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Shakedown theorems in Contact Mechanics

Nicolas Antoni [a], Quoc-Son Nguyen [b,*]

[a] *Laboratoire de la conception et ingénierie de l'environnement, Université de Savoie, 73011 Chambery cedex, France*
[b] *Laboratoire de mécanique des solides, CNRS – UMR7649, École polytechnique, 91128 Palaiseau cedex, France*

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Abstract

In this Note, the phenomenon of cumulative slip of solids under cyclic loads is considered. The topic concerns with the relative displacements of two solids maintained in contact by friction. This problem of the daily-life mechanics can be compared to the shakedown of elastic plastic solids under cyclic loads. A transcription of the plastic shakedown theorems is given here for the problem of frictional contact. The statement of Melan theorem is first given for the cumulative-slip problem under some restrictive assumptions. It suggests again the introduction of a safety coefficient with respect to slips. The safety coefficient can be computed from two static and kinematic approaches in min–max duality and leads again to Koiter theorem. As in plasticity, these results are available only for associated laws and do not hold for Coulomb law of friction, except in some very particular situations. These results can be understood mathematically as a particular case of shakedown theorems in plasticity, when the plastic strain is localized on a surface. For Coulomb friction, numerical simulations by direct step-by-step calculations show that the asymptotic behaviour of the response with or without slip-shakedown could be obtained very quickly after some cycles. ***To cite this article: N. Antoni, Q.-S. Nguyen, C. R. Mecanique 336 (2008).***

Résumé

Théorèmes d'adaptation en Mécanique du Contact. On propose dans cette Note une analyse du phénomène de micro-reptation des solides sous chargements cycliques. Il s'agit du glissement relatif cumulé entre deux solides maintenus en contact par frottement. Ce problème de la mécanique appliquée s'apparente au phénomène de l'adaptation ou du rochet des solides élasto-plastiques sous chargements cycliques, le vecteur glissement jouant le rôle de la déformation plastique. Dans le même esprit qu'en plasticité classique, on donne ici un théorème d'adaptation du glissement lorsque la loi de frottement est une loi standard. Ce théorème conduit à l'introduction d'un coefficient de sécurité aux glissements. Ce coefficient est défini par deux approches statique et cinématique en dualité min–max. Comme en plasticié incrémentale, ces résultats ne sont pas valables pour des lois non standard, en particulier pour le frottement de Coulomb. Dans ce cas, le recours aux simulations numériques s'impose. Nos simulations sur des exemples plus ou moins complexes montre que la vitesse de convergence vers un état asymptotique avec adaptation ou sans adaptation des glissements relatifs est souvent obtenue après quelques cycles seulement, c'est-à-dire très rapide comparé aux situations rencontrées en Plasticité. ***Pour citer cet article : N. Antoni, Q.-S. Nguyen, C. R. Mecanique 336 (2008).***

* Corresponding author.
E-mail address: son@lms.polytechnique.fr (Q.-S. Nguyen).

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Mots-clés : Solides et structures ; Chargements cycliques ; Comportement asymptotique ; Micro-reptation ; Théorème d'adaptation du glissement ; Coefficient de sécurité aux glissements ; Approches statique et cinématique

1. The problem of cumulative slips of solids in frictional contact

The problem of micro-slips of two solids maintained in contact with friction under pre-stress fields and cyclic loads is often observed in the daily life. In particular, the micro-slips may lead to undesired effects such as an unbounded cumulative slip of the solids in contact or to the damage and fatigue by alternating slips. The micro-slips may also shake down in the sense that the slip vector at any point of the contact surface tends to a finite limit at large time. Conditions ensuring the shakedown of the system are often searched for in the engineering design to avoid the undesired effects.

Some interesting discussions on the subject can be found in the recent literature, cf. Antoni and co-workers [1–3]. In their works, the phenomenon of slip motion has been studied principally by numerical analysis in several examples of engine components under periodic thermo-mechanical loadings and the analogy of this phenomenon with the elastic-plastic response of structures has been pointed out, especially concerning the possibility of shakedown under cyclic loads. This discussion is completed here by a shakedown analysis and leads to some shakedown theorems when a standard law of friction is assumed in the same spirit as in classical plasticity, [4–7].

2. A sufficient condition for slip-shakedown

2.1. The governing equations

A system of two elastic solids is considered to describe the relative displacement problem. A solid V_1 is assumed to be elastic (or rigid) and maintained by implied displacements $\mathbf{u_d} = \mathbf{0}$ on a portion S_1^u of its boundary. A second elastic solid of volume V_2 is maintained only by a contact with friction on a common portion S_c of the boundaries. The two solids are submitted to a thermo-mechanical loadings composed of the fields of volume force $\mathbf{f}(t)$ and surface forces $\mathbf{F}(t),\ i = 1, 2$ on the complementary portions of the boundaries ($S_1^F = S_1 - S_1^u - S_c$ and $S_2^F = S_2 - S_c$) and of the fields of temperature elevation $\theta(t)$ for $0 \leqslant t \leqslant +\infty$. The existence of a pre-stress field σ^I in V_i is also admitted.

It is also assumed that the contact is ensured for all $0 \leqslant t \leqslant +\infty$. This means that the normal relative displacement is null for all t. The tangent relative displacement of V_2 with respect to V_1 on S_c is denoted as $\mathbf{U}(t)$ and the tangent force acting by V_2 on V_1 is $\mathbf{T}(t)$. Starting from a given initial value $\mathbf{U}(0)$, the variation of the tangent relative displacement $\mathbf{U}(t)$ is governed by friction laws under the considered loading.

On the contact surface S_c, a standard law of friction is assumed. This means that the friction force T must satisfy $T(x,t) \in C(x,t)$ where $C(x,t)$ denotes a given convex domain of admissible force and that the displacement rate $\dot{U}$ must satisfy the normality law for all $x \in S_c$ and for all t:

$$(T - \hat{T}) \cdot \dot{U} \geqslant 0 \quad \forall \hat{T} \in C \tag{1}$$

The dissipation by friction per unit surface is a positive homogeneous convex function of the displacement rate

$$T \cdot \dot{U} = \max_{\hat{T} \in C} \hat{T} \cdot \dot{U} = D(\dot{U}), \quad \text{with } D(\dot{U}) \geqslant c\|\dot{U}\| \tag{2}$$

The given domain of admissible force may depend on time, for example when the yield values of force depend on a prescribed time-varying temperature. For example, this domain can be defined as $\|T(x,t)\| \leqslant k(x,t)$ where $k(x,t)$ is a given function for all x and all t. The anisotropy of the friction can also be taken into account from the geometry of C. However, it is well known that the assumption of such a standard law of friction is a restrictive condition. For instance, Coulomb's friction does not satisfy this assumption since the associated admissible domain C depends on the normal reaction N which is not a given data.

It is also assumed that the contact surface S_c is time-independent. Because of the slip motion, the contact surface is varying with the relative displacement in most examples. Since we are interested principally in the shakedown behaviours i.e. in limited displacements only, such an assumption may be considered as meaningful.

Finally, the governing equations of the system are, for all $t \geqslant 0$, in a dynamic transformation:

$$\begin{cases} \nabla \cdot \sigma + f = \rho \ddot{u} \\ \sigma = \sigma^I + L : (\nabla u - \alpha \theta \delta) \quad \text{in } V = V_1 \cup V_2 \\ \sigma \cdot n = F \quad \text{on } S^F, \qquad u = 0 \quad \text{on } S_1^u \\ [u] \cdot n = 0, \quad U = [u] \quad \text{on } S_c \\ T = \sigma \cdot n - (n \cdot \sigma \cdot n) n \\ T \in C, \quad (T - \hat{T}) \cdot \dot{U} \geqslant 0 \quad \forall \hat{T} \in C \\ + \text{Initial conditions in position and velocity} \end{cases} \tag{3}$$

where α denotes the thermal dilatation coefficients, $[u] = u_2 - u_1$ is the relative displacement on S_c and n denotes the outward normal vector to V_1 on S_c and to V on S^F.

If $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ denote two solutions of (3) starting from two different initial conditions, then

$$\int_V \big((\epsilon_2 - \epsilon_1) : L : (\dot{\epsilon}_2 - \dot{\epsilon}_1) + \rho (\ddot{u}_2 - \ddot{u}_1)(\dot{u}_2 - \dot{u}_1) \big) \, \mathrm{d}V = - \int_{S_c} (T_2 - T_1) \cdot (\dot{U}_2 - \dot{U}_1) \, \mathrm{d}S \leqslant 0$$

It follows that their distance in the sense of energy $\ell(t)$ defined by

$$\ell(t)^2 = \int_V (\epsilon_2 - \epsilon_1) : L : (\epsilon_2 - \epsilon_1) \, \mathrm{d}V + \int_V \rho (\dot{u}_2 - \dot{u}_1)^2 \, \mathrm{d}V \tag{4}$$

must be a decreasing function of time i.e. there is a contraction of the distance of two different solutions.

It is interesting to introduce some associated problems in view of the analysis. The following system of equations is considered to describe the loading data

$$\begin{cases} \nabla \cdot \sigma + f = \rho \ddot{u} \\ \sigma = \sigma^I + L : (\nabla u - \alpha \theta \delta) \quad \text{in } V \\ \sigma \cdot n = F \quad \text{on } S^F, \qquad u = 0 \quad \text{on } S_1^u \\ [u] = 0 \quad \text{on } S_c \end{cases} \tag{5}$$

in which the data are $\mathbf{f}(t)$, $\mathbf{F}(t)$, σ^I and the solution in displacement of this system is denoted as $\mathbf{u}^{e\ell}$, the thermo-elastic answer.

A second associated problem is defined by the system:

$$\begin{cases} \nabla \cdot \sigma = 0 \\ \sigma = L : \nabla \quad \text{in } V \\ \sigma \cdot n = 0 \quad \text{on } S^F, \qquad u = 0 \quad \text{on } S_1^u \\ [u] \cdot n = 0, \quad \sigma_t = T^* \quad \text{on } S_c \end{cases} \tag{6}$$

in which the data are the tangent force $\mathbf{T}^*$. The solution in displacement and stress is denoted as $\mathbf{u}^{T*}$, σ^{T*}.

It is also interesting to introduce some definitions. Let SF be the space of self-forces τ defined on S_c such that there exists at least a self-stress field σ^τ defined in V and admits τ as the associated tangent force on S_c

$$\begin{cases} \nabla \cdot \sigma^\tau = 0 \quad \text{in } V \\ \sigma^\tau \cdot n = 0 \quad \text{on } S^F \\ [\sigma^\tau] \cdot n = 0 \quad \text{on } S_c \\ \tau = \sigma^\tau \cdot n - (n \cdot \sigma^\tau \cdot n) n \quad \text{on } S_c \end{cases} \tag{7}$$

A relative rigid displacement of the solids is eventually possible. Let RS be the space of rigid slips i.e. of slip fields on S_c which are the trace of relative rigid displacements of the two solids. By definition, $G \in RS$ if there exists at least a displacement field $\mathbf{u}$ such that

$$\begin{cases} \epsilon(u) = 0 \quad \forall x \in V, \qquad u = 0 \quad \text{on } S_u^1 \\ [u] \cdot n = 0, \quad [u] = G \quad \text{on } S_c \end{cases} \tag{8}$$

In particular, the following orthogonality relation holds:

$$\begin{cases} \int_{S_c} T \cdot G \, \mathrm{d}S = 0, & \forall T \in SF, \ \forall G \in SR \\ \int_{S_c} T \cdot \hat{G} \, \mathrm{d}S = 0, & \forall \hat{G} \in RS \quad \Rightarrow \quad T \in SF \\ \int_{S_c} \hat{T} \cdot G \, \mathrm{d}S = 0, & \forall \hat{T} \in SF \quad \Rightarrow \quad G \in RS \end{cases} \tag{9}$$

2.2. A slip-shakedown theorem

In a similar spirit as Melan's theorem in Plasticity, the following theorem holds:

Theorem 1. *If there exists a time-independent tangent force field* $\mathbf{T}^* \in SF$ *and a coefficient* $m > 1$ *such that the tangent force* $\tilde{\mathbf{T}}(t) = m(\mathbf{T}^* + \mathbf{T}^{e\ell}(t))$ *satisfies for all* $t \geqslant 0$ *and for all* $x \in S_c$ *the condition* $\tilde{T}(x,t) \in C(x,t)$*, then there is necessarily a shakedown of the tangent slips whatever the initial conditions.*

The proof follows as in Plasticity. Since $(\mathbf{u}(t), \sigma(t))$ and $(\mathbf{u}^*(t), \sigma^*(t))$, with $\mathbf{u}^* = \mathbf{u}^{T*} + \mathbf{u}^{e\ell}$ and $\sigma^* = \sigma^{T*} + \sigma^{e\ell}$, are two solutions of (3) starting from two different initial conditions, the following expression holds:

$$\int_{S_c} (T - T^*) \cdot \dot{U} \, \mathrm{d}S + \int_V \big((\sigma - \sigma^*) : L^{-1} : (\dot{\sigma} - \dot{\sigma}^*) + \rho(\ddot{u} - \ddot{u}^*)(\dot{u} - \dot{u}^*)\big) \, \mathrm{d}V = 0 \tag{10}$$

On the other hand, from the assumption concerning the field $\tilde{\mathbf{T}}$, the following relation holds

$$(T - T^*) \cdot \dot{U} \geqslant \frac{m-1}{m} T \cdot \dot{U} \tag{11}$$

and leads to

$$\frac{m-1}{m} \int_{S_c} T \cdot \dot{U} \, \mathrm{d}S \leqslant - \int_V \big((\sigma - \sigma^*) : L^{-1} : (\dot{\sigma} - \dot{\sigma}^*) + \rho(\ddot{u} - \ddot{u}^*)(\dot{u} - \dot{u}^*)\big) \, \mathrm{d}V$$

Thus, for all time $\tau > 0$

$$\frac{m-1}{m} \int_o^\tau \int_{S_c} T \cdot \dot{U} \, \mathrm{d}S \, \mathrm{d}t \leqslant \int_V \frac{1}{2}\Big(\big(\sigma(x,0) - \sigma^*(x,0)\big) : L^{-1} : \big(\sigma(x,0) - \sigma^*(x,0)\big) + \rho\big(\dot{u}(x,0) - \dot{u}^*(x,0)\big)^2\Big) \, \mathrm{d}V \tag{12}$$

It is concluded that the dissipation work $W^d(\tau)$ remains bounded

$$W^d(\tau) = \int_o^\tau \int_{S_c} T \cdot \dot{U} \, \mathrm{d}S \, \mathrm{d}t < M \quad \forall \tau \tag{13}$$

whatever be the initial conditions. Thus, taking account of (2), it follows that

$$\int_o^\tau \int_{S_c} \|\dot{U}\| \, \mathrm{d}S \, \mathrm{d}t < C \quad \forall \tau \tag{14}$$

It is concluded that, cf. [8], the solution $\mathbf{U}(t)$ must converge to a time-independent field at large time i.e. there is necessarily a shakedown of the slips.

3. A safety coefficient for slip-shakedown

As in the theory of plasticity [8], a safety coefficient for the shakedown of the relative displacement can be now introduced. By definition, the safety coefficient m_s is the maximum of m ensuring the assumptions of Theorem 1:

Definition 1. The static expression of the safety coefficient is:

$$\begin{cases} m_s = \max_{\ T^* \in SF}\ m \\ \text{such that } \tilde{\mathbf{T}}(t) = m(\mathbf{T}^* + \mathbf{T}^{e\ell}(t)) \\ \text{satisfies for all } t \geqslant 0 \text{ and for all } x \in S_c \\ \text{the condition } \tilde{T}(x,t) \in C(x,t) \end{cases} \tag{15}$$

It is clear that if $m_s > 1$, then from Theorem 1, there is a shakedown of the slip motion.

3.1. Static approach and min–max duality

The definition (15) leads to the resolution of a problem of optimization under convex constraints. As in plastic shakedown theory [6], an associated Lagrangian Λ can be introduced:

$$\begin{aligned} \Lambda(m, \mathbf{T}^*, \tilde{\mathbf{T}}, \mathbf{g}) &= m + \int_o^{+\infty}\int_{S_c} \left(\tilde{T}(x,t) - m\left(T^*(x) + T^{e\ell}(x,t)\right)\right) \cdot g(x,t)\,\mathrm{d}S\,\mathrm{d}t \\ &= m\left(1 - \int_S T^*(x) \cdot G(x)\,\mathrm{d}S - \int_o^{+\infty}\int_S T^{e\ell}(x,t) \cdot g(x,t)\,\mathrm{d}S\,\mathrm{d}t\right) \\ &\quad + \int_o^{+\infty}\int_S \tilde{T}(x,t) \cdot g(x,t)\,\mathrm{d}S\,\mathrm{d}t \end{aligned} \tag{16}$$

where $\mathbf{g}(x,t)$ is a slip rate defined for $x \in S_c,\ t \in [0, +\infty[$ and represents the Lagrange multipliers associated with the constraints $\tilde{T}(x,t) = m(T^*(x) + T^{e\ell}(x,t))$.

It is clear that

$$m_s = \max_{m, \mathbf{T}^*, \tilde{\mathbf{T}}} \min_{\mathbf{g}} \Lambda(m, \mathbf{T}^*, \tilde{\mathbf{T}}, \mathbf{g}) \tag{17}$$

under the constraints $\mathbf{T}^* \in SF$ and $\tilde{T}(x,t) \in C(x,t)$ for all $t \geqslant 0$ and for all $x \in S_c$.

3.2. Kinematic approach

It is also well known that the resolution of the dual problem:

$$m_k = \min_{\mathbf{g}} \max_{m, \mathbf{T}^*, \tilde{\mathbf{T}}} \Lambda(m, \mathbf{T}^*, \tilde{\mathbf{T}}, \mathbf{g}) \tag{18}$$

under the same constraints leads to the definition of an expression of the safety coefficient by the kinematic approach. Indeed, it is clear that $m_k \geqslant m_s$ and the equality $m_k = m_s$ can be expected as usual in min–max duality.

The first operation, which is the maximization of Λ with respect to the variables m, $\mathbf{T}^* \in SF$, $\tilde{\mathbf{T}}$ leads to the following results:

$$\begin{cases} \max_{m, \mathbf{T}^*, \tilde{\mathbf{T}}} \Lambda(m, \mathbf{T}^*, \tilde{\mathbf{T}}, \mathbf{g}) = \int_o^{+\infty} \int_{S_c} D(g(x,t))\,\mathrm{d}S\,\mathrm{d}t \\ \text{if } \int_o^{+\infty}\int_{S_c} T^{e\ell}(x,t) \cdot g(x,t)\,\mathrm{d}S\,\mathrm{d}t + \int_{S_c} T^*(x) G(x)\,\mathrm{d}S = 1 \\ \text{and if } \int_{S_c} \delta T^*(x) G(x)\,\mathrm{d}S = 0 \quad \forall \delta T^* \in SF \\ \text{otherwise } \max_{m, \mathbf{T}^*, \tilde{\mathbf{T}}} \Lambda(m, \mathbf{T}^*, \tilde{\mathbf{T}}, \mathbf{g}) = +\infty \\ \text{with } G(x) = \int_0^{+\infty} g(x,t)\,\mathrm{d}t \end{cases} \tag{19}$$

From (9), the last condition in (19) gives in particular $\mathbf{G} \in RS$. The following statement results:

Theorem 2. *The kinematic expression of the safety coefficient satisfies*

$$\begin{cases} m_k = \min_{\mathbf{g}} \int_o^{+\infty} \int_{S_c} D(g(x,t))\,\mathrm{d}S\,\mathrm{d}t \\ \textit{among the slip rates } g(x,t) \textit{ such that} \\ \mathbf{G} = \int_0^{+\infty} \mathbf{g}\,\mathrm{d}t \in RS \textit{ and that } \int_o^{+\infty} \int_{S_c} T^{e\ell}(x,t) \cdot g(x,t)\,\mathrm{d}S\,\mathrm{d}t = 1 \end{cases} \tag{20}$$

As in plasticity, the kinematic approach shows clearly two mechanisms of non-shakedown. An alternating slip mechanism corresponds to a resulting slip $\mathbf{G} = 0$ while a cumulative relative displacement admits a resulting slip $\mathbf{G} \neq \mathbf{0}$.

4. Non-associated laws and Coulomb friction

As in Plasticity, the derived results are available only for associated laws of friction. Thus, it cannot be applied to Coulomb friction in full generality. Some discussions have been given recently in the literature by Klarbring et al. [9] under the assumption of total un-coupling between the normal and tangent displacements (for example in the problem of contact with friction of a punch on a half-plane). In this case, the un-coupling property leads again to an associated law of the form (2) and thus Melan theorem holds.

Since general theoretical results are still lacking for the shakedown analysis of non-associated laws in Plasticity as in Frictional Contact, it may be interesting to consider the numerical simulations, cf. [1,2,10]. The phenomenon of shakedown or non-shakedown can be observed from the numerical response at large time in a step-by-step simulation along the loading path. It has been pointed out in several examples that the rate of convergence is rather quick compared to the rate of plastic shakedown. The asymptotic behaviour is obtained within a very small number of cycles.

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