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Topology optimization in damage governed low cycle fatigue

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Abstract

Topology optimization is applied here to discuss an optimization problem of fatigue resistance. Fatigue lifetime is maximized by optimizing the shape of a structure in cyclic plasticity combined with Lemaitre damage law. The topology optimization algorithm is detailed. A 3D numerical example is given. ***To cite this article: B. Desmorat, R. Desmorat, C. R. Mecanique 336 (2008).***

Résumé

Optimisation topologique en fatigue oligo-cyclique. L'optimisation topologique est utilisée pour traiter un problème d'optimisation de résistance en fatigue. La durée de vie en fatigue est maximisée en optimisant la forme d'une structure en plasticité cyclique combinée à la loi d'endommagement de Lemaitre. L'algorithme d'optimisation est détaillé. Un exemple numérique 3D est présenté. ***Pour citer cet article : B. Desmorat, R. Desmorat, C. R. Mecanique 336 (2008).***

Keywords: Fatigue; Topology optimization; Low cycle fatigue

Mots-clés : Fatigue ; Optimisation topologique ; Fatigue oligocyclique

1. Introduction

Topology optimization is often restricted to elastic behavior and monotonic loading. Classical algorithms are now efficient to handle the complex 3D cases [1]. In order to maximize the global rigidity of an elastic structure, one possible formulation is the minimization of the complementary elastic energy at both the local and global scales (alternate directions algorithm [2]). The local minimization is an important feature to keep in fatigue as, if a damage law with an energetic basis is used, it can lead to the minimization of local plasticity and damage. Ensuring this local minimization over a cycle in fatigue leads to the maximization of the lifetime. These reasons make us consider and extend the algorithm of alternate directions to low cycle fatigue in three steps: define a cyclic elasto-plasticity law that

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derives from a state potential, relate this potential to fatigue life and define a fatigue life maximization problem (on which a generalized alternate directions algorithm is used). The design problem is then stated as follows: determine the lightest structure that can achieve a given lifetime.

2. Cyclic elasto-plasticity law

Let us consider an isotropic and homogeneous material in the framework of generalized standard materials under the assumption of small strains and displacements. The behavior is elasto-plastic with linear kinematic hardening, so that the initial set of constitutive equations may be integrated in cyclic plasticity where the assumptions of proportional loading and of a symmetric stabilized cycle have been made (see for instance [3]).

The stress amplitude $\underline{\Delta\sigma}$ and the total strain amplitude $\underline{\Delta\epsilon}$ are then related by the nonlinear relation $\underline{\Delta\epsilon} = \frac{\partial\psi^{\text{cyclic}}}{\partial\underline{\Delta\sigma}}$ with

$$\psi^{\text{cyclic}}(\underline{\Delta\sigma}) = \frac{1}{2}\left[\frac{\Delta\sigma_{\text{eq}}^2 R_\nu}{E} + \frac{\langle\Delta\sigma_{\text{eq}} - 2\sigma_Y\rangle_+^2}{C}\right] \tag{1}$$

with $\Delta\sigma_{\text{eq}} = \sqrt{\frac{3}{2}\Delta\sigma_{ij}^D\Delta\sigma_{ij}^D}$, $\Delta\sigma_H = \frac{1}{3}\Delta\sigma_{kk}$, $R_\nu = \frac{2}{3}(1+\nu) + 3(1-2\nu)[\frac{\Delta\sigma_H}{\Delta\sigma_{\text{eq}}}]^2$ (triaxiality function), where E and ν are the elasticity parameters, C is the plastic modulus and $(.)^D$ denotes the deviatoric part of a tensor.

The stress-strain cyclic law reads:

$$\Delta\epsilon_{ij} = \frac{1+\nu}{E}\Delta\sigma_{ij} - \frac{\nu}{E}\Delta\sigma_{kk}\delta_{ij} + \frac{3}{2C}\frac{\Delta\sigma_{ij}^D}{\Delta\sigma_{\text{eq}}}\langle\Delta\sigma_{\text{eq}} - 2\sigma_Y\rangle_+ \tag{2}$$

This cyclic elasto-plasticity law can be interpreted as a nonlinear elasticity law relating the amplitude of stress $\underline{\Delta\sigma}$ (during one cycle) to the amplitude of total strain $\underline{\Delta\epsilon}$. In the following sections, we will use the notation $\psi^{\text{cyclic}}(\underline{\Delta\sigma}) = \psi_e^{\text{cyclic}}(\underline{\Delta\sigma}) + \psi_p^{\text{cyclic}}(\underline{\Delta\sigma})$ with (e and p standing for elastic and plastic contributions):

$$\psi_e^{\text{cyclic}}(\underline{\Delta\sigma}) = \frac{\Delta\sigma_{\text{eq}}^2 R_\nu}{2E}, \qquad \psi_p^{\text{cyclic}}(\underline{\Delta\sigma}) = \frac{\langle\Delta\sigma_{\text{eq}} - 2\sigma_Y\rangle_+^2}{2C} \tag{3}$$

3. Damage governed low cycle fatigue

Damage is next classically described by the state variable D which models at mesoscale a loss of resisting area due to micro-cracks or micro-cavities ($0 \leqslant D < 1$) [4]. Failure of the representative volume element occurs when D reaches its critical value D_c (of the order of magnitude of 0.2–0.3 for metals). Isotropic damage is coupled to elasticity and plasticity by means of the effective stress tensor $\tilde{\sigma}_{ij} = \frac{\sigma_{ij}}{1-D}$, i.e. the stress tensor σ_{ij} is replaced by $\tilde{\sigma}_{ij}$ in the elasticity law as well as in the yield criterion. Lemaitre's damage evolution law is used:

$$\dot{D} = \left(\frac{Y}{S}\right)^s \dot{p} \quad \text{with } Y = \frac{\sigma_{\text{eq}}^2 R_\nu}{2E} \tag{4}$$

in which Y is the strain energy density release rate, S, s and D_c are material parameters, and σ_{eq} is the von Mises stress.

In order to evaluate the number of cycles to crack initiation, a reliable damage increment over one cycle $\frac{\delta D}{\delta N} = \int_{1\text{ cycle}} \dot{D}\,\mathrm{d}t$ can be determined with the assumption of uncoupled elasto-plasticity and damage (i.e. of damage estimated by post-processing an elasto-plastic computation) [5].

Assuming that the strain energy release rate does not vary much between the applied load inducing the reach of the yield stress and the minimum and maximum loads inducing the maximum von Mises stress, the damage increment over one cycle $\frac{\delta D}{\delta N}$ reads (for a symmetric periodic loading):

$$\begin{cases} \dfrac{\delta D}{\delta N} = \left(\dfrac{\Delta\sigma_{\text{eq}}^2 R_\nu}{8ES}\right)^s \dfrac{\delta p}{\delta N} & \text{if } \Delta\sigma_{\text{eq}} > 2\sigma_Y \\ \dfrac{\delta D}{\delta N} = 0 & \text{otherwise} \end{cases} \tag{5}$$

in which $\frac{\delta p}{\delta N} = \int_{1\ \text{cycle}} \dot{p}\,\mathrm{d}t$ is the increase of accumulated plastic strain during one cycle, R_ν is the triaxiality function and where the yield stress σ_Y acts as the asymptotic fatigue limit in Wöhler diagram. For cyclic loading, the number of cycles to crack initiation N_R is obtained for $D = D_c$ after integration over the number of cycles. Considering Eq. (3) and with $\frac{\delta p}{\delta N} = \frac{2^{3/2}}{C^{1/2}}(\psi_p^{\text{cyclic}})^{1/2}$, one obtains:

$$N_R = \frac{D_c}{(\frac{\delta D}{\delta N})} = \frac{A}{(\psi_e^{\text{cyclic}}(\underline{\Delta\sigma}))^s(\psi_p^{\text{cyclic}}(\underline{\Delta\sigma}))^{1/2}} \quad \text{with } A = 2^{2s-3/2}\sqrt{C}\,S^s D_c \tag{6}$$

4. Fatigue life maximization problem

4.1. Cyclic elasto-plasticity problem

Consider a 3D medium Ω. The external boundary is split into two surfaces: Γ_0 and Γ_1. On Γ_0 is imposed a zero displacement, on Γ_1 a periodic surface load $F(t)$ which vary symmetrically with an amplitude ΔF. The cyclic elasto-plasticity problem is defined as a nonlinear elasticity problem (P):

$$\text{(P)} \quad \begin{cases} \Delta\sigma_{ij,j} = 0 \quad \text{in } \Omega, \quad \Delta\epsilon_{ij} = \frac{\partial\psi^{\text{cyclic}}(\underline{\Delta\sigma})}{\partial\Delta\sigma_{ij}}, \quad \Delta\epsilon_{ij} = \frac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i}) \\ \Delta\sigma_{ij}n_j = \Delta F_i \quad \text{on } \Gamma_1, \quad \Delta u_i = 0 \quad \text{on } \Gamma_0 \end{cases} \tag{7}$$

This problem is put into a variational form, in which $E_c = \int_\Omega \psi^{\text{cyclic}}(\underline{\Delta\sigma})\,\mathrm{d}V$ is the complementary energy.

4.2. Optimization problem

In order to increase the number of cycles to crack initiation in the structure, consider the following minimization problem, with respect to material and void distribution:

$$\min\left[\int_\Omega \psi_e^{\text{cyclic}}(\underline{\Delta\sigma})\,\mathrm{d}V + \int_\Omega \psi_p^{\text{cyclic}}(\underline{\Delta\sigma})\,\mathrm{d}V\right]$$

This optimization problem is the classical compliance optimization in the case of linear elasticity (i.e. if $\psi_p^{\text{cyclic}}(\underline{\Delta\sigma}) = 0$ everywhere in the structure) which corresponds to the maximization of the global rigidity of an elastic structure [6,1]. In its general form, this optimization problem is similar to the problem of minimizing the complementary energy of a structure made of nonlinear elastic material [7,8]. This optimization problem has to be formulated in an integral form (and not in a local form) because the distribution of material is not a priori known, and then the location of the failure point in which damage will reach its critical value is also not a priori known.

To solve this optimization problem (known to be ill-posed even in the linear elasticity setting), we will consider a distributed fictitious density β allowed to vary between $\beta_{\min} > 0$ and 1 ($\beta_{\min} \neq 0$ is set for numerical reasons). Introduce β in the state potential ψ_β as following:

$$\psi_\beta = \frac{1}{\beta^n}\psi^{\text{cyclic}} \tag{8}$$

This leads to the definition of an equivalent Young's modulus $E^* = \beta^n E$, an equivalent plastic modulus $C^* = \beta^n C$ and to an unchanged elastic limit. Introduce a cost term in the objective function in order to limit the total quantity of material in the optimal design. The optimization problem then reads:

$$\min_{\beta\in[\beta_{\min},1]}\left[\int_\Omega \psi_\beta(\underline{\Delta\sigma})\,\mathrm{d}V + \int_\Omega \text{cost}(\beta)\,\mathrm{d}V\right]$$

This problem is put into the form of a double minimization with the use of the variational formulation of the local problem (P):

$$\min_{\beta\in[\beta_{\min},1]}\ \min_{\underline{\Delta\tau}\in\Sigma_{ad}}\left[\int_\Omega \psi_\beta(\underline{\Delta\tau})\,\mathrm{d}V + \int_\Omega \text{cost}(\beta)\,\mathrm{d}V\right]$$

Remark 1. β has no specific physical meaning and *is not related* to a damaged configuration. We are looking forward to optimizing the distribution of material in a fixed domain and thus want to obtain values of β mostly close to 0 and 1 (black and white design), which explains the use of the power n ($n \approx 3$).

4.3. Optimization algorithm

The optimization procedure (initially introduced in the framework of linear elasticity in [2]) consists in:

- an initialization of iteration ($q = 0$): $\beta^{(0)} \Rightarrow \underline{\Delta\sigma}^{(0)}$,
- local minimizations $\min_{\beta\in[\beta_{\min},1]}(\frac{1}{\beta^n}\psi^{\text{cyclic}}(\underline{\Delta\sigma}^{(q)}) + \text{cost}(\beta))$ with respect to β with fixed stress $\underline{\Delta\sigma}^{(q)}$ which give $\beta^{(q+1)}$ for iteration $q + 1$,
- a global minimization with respect to $\underline{\Delta\sigma}$ with fixed optimization parameters $\beta^{(q+1)}$ which gives $\underline{\Delta\sigma}^{(q+1)}$ (finite element computation and use of complementary energy theorem),
- iterations on local and global minimizations ($q \leftarrow q + 1$).

After one set of local minimizations followed by a global minimization, the criterion decreases. The criterion being a positive quantity, the algorithm is convergent.

4.4. Local minimizations with fixed stress field

The cost functional is chosen proportional to the distributed fictitious density: $\text{cost}(\beta) = k\beta$, where the parameter k is a *constant* chosen by the user. The local minimization problem reads then:

$$\min_{\beta\in[\beta_{\min},1]}\left[\frac{B}{\beta^n} + k\beta\right] \quad \text{with the } \beta\text{-independent constant } B = \frac{1}{2}\left(\frac{\Delta\sigma_{\text{eq}}^2 R_\nu}{E} + \frac{\langle\Delta\sigma_{\text{eq}} - 2\sigma_Y\rangle_+^2}{C}\right)$$

Let $\beta_0 = (\frac{nB}{k})^{1/(n+1)}$. The principle of the minimization is identical to the classical Solid Isotropic Material with Penalization (SIMP) case:

$$\begin{cases} \beta = \beta_{\min} & \text{if } \beta_0 \leqslant \beta_{\min} \\ \beta = \beta_0 & \text{if } \beta_{\min} \leqslant \beta_0 \leqslant 1 \\ \beta = 1 & \text{if } 1 \leqslant \beta_0 \end{cases}$$

4.5. Fatigue life maximization problem

During the local minimizations steps (with fixed stress amplitude), the locally heavily loaded points in the structure have an optimal fictitious density equal to 1. Locally, the maximization of lifetime for a fixed stress amplitude (i.e. minimization of $(\psi_e^{\text{cyclic}})^s(\psi_p^{\text{cyclic}})^{1/2}$) is achieved for a fictitious density equal to 1. Thus, with the optimization procedure, the heavily loaded points where the crack initiation will occur have a fictitious density optimal with respect to fatigue life.

The design problem of finding the lightest structure that achieves a given lifetime is solved using the following procedure: determine the lightest structure that can achieve a given lifetime. The numerical procedure is: (i) choose an initial value for the cost parameter k, (ii) apply the optimization algorithm, (iii) if the number of cycles to rupture is lower than the targeted one, allow for more material in the design domain by decreasing the value of k and run the optimization procedure again (increase k otherwise). When the targeted number of cycles to crack initiation is obtained, the optimal distribution of material is given by the last optimization procedure.

5. Numerical example

Consider as an example a beam clamped on both lateral sides. A surface load is imposed on the upper surface at its center on 20% of the length of the bar (Fig. 1). The load varies symmetrically between 100 and −100 MPa. The material parameters are: Young's modulus $E = 200\,000$ MPa, Poisson's ratio $\nu = 0.3$, plastic modulus $C = 6000$ MPa, yield stress $\sigma_Y = 180$ MPa, damage parameters $S = 2.8$ MPa, $s = 2$ and $D_c = 0.2$ (those of a 2-1/4 CrMo steel [5]).

Table 1 shows for different optimized geometries the results of the proposed design procedure:

Table 1
Numerical results

Tableau 1
Résultats numériques

Cost parameter k	1.0	0.6	0.2
Material volume	48 %	52 %	69 %
Number of cycles to crack initiation	1580	5603	83 620
(Normalized) Optimized energy $\int_\Omega \psi_\beta(\underline{\Delta\sigma})\,\mathrm{d}x$	1.0	0.75	0.53

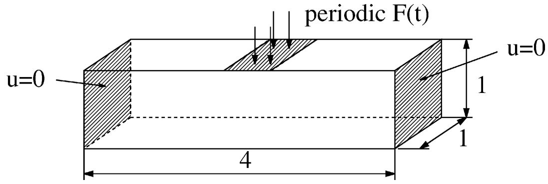

Fig. 1. Sketch of the 3D medium and boundary conditions.

Fig. 1. Description du milieu 3D et des conditions aux limites considérés.

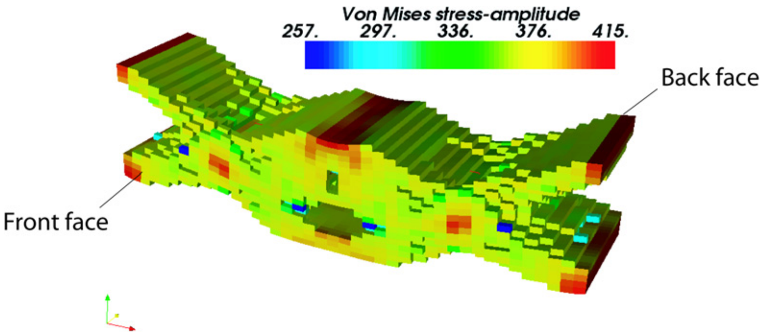

Fig. 2. Von Mises stress-amplitude (MPa) on the optimized deformed geometry at crack initiation (material volume equal to 48%).

Fig. 2. Amplitude de contrainte de von Mises (MPa) sur la géométrie déformée à initiation de fissure (pour un volume de matière de 48 %).

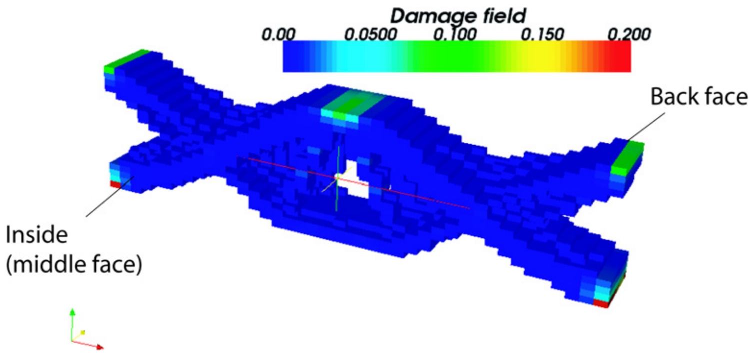

Fig. 3. Damage field on half of the optimized geometry at crack initiation for a material volume equal to 48% (middle cutting plane).

Fig. 3. Endommagement sur la moitié de la structure à initiation de fissure pour un volume de matière de 48 % (plan de coupe central).

– the increase of the number of cycles to rupture with an increasing total material volume,
– the increase of the total material volume with an decreasing cost parameter k,
– the increase of the energy term $\int_\Omega \psi_\beta(\underline{\Delta\sigma})\,\mathrm{d}x$ (normalized with respect to the case $k = 1$) of the optimized criterion with a decreasing total material volume.

Figs. 2 and 3 show the von Mises stress amplitude (more than 90% of the optimized structure is plastified with a maximum plastic strain amplitude of 1.3%) and the damage field on the optimized geometry for the number of cycles to crack initiation in the case of a material volume equal to 48%. The critical damage is obtained at the lower clamped parts of the structure.

6. Conclusion

Continuum Damage Mechanics can be used for the low cycle fatigue design of a structure by considering Lemaitre damage evolution law: damage is governed by the plasticity (through the accumulated plastic strain rate) and is enhanced by the stress level (through the elastic energy density). The damage increment per cycle in fatigue is then expressed as a function of the associated complementary energy density. Both the state potential and the local damage increment are minimized by the proposed topology optimization procedure. An optimized structure shape is gained in fatigue under symmetric cyclic loading, as illustrated in a three-dimensional cantilever beam.

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