de la surface de plasticité pour un matériau à matrice de von Mises et à cavités ellipsoïdales aplaties orientées aléatoirement. Le cas particulier où les cavités ellipsoïdales sont des fissures circulaires est particulièrement étudié. L'étude est basée sur le problème de la cellule unitaire constituée d'un volume ellipsoïdal aplati contenant en son centre une cavité ellipsoïdale aplatie confocale. Des bornes sont recherchées afin de vérifier la validité de l'estimation proposée. ***Pour citer cet article : P.-G. Vincent, Y. Monerie, C. R. Mecanique 336 (2008).***

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Mots-clés : Milieux poreux ; Critère de plasticité ; Plasticité parfaite ; Milieux fissurés

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Le matériau étudié est composé d'une matrice rigide plastique parfaite (critère de von Mises) et d'un grand nombre de cavités ellipsoïdales aplaties (ou de fissures) distribuées et orientées aléatoirement (orientations équiprobables, isotropie macroscopique). Le critère de plasticité macroscopique est défini par un potentiel en taux déformation $\Phi(\dot{E})$.

Borne supérieure de $\Phi(\dot{E})$: Des conditions de taux de déformation homogène au bord sur une cellule unitaire de type ellipsoïde aplati de révolution creux et une extension (8) du champ de vitesse de [1] au cas d'un taux de

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déformation effectif $\dot{E}$ *quelconque* et non plus seulement axisymétrique, fournissent une borne supérieure (13) du potentiel effectif de cette cellule unitaire. L'hypothèse de Voigt et une moyenne sur les orientations conduisent à une borne rigoureuse (15) pour $\Phi(\dot{E})$.

Une estimation de $\Phi(\dot{E})$: la borne obtenue ne conduisant pas a priori à une expression analytique du critère de plasticité macroscopique, une estimation est proposée. A l'échelle de la cellule unitaire, différentes approximations proposées par [1] sont utilisées. Le critère sur cellule unitaire obtenu (19) est similaire à celui de [1] avec des termes de cisaillement additionnels. L'hypothèse de Reuss et une moyenne sur les orientations conduisent à un critère macroscopique approché (22) de type Gurson.

Résultats : la comparaison du critère approché (22) et de la borne (15) à des simulations numériques par éléments finis montre — dans le cas des pores ellipsoïdaux aplatis — la pertinence de l'estimation (22), notamment en hydrostatique, et le caractère peu majorant de la borne (15), notamment pour les faibles fractions volumiques (Fig. 3). Dans le cas du milieu fissuré, la nouvelle borne proposée améliore grandement celle de [2] en hydrostatique et l'estimation (22) semble convenable pour les chargements purement déviatoriques.

1. Studied material, unit cell, notations and recall of the variational principle

Let us consider a domain Ω_t (volume $|\Omega_t|$) containing an infinity of ellipsoidal voids (domain ω_t) randomly distributed and oriented (isotropy in the distribution and in the orientations) in an isotropic rigid ideal-plastic matrix (domain $M_t = \Omega_t - \omega_t$). The constitutive behavior of the matrix is supposed to be governed by a von Mises strain rate potential ϕ_{VM} (stress potential ψ_{VM}):

$$\dot{\epsilon} = \frac{\partial \psi_{\mathrm{VM}}}{\partial \sigma}, \qquad \sigma = \frac{\partial \phi_{\mathrm{VM}}}{\partial \dot{\epsilon}}, \qquad \phi_{\mathrm{VM}}(\dot{\epsilon}) + \psi_{\mathrm{VM}}(\sigma) = \sigma : \dot{\epsilon}$$
$$\phi_{\mathrm{VM}} = \{\sigma_o \dot{\epsilon}_{\mathrm{eq}} \text{ if } \dot{\epsilon}_m = 0,\ +\infty \text{ otherwise}\} \tag{1}$$

where σ (resp. $\dot{\epsilon}$) is the stress field (resp. the strain rate), $\sigma_{\mathrm{eq}} = \sqrt{3/2\sigma_d : \sigma_d}$ is the equivalent von Mises stress (deviatoric part $\sigma_d = \sigma - \sigma_m I$, and hydrostatic part $\sigma_m = 1/3\,\mathrm{tr}(\sigma)$), and $\dot{\epsilon}_{\mathrm{eq}} = \sqrt{2/3\dot{\epsilon}_d : \dot{\epsilon}_d}$ is the equivalent strain rate (same notations for $\dot{\epsilon}_d$ and $\dot{\epsilon}_m$).

Denoting by $\langle \bullet \rangle_{\mathcal{D}} = \frac{1}{|\Omega_t|}\int_{\mathcal{D}} \bullet \,\mathrm{d}x$ the average symbol over any domain $\mathcal{D}$, the volume average of σ and $\dot{\epsilon}$ are respectively defined by $\Sigma = \langle \sigma \rangle_{M_t}$, and $\dot{E} = \frac{1}{|\Omega_t|}\int_{\partial \Omega_t} \frac{1}{2}(v \otimes N + N \otimes v)\,\mathrm{d}S$, where v is the velocity field, N is the outer unit normal vector to Ω_t, and $\partial\Omega_t$ denotes the outer boundary of Ω_t. Extending the velocity field v in an arbitrary way over the cavities [3], and defining σ as zero in ω_t, these macroscopic quantities are found as $\Sigma = \langle \sigma \rangle_{\Omega_t}$ and $\dot{E} = \langle \dot{\epsilon} \rangle_{\Omega_t}$. The unknown macroscopic potentials are also defined as: $\Phi(\dot{E}) = \langle \phi(\dot{\epsilon}) \rangle_{M_t}$ and $\Psi(\Sigma) = \langle \psi(\sigma) \rangle_{M_t}$.

Assuming that the boundary conditions on $\partial\Omega_t$ take the form of uniform strain rates ($v = \dot{E} \cdot x$) or uniform stresses ($\sigma \cdot N = \Sigma \cdot N$), the following variational problems are respectively obtained [4]:

$$\Phi^+(\dot{E}) = \inf_{v \in \mathcal{K}} \langle \phi_{\mathrm{VM}}(\dot{\epsilon}(v)) \rangle_{M_t} \quad \text{with } \mathcal{K} = \{v \mid v = \dot{E} \cdot x \text{ on } \partial\Omega_t\} \tag{2}$$
$$\Psi^-(\Sigma) = \inf_{\tau \in \mathcal{S}} \langle \psi_{\mathrm{VM}}(\tau) \rangle_{M_t} \quad \text{with } \mathcal{S} = \{\tau \mid \tau \cdot N = \Sigma \cdot N \text{ on } \partial\Omega_t \text{ and } \mathrm{div}(\tau) = 0 \text{ in } M_t\}$$

Denoting by $\Phi^-(\dot{E})$ the dual potential of $\Psi^-(\Sigma)$, the following inequalities hold:

$$\Phi^-(\dot{E}) \leqslant \Phi(\dot{E}) \leqslant \Phi^+(\dot{E}) \tag{3}$$

This work provides: (i) a rigorous upper bound for $\Phi^+(\dot{E})$ (Section 2); (ii) an analytical estimate for $\Phi(\dot{E})$ (Section 3); and (iii) a numerical estimate of a lower bound of $\Phi^-(\dot{E})$ in the hydrostatic case (Section 4).

For that purpose, a unit cell composed of an ellipsoidal oblate domain Ω (semi-axis $a_2 \leqslant b_2$) containing an ellipsoidal oblate confocal cavity ω (semi-axis $a_1 \leqslant b_1$) is considered (Fig. 1). Following [1], the foci are located on a circle whose radius is referred as $c = \sqrt{b_1^2 - a_1^2} = \sqrt{b_2^2 - a_2^2}$, the Cartesian (e_x, e_y, e_z) and cylindrical (e_ρ, e_ϕ, e_z) basis are invoked, and the following notations are used:

$$\rho = c\cosh\lambda \sin\beta, \quad \phi = \phi, \quad z = c\sinh\lambda\cos\beta, \quad \lambda \in [0, +\infty[, \quad \beta \in [0, \pi], \quad \phi \in [0, 2\pi] \tag{4}$$
$$f = (a_1 b_1^2)/(a_2 b_2^2) \text{ (porosity)}; \quad e_2 = c/b_2, \quad e_1 = c/b_1, \quad e = c/b \text{ (excentricities)} \tag{5}$$
$$a = c\sinh\lambda, \quad b = c\cosh\lambda \tag{6}$$

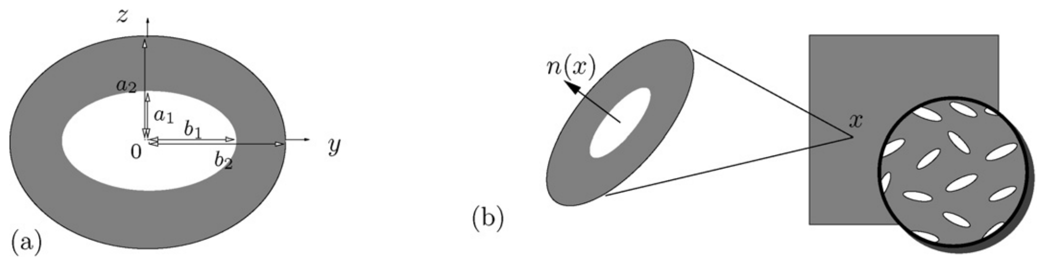

Fig. 1. (a) Unit cell. (b) Studied material: domain containing an infinity of randomly oriented ellipsoidal voids.

Fig. 1. (a) Cellule unitaire. (b) Matériau d'étude : domaine contenant une infinité de cavités ellipsoïdales orientées aléatoirement.

The iso-λ surfaces are confocal ellipsoids which foci are on the circle $\rho = c$, $z = 0$ and which minor and major semi-axes are given by a and b. The matrix (domain $M = \Omega - \omega$) is thus delimited by the iso-λ_1 and λ_2 surfaces. Note that the case of a penny-shaped crack is obtained for $f = 0$, $e_1 = 1$ and e_2 arbitrary. Considering the von Mises criterion (1), the uniform strain rate boundary condition on the outer ellipsoid $\partial\Omega$, and inequalities (3), the macroscopic strain rate potential of the unit cell (subscript U) verifies:

$$\begin{aligned} &\Phi_U^-(\dot{E}) \leqslant \Phi_U(\dot{E}) \leqslant \Phi_U^+(\dot{E}), \quad \text{with} \\ &\Phi_U^+(\dot{E}) = \inf_{v\in\mathcal{K}_i(\dot{E})} \frac{\sigma_o}{|\Omega|}\int_M \dot{\epsilon}_{\text{eq}}\big(v(x)\big)\,\mathrm{d}x \\ &\mathcal{K}_i(\dot{E}) = \big\{v \mid v = \dot{E}\cdot x \text{ on } \partial\Omega \text{ and } \operatorname{div}\big(v(x)\big) = 0 \text{ in } M\big\} \end{aligned} \tag{7}$$

2. Upper bound for the yield criterion

2.1. Choice of a velocity field for the unit cell

The chosen velocity field v extends the field v^{Go} of [1] to the case of a completely general macroscopic strain rate $\dot{E}$. This property (concerning the fact that $\dot{E}$ is under a general form and not only an axisymmetric form) is a key point for the study of the material with randomly oriented cavities. This velocity field is based on a splitting of $\dot{E}$ into its axisymmetric $\dot{E}^{\text{A}}$ and non-axisymmetric $\dot{E}^{\text{NA}}$ parts:

$$v(x) = v^{\text{Go}}(x) + v^{\text{NA}}(x) \quad \text{with } v^{\text{Go}}(x) = \dot{E}^{\text{A}}\cdot x,\ \forall x\in\partial\Omega,\ \text{and } v^{\text{NA}}(x) = \dot{E}^{\text{NA}}\cdot x,\ \forall x\in M \tag{8}$$

$$\{\dot{E}^{\text{A}}\} = \big((\dot{E}_{xx}+\dot{E}_{yy})/2,\ (\dot{E}_{xx}+\dot{E}_{yy})/2,\ \dot{E}_{zz}, 0, 0, 0\big)^T$$

$$\{\dot{E}^{\text{NA}}\} = \big((\dot{E}_{xx}-\dot{E}_{yy})/2,\ (\dot{E}_{yy}-\dot{E}_{xx})/2,\ 0,\ \sqrt{2}\dot{E}_{xy},\ \sqrt{2}\dot{E}_{xz},\ \sqrt{2}\dot{E}_{yz}\big)^T$$

The fields v^{Go} and v^{NA} are incompressible fields, which ensures that v is in $\mathcal{K}_i(\dot{E})$. The field v^{Go} is briefly recalled:

$$v^{\text{Go}} = Av^{(A)} + Bv^{(B)} \quad \text{with } v^{(A)} = R(\lambda)\rho e_\rho + Z(\lambda)z e_z,\ v^{(B)} = -\frac{x}{2}e_x - \frac{y}{2}e_y + z e_z \tag{9}$$

$$R(\lambda) = -ac/b^2 + \text{Arcsin}(c/b) \quad \text{and} \quad Z(\lambda) = 2c/a - 2\text{Arcsin}(c/b) \tag{10}$$

$$A = \text{tr}(\dot{E})/(2R_2+Z_2) \quad \text{and} \quad B = \dot{E}_{zz} - AZ_2 \quad \text{where: } R_2 = R(\lambda_2),\ Z_2 = Z(\lambda_2) \tag{11}$$

2.2. Upper bound for the effective strain rate potential of the unit cell

Using the field (8) and the Cauchy–Schwarz inequality, the effective potential $\Phi_U(\dot{E})$ is rigorously bounded by:

$$\Phi_U(\dot{E}) \leqslant \Phi_U^+(\dot{E}) \leqslant \frac{\sigma_o}{|\Omega|}\int_M \dot{\epsilon}_{\text{eq}}\big(v(x)\big)\,\mathrm{d}\Omega \leqslant \sigma_o \int_{\lambda_1}^{\lambda_2} \sqrt{J(\lambda)}\sqrt{\int_{\phi=0}^{2\pi}\int_{\beta=0}^{\pi} \dot{\epsilon}_{\text{eq}}^2(x) J_{\lambda\beta\phi}\,\mathrm{d}\beta\,\mathrm{d}\phi}\,\mathrm{d}\lambda \tag{12}$$

where $J_{\lambda\beta\phi} = b(a^2\sin^2\beta + b^2\cos^2\beta)\sin\beta/|\Omega|$, and $J(\lambda) = \int_{\phi=0}^{2\pi}\int_{\beta=0}^{\pi} J_{\lambda\beta\phi}\,\mathrm{d}\beta\,\mathrm{d}\phi = \frac{4}{3}\pi b(2a^2+b^2)/|\Omega|$. After some algebra, and remarking that $\int_{\phi=0}^{2\pi}\int_{\beta=0}^{\pi}\dot\epsilon(v^{\mathrm{Go}}(x)) : \dot E^{\mathrm{NA}} J_{\lambda\beta\phi}\,\mathrm{d}\beta\,\mathrm{d}\phi$ nullifies, inequality (12) can be rewritten:

$$\Phi_U(\dot E) \leqslant \Phi_U^+(\dot E) \leqslant \sigma_o \int_{\lambda_1}^{\lambda_2} \sqrt{J(\lambda)}\sqrt{\{\dot E\}^T\cdot Q(\lambda)\cdot\{\dot E\}}\,\mathrm{d}\lambda = \Phi_U^{++}(\dot E) \quad \text{with:} \tag{13}$$

$$Q(\lambda) = \frac{2\pi b}{|\Omega|}T^T\cdot\begin{pmatrix} I_1 & I_2 \\ I_2 & I_3\end{pmatrix}\cdot T + J(\lambda)Q^{\mathrm{NA}}, \quad I_1 = \frac{8}{9}\big(3b^2R^2 + 2a^2(R^2+RZ+Z^2)\big)$$
$$I_2 = (4/3)(-b^2R + a^2Z), \quad I_3 = (4a^2+2b^2)/3, \quad \tilde\alpha = (2R_2+Z_2)^{-1} = (a_2b_2^2)/(2c^3)$$

where T is a 2×6 matrix and Q^{NA} is a symmetric 6×6 matrix, whose non-zero coefficients are: $T_{11} = T_{12} = T_{13} = \tilde\alpha$, $T_{21} = T_{22} = -\tilde\alpha Z_2$, $T_{23} = 1 - \tilde\alpha Z_2$, $Q_{11}^{\mathrm{NA}} = Q_{22}^{\mathrm{NA}} = -Q_{12}^{\mathrm{NA}} = -Q_{21}^{\mathrm{NA}} = \frac{1}{3}$ and $Q_{44}^{\mathrm{NA}} = Q_{55}^{\mathrm{NA}} = Q_{66}^{\mathrm{NA}} = \frac{2}{3}$.

2.3. Upper bound for the effective strain rate potential of the studied material

Fig. 1 shows that at each point x in Ω_t corresponds a randomly oriented unit cell. So, the use of the Voigt assumption in the domain Ω_t (i.e. $\dot\epsilon(x) = \dot E$, $\forall x \in \Omega_t$) leads to:

$$\Phi^+(\dot E) \leqslant \oint \Phi_U^+(\mathcal{R}, \dot E)\,\mathrm{d}\mathcal{R} \tag{14}$$

where $\mathcal{R}$ denotes a dependence on the orientation of the quantity under consideration and $\oint$ is the average over all possible orientations.

Remark 1. This result can also be obtained by considering that Ω_t is a Hashin assemblage of randomly oriented ellipsoidal cells (see Fig. 2). A proof that one call fill space with randomly oriented ellipsoidal inclusions is given in [5]. Moreover the velocity field (8) satisfies boundary conditions of uniform deformation on the outer surface of the hollow ellipsoid $v = \dot E\cdot x$ on $\partial\Omega$. These particular boundary conditions allow for the extension of this field into a velocity field which is continuous over the entire assemblage of hollow ellipsoids.

The inequalities (3), (13), and (14), and the Cauchy–Schwarz inequality thus give the following rigorous upper bound for the cracked media:

$$\Phi(\dot E) \leqslant \oint \Phi_U^{++}(\mathcal{R}, \dot E)\,\mathrm{d}\mathcal{R} \leqslant \sigma_o \int_{\lambda_1}^{\lambda_2}\sqrt{J(\lambda)}\sqrt{\oint \{\dot E\}^T\cdot Q(\mathcal{R},\lambda)\cdot\{\dot E\}\,\mathrm{d}\mathcal{R}}\,\mathrm{d}\lambda \tag{15}$$

Following [6], the quantity $\oint\{\dot E\}^T\cdot Q(\mathcal{R},\lambda)\cdot\{\dot E\}\,\mathrm{d}\mathcal{R}$ is equal to $\{\dot E\}^T\cdot P_{JK}(Q(\lambda))\cdot\{\dot E\} = 3Q_J\dot E_m^2 + \frac{3}{10}Q_K\dot E_{\mathrm{eq}}^2$, where P_{JK} denotes the projection on the usual basis of isotropy of stiffness tensors. After some algebra, one obtains:

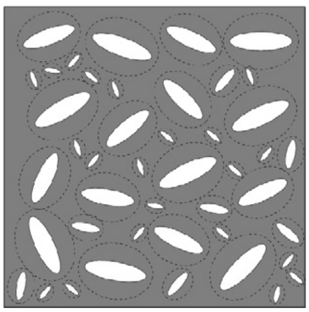

Fig. 2. Assemblage of ellipsoids.

Fig. 2. Assemblage d'ellipsoïdes.

$Q_J = \frac{4}{9}\frac{b\pi}{|\Omega|}[b^2(-1+6R\tilde{\alpha}+3Z_2\tilde{\alpha})^2 + 2a^2(1-6Z_2\tilde{\alpha}+12R^2\tilde{\alpha}^2+12Z^2\tilde{\alpha}^2+9Z_2^2\tilde{\alpha}^2+6Z\tilde{\alpha}(1+2R\tilde{\alpha}-3Z_2\tilde{\alpha}))]$ and $Q_K = \frac{10}{3}J(\lambda)$. The macroscopic yield surface is obtained numerically: $\Sigma = \frac{\partial \Phi}{\partial \dot{E}}$.

Remark 2. In particular, the right member of (15) leads to a purely deviatoric point of the form $\Sigma_{\text{eq}} = (1-f)\sigma_o$ which corresponds exactly to the Voigt upper bound. Moreover, a lower bound of $\Phi(\dot{E})$ can be obtained using the *Reuss assumption*:

$$\Psi^-(\Sigma) \leqslant \oint \Psi_U^{-+}(\mathcal{R}, \Sigma)\,\mathrm{d}\mathcal{R} \tag{16}$$

where Ψ_U^{-+} is the stress potential related to a stress field τ defined on the unit cell and such that $\tau \cdot N = \Sigma \cdot N$ on $\partial\Omega$. Denoting by Φ_U^{-+} the dual potential of Ψ_U^{-+}, one obtains:

$$\oint \Phi_U^{-+}(\mathcal{R}, \dot{E})\,\mathrm{d}\mathcal{R} \leqslant \Phi(\dot{E}) \leqslant \oint \Phi_U^{++}(\mathcal{R}, \dot{E})\,\mathrm{d}\mathcal{R} \tag{17}$$

3. Estimate for the yield criterion

3.1. Estimate for the yield criterion of the unit cell

Adopting the same approximations and the same sequence of changes of variables as [1] on (12) leads to an estimate for Φ_U^+. The sequence of changes of variables reads as: $x = c^3/(ab^2) = 1/(\sinh\lambda\cosh^2\lambda)$, $y = (\chi x)/(x + 3\chi/4)$ with $\chi = \sqrt{\pi^2 + 32/3}$, and finally $z = y_2/y$ (y_2 denotes the value of y for $\lambda = \lambda_2$). The quantity $\cos^2\beta$ is replaced by 1/3 and the quantity $(\dot{\epsilon}_{\text{eq}}(v^{\text{Go}}))^2$ is approximated by $(A\bar{F} + B\bar{G})^2 y^2 + B^2\bar{H}^2 = A'^2 y^2 + B'^2$ where $\bar{F}, \bar{G}, \bar{H}$ are constants given in [1]. One gets:

$$\Phi_U^+(\dot{E}) \approx \sigma_o' \int_{y_2/y_1}^{1} \sqrt{\frac{A''^2}{z^2} + B'^2 + (\dot{E}_{\text{eq}}^{\text{NA}})^2}\,\mathrm{d}z \quad \text{with } \sigma_o' = \frac{16\pi c^3 \sigma_o}{9|\Omega| y_2} \text{ and } A'' = A' y_2 \tag{18}$$

Moreover, it can be shown (the proof is not given here) that the yield surface related to a potential of the form $W(A, B_i) = \sigma_o \int_f^1 \sqrt{\frac{A^2}{y^2} + \sum_{i=1}^N B_i^2}\,\mathrm{d}y$ writes: $\sum_{i=1}^N (\frac{1}{\sigma_o}\frac{\partial W}{\partial B_i})^2 + 2f\cosh(\frac{1}{\sigma_o}\frac{\partial W}{\partial A}) - 1 - f^2 = 0$. The use of this lemma gives, after some calculations, the following estimate of the macroscopic yield criterion:

$$\left(\frac{\Sigma_n - \Sigma_p + \eta\Sigma_h}{\sigma_o/\sqrt{C}}\right)^2 + \frac{3\tau_p^2}{\sigma_o^2} + \frac{3\tau_n^2}{\sigma_o^2} + 2(g+1)(g+f)\cosh\left(\frac{\kappa}{\sigma_o}\Sigma_h\right) - (g+1)^2 - (g+f)^2 = 0 \tag{19}$$

where $\Sigma_p = (\Sigma_{xx} + \Sigma_{yy})/2$, $\Sigma_n = \Sigma_{zz}$, $\Sigma_h = 2\alpha_2\Sigma_p + (1-2\alpha_2)\Sigma_n$, $\tau_p^2 = \Sigma_{xy}^2 + (\Sigma_{xx} - \Sigma_{yy})^2/4$, and $\tau_n^2 = \Sigma_{xz}^2 + \Sigma_{yz}^2$. The coefficients $C, \eta, g, \kappa, \alpha_2$ can be found in [1] or in [3]. The coefficients given in [3] (denoted by a *) are retained in the present estimate for their accuracy in the case of the axisymmetric loading:

$$\alpha_1^* = \left[-e_1(1-e_1^2) + (1-e_1^2)^{1/2}\operatorname{Arcsin}(e_1)\right]/(2e_1^3), \quad \alpha_2^* = (1-e_2^2)(1-2e_2^2)/(3-6e_2^2+4e_2^4)$$

$$g^* = e_2^3/(1-e_2^2)^{1/2}, \quad g_f = g^*/(g^*+f), \quad g_1 = g^*/(g^*+1), \quad H = 2|\alpha_1^* - \alpha_2^*|$$

$$(\kappa^*)^{-1} = 2/3 + \left[(2/3)(g_f - g_1) + (2/5)(g_f^{5/2} - g_1^{5/2})(4/3 - g_f^{5/2} - g_1^{5/2})\right]/\ln(g_f/g_1)$$

$$\eta^* = \frac{\kappa^*(1-f)(g^*+1)(g^*+f)\sinh(\kappa^* H)}{(g^*+1)^2 + (g^*+f)^2 + (g^*+1)(g^*+f)(\kappa^* H\sinh(\kappa^* H) - 2\cosh(\kappa^* H))}$$

$$C^* = -(\kappa^*/\eta^*)(g^*+1)(g^*+f)\sinh(\kappa^* H)/\left(\eta^* H - (1-f)\right)$$

The form of the obtained criterion (19) is exactly the same as in [1] with additional shear terms.

3.2. Estimate for the yield criterion of the studied material

As underlined in Remark 2, an estimate of the overall yield criterion is sought using the Reuss assumption (i.e. $\forall x \in \Omega_t$, $\sigma(x) = \Sigma$). For any macroscopic stress Σ and any normal to the unit cell $n(x)$ (Fig. 1), there is a second order tensor $Q^{(n)}$ such that $\Sigma^{(n)} = Q^{(n)} \cdot \Sigma \cdot Q^{(n)T}$ and the approximate criterion (19) reads:

$$\forall x \in \Omega_t: \quad \Sigma^{(n)} : S : \Sigma^{(n)} + 2(g+1)(g+f)\cosh(T : \Sigma^{(n)}) - (g+1)^2 - (g+f)^2 \leqslant 0 \tag{20}$$

The fourth order tensor S and the second order tensor T are easily derived from (19). Moreover there is another second order tensor such that $\Sigma = R \cdot \bar{\Sigma} \cdot R^T$ with $\{\bar{\Sigma}\}^T = (\Sigma_1, \Sigma_2, \Sigma_3, 0, 0, 0)$. So there are $R^{(n)}$ second order tensors such that $\Sigma^{(n)} : S : \Sigma^{(n)} = \bar{\Sigma} : S^{(n)} : \bar{\Sigma}$ and $T : \Sigma^{(n)} = T^{(n)} : \bar{\Sigma}$ with $S^{(n)}_{mnop} = R^{(n)T}_{mi} R^{(n)T}_{nj} R^{(n)T}_{ok} R^{(n)T}_{pl} S_{ijkl}$ and $T^{(n)}_{kl} = R^{(n)T}_{ki} T_{ij} R^{(n)}_{jl}$. The approximate criterion (19) is rewritten as:

$$\forall x \in \Omega_t: \quad \bar{\Sigma} : S^{(n)} : \bar{\Sigma} + 2(g+1)(g+f)\cosh(T^{(n)} : \bar{\Sigma}) - (g+1)^2 - (g+f)^2 \leqslant 0 \tag{21}$$

The orientational average of (21) implies that $\oint \bar{\Sigma} : S^{(n)} : \bar{\Sigma} - (g+1)^2 - (g+f)^2$ is upper bounded by $-2(g+1) \times (g+f) \oint \cosh(T^{(n)} : \bar{\Sigma})$ and thus by $-2(g+1)(g+f)\cosh(\oint T^{(n)} : \bar{\Sigma})$. The latter is equal to $\cosh(\frac{\kappa}{\sigma_o}\Sigma_m)$ [7], and as in Section 2.3, the term $\oint \bar{\Sigma} : S^{(n)} : \bar{\Sigma}$ is equal to $P_{JK}(S^{(n)})$. The resulting macroscopic criterion reads:

$$C\eta^2\left(\frac{\Sigma_m}{\sigma_o}\right)^2 + D\left(\frac{\Sigma_{\mathrm{eq}}}{\sigma_o}\right)^2 + 2(g+1)(g+f)\cosh\left(\frac{\kappa}{\sigma_o}\Sigma_m\right) - (g+1)^2 - (g+f)^2 = 0 \tag{22}$$

where $D = \frac{4}{15}C(\frac{3}{4} + \frac{\eta^2}{3}(1-3\alpha_2)^2 + \eta(1-3\alpha_2)) + \frac{4}{5}$, and the retained coefficients C, η, α_2, g, κ are those given in [3] (denoted by a *, as in the previous section).

In the case of the spherical voids, it is easy to check that $C^* = 1$, $\eta^* = 0$, $\kappa^* = 3/2$ and $g^* = 0$, so that the criterion (22) is exactly the original Gurson criterion. It has also been seen that, in the case of the spherical voids, the criterion (22) gives the same result as the upper bound (15).

4. Results

Finite element calculations on the unit cell are performed with the condition $\sigma \cdot N = \Sigma_m I \cdot N$ on $\partial\Omega_t$. The hydrostatic stresses Σ_m obtained with these numerical simulations allow us to approximate the purely hydrostatic points given by the lower bound (17) and concerning the studied material (Reuss assumption).

Fig. 3 shows that the final estimate (22) does not violate the upper bound (15) and that the purely hydrostatic points given by the final estimate (22) are greater than those of the numerical simulations (except when f is weak). In the

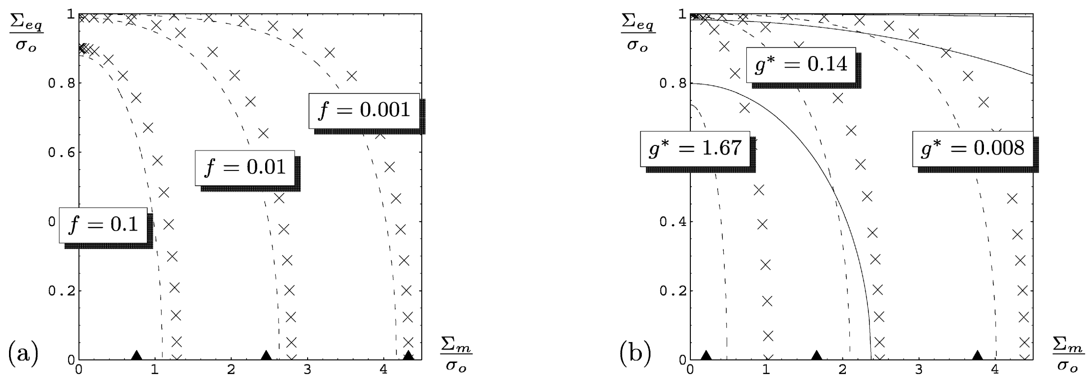

Fig. 3. Yield criteria for randomly oriented oblate ellipsoidal cavities (a) ($a_1/b_1 = 1/5$) or penny-shaped cracks (b) in a von Mises matrix. Crosses: numerical upper bound (15). Triangles: numerical points (lower bound). Dashed lines: final estimate (22). Solid lines: result given in [2] (upper bound when $g^* \leqslant 1$).

Fig. 3. Critères de plasticité pour cavités ellipsoïdales aplaties orientées aléatoirement (a) ($a_1/b_1 = 1/5$) ou fissures circulaires orientées aléatoirement (b) dans une matrice de von Mises. Croix : borne supérieure numérique (15). Triangles : points numériques (borne inférieure). Pointillés : estimation proposée (22). Traits pleins : résultat donné dans [2] (borne supérieure lorsque $g^* \leqslant 1$).

case of cracks, the upper bound given in [2] can also be plotted. For being more precise, this result of [2] is only an upper bound when $g^* \leqslant 1$ and is obtained by the way of the variational approach of [8] or equivalently by the way of the so-called 'modified secant method' (see [9]). The purely equivalent point Σ_{eq} of the final estimate (22) is slightly greater than the one that is given by the upper bound [2]. However, the purely equivalent point given by the final estimate (22) is smaller than a simple upper bound using the Voigt assumption (which in the case of cracks is equal to σ_o). The purely hydrostatic point of the upper bound given in [2] is much larger than the purely hydrostatic point of the upper bound (15).

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