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Influence of boundary layers over the rate of convergence in a penalization method for a 1-D wave equation

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Abstract

For domain penalization methods, there can be a gap between the expected speed of convergence and the observed one, by numerical means. Such a gap has been observed by Paccou, et al. [A. Paccou, G. Chiavassa, J. Liandrat, K. Schneider, A penalization method applied to the wave equation, C. R. Mecanique 333 (2005) 79–85], concerning the penalization of a wave equation. We answer here one of their questions by proving that the observed lack in convergence speed is clearly related to the formation of boundary layers on one side of the boundary. ***To cite this article: B. Fornet, C. R. Mecanique 336 (2008).***

Résumé

Influence des couches limites sur la vitesse de convergence dans une méthode de pénalisation. Pour les méthodes de pénalisation de domaine, il est possible qu'il y ait un écart entre la vitesse de convergence attendue et celle observée numériquement. Un tel écart a été mis en évidence par Paccou, et al. [A. Paccou, G. Chiavassa, J. Liandrat, K. Schneider, A penalization method applied to the wave equation, C. R. Mecanique 333 (2005) 79–85], lors de la pénalisation d'une équation des ondes. On répond ici à une question posée dans, en prouvant que le défaut de vitesse de convergence observé est clairement provoqué par la formation de couches limites, localisées d'un seul côté du bord. ***Pour citer cet article : B. Fornet, C. R. Mecanique 336 (2008).***

Keywords: Computational fluid mechanics; Domain penalization method; Boundary layers; Wave equation

Mots-clés : Mécanique des fluides numérique ; Méthode de pénalisation de domaine ; Couches limites ; Équation des ondes

1. Introduction

Penalization methods are frequently used in numerical simulation of fluid dynamics, when a boundary is involved, for example we can refer to [1] by Angot, Bruneau and Fabrie. Roughly speaking, the main idea of this kind of approach is to immerse the original domain into a geometrically bigger and simpler one called a fictitious domain.

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The main interest is that, for the obtained singularly perturbed problem, the discretization is not boundary-fitted to the original domain.

In [2], written in collaboration with Guès, in view of future applications, the authors give two results concerning the penalization of mixed semi-linear hyperbolic problems with dissipative boundary conditions. The quality of the two methods proposed in [2] are compared based on the boundary layers they generate. However, it was not clear whether the boundary layers forming were really detrimental from a numerical point of view.

The goal of this Note is then, taking as a basis the numerical study of the convergence made in [3], to show that the numerical rate of convergence, not as good as awaited, observed in [3] can be explained by the formation of *boundary layers*.

As in [3], we will investigate the quality of the approximation of the solution U of the 1-D wave equation (1) by a given method of penalization:

$$\begin{cases} \partial_{tt}U - c^2\partial_{xx}U = 0, \quad (x,t) \in (0,\pi) \times \mathbb{R}^+ \\ U|_{x=0} = U|_{x=\pi} = 0 \\ U|_{t=0}(x) = \sin(x) \\ \partial_t U|_{t=0} = 0 \end{cases} \tag{1}$$

As $\varepsilon \to 0^+$, we analyze the approximation of U by U^ε on $x \in (0,\pi)$, where $U^\varepsilon = U^{\varepsilon+}\mathbf{1}_{0<x<\pi} + U^{\varepsilon-}\mathbf{1}_{x<0}$ is defined as the solution of the following hyperbolic transmission problem:

$$\begin{cases} \partial_{tt}U^{\varepsilon+} - c^2\partial_{xx}U^{\varepsilon+} = 0, \quad (x,t) \in (0,\pi) \times \mathbb{R}^+ \\ \partial_{tt}U^{\varepsilon-} - c^2\partial_{xx}U^{\varepsilon-} + \frac{1}{\varepsilon^2}U^{\varepsilon-} = 0, \quad (x,t) \in]-\infty,0[\times \mathbb{R}^+ \\ U^{\varepsilon+}|_{x=0} - U^{\varepsilon-}|_{x=0} = 0 \\ \partial_x U^{\varepsilon+}|_{x=0} - \partial_x U^{\varepsilon-}|_{x=0} = 0 \\ U^{\varepsilon+}|_{x=\pi} = 0 \\ U^{\varepsilon\pm}|_{t=0}(x) = \sin(x) \\ \partial_t U^{\varepsilon\pm}|_{t=0} = 0 \end{cases} \tag{2}$$

We prove the following result, observed numerically in [3]:

Theorem 1.1. *For all* $0 < \varepsilon < 1$ *and* $T > 0$ *there holds*:

$$\|U^\varepsilon - U\|_{L^\infty(]0,T[:L^2((-\infty,\pi)))} = \mathcal{O}(\varepsilon)$$

The proof of this theorem incorporates an asymptotic analysis of the boundary layers forming, at any order.

2. Proof of Theorem 1.1

We will now construct formally an approximate solution $(U^{\varepsilon+}_{\text{app}}, U^{\varepsilon-}_{\text{app}})$ of the solution $(U^{\varepsilon+}, U^{\varepsilon-})$ of the transmission problem (2). We shall construct this approximate along the following ansatz:

$$U^{\varepsilon+}_{\text{app}} = \sum_{j=0}^{M} U^+_j(t,x)\varepsilon^j$$

$$U^{\varepsilon-}_{\text{app}} = \sum_{j=0}^{M} U^-_j\left(t,x,\frac{x}{\varepsilon}\right)\varepsilon^j$$

where the profiles $U^-_j(t,x,z) := \underline{U^-_j}(t,x) + U^{*-}_j(t,z)$, with

$$\lim_{z\to-\infty} \mathrm{e}^{-\alpha z}U^{*-}_j = 0$$

for some $\alpha > 0$. The layer profiles U^{*-}_j serve the purpose of describing quick fluctuations of the solution as $\varepsilon \to 0^+$.

We will only focus on the construction of

$$U^\varepsilon_{\text{app}} := U^{\varepsilon+}_{\text{app}}\mathbf{1}_{x\in(0,\pi)} + U^{\varepsilon-}_{\text{app}}\mathbf{1}_{x<0}$$

Plugging $U_{\text{app}}^{\varepsilon\pm}$ into problem (2) and identifying the terms with same power of ε, we obtain the following equation:

$$\underline{U_0^-} = 0$$

moreover, $U_0^{*-} = 0$ as it is solution of the problem:

$$\begin{cases} U_0^{*-} - c^2 \partial_{zz} U_0^{*-} = 0, \quad \{z < 0\} \\ \partial_z U_0^{*-}|_{z=0} = 0 \\ \lim_{z \to -\infty} U_0^{*-} = 0 \end{cases}$$

The function $U_{\text{app}}^{\varepsilon+}$ converges towards U_0^+ as $\varepsilon \to 0^+$. As awaited, U_0^+ is the solution of the well-posed 1-D wave equation:

$$\begin{cases} \partial_{tt} U_0^+ - c^2 \partial_{xx} U_0^+ = 0, \quad (x,t) \in (0,\pi) \times \mathbb{R}^+ \\ U_0^+|_{x=0} = \underline{U_0^-}|_{x=0} + U_0^{*-}|_{z=0} = 0 \\ U_0^+|_{x=\pi} = 0 \\ U_0^+|_{t=0}(x) = \sin(x) \\ \partial_t U_0^+|_{t=0} = 0 \end{cases}$$

Let us now proceed with the construction of the next profiles. First, remark that, not only $\underline{U_0^-} = 0$, but for all $j \geqslant 1$, there holds:

$$\underline{U_j^-} = 0$$

The profile U_1^{*-} satisfies the well-posed equation:

$$\begin{cases} U_1^{*-} - c^2 \partial_{zz} U_1^{*-} = 0 \quad \{z < 0\} \\ \partial_z U_1^{*-}|_{z=0} = \partial_x U_0^+|_{x=0} \\ \lim_{z \to -\infty} U_1^{*-} = 0 \end{cases}$$

as a result, we get that:

$$U_1^{*-} = c \partial_x U_0^+|_{x=0} \, \mathrm{e}^{\frac{z}{c}}$$

We will now prove, by induction, that the construction of the profiles can go on at any order, which means that for all $M \in \mathbb{N}$ fixed beforehand, we are able to construct $U_{\text{app}}^{\varepsilon}$ satisfying:

$$\begin{cases} \partial_{tt} U_{\text{app}}^{\varepsilon+} - c^2 \partial_{xx} U_{\text{app}}^{\varepsilon+} = \varepsilon^M R^{\varepsilon+} \quad (x,t) \in (0,\pi) \times \mathbb{R}^+ \\ \partial_{tt} U_{\text{app}}^{\varepsilon-} - c^2 \partial_{xx} U_{\text{app}}^{\varepsilon-} + \frac{1}{\varepsilon^2} U_{\text{app}}^{\varepsilon-} = \varepsilon^M R^{\varepsilon-}, \quad (x,t) \in]-\infty, 0[\times \mathbb{R}^+ \\ U_{\text{app}}^{\varepsilon+}|_{x=0} - U_{\text{app}}^{\varepsilon-}|_{x=0} = 0 \\ \partial_x U_{\text{app}}^{\varepsilon+}|_{x=0} - \partial_x U_{\text{app}}^{\varepsilon-}|_{x=0} = 0 \\ U_{\text{app}}^{\varepsilon+}|_{x=\pi} = 0 \\ U_{\text{app}}^{\varepsilon\pm}|_{t=0}(x) = \sin(x) \\ \partial_t U_{\text{app}}^{\varepsilon\pm}|_{t=0} = 0 \end{cases}$$

where $R^{\varepsilon+} \in L^2((0,\pi) \times \mathbb{R}^+)$ and $R^{\varepsilon-} \in L^2(]-\infty, 0[\times \mathbb{R}^+)$.

Let us assume that the boundary layers profiles have been computed up to order j. The profile U_j^+ is then defined as the unique solution of the following 1-D wave equation:

$$\begin{cases} \partial_{tt} U_j^+ - c^2 \partial_{xx} U_j^+ = 0, \quad (x,t) \in (0,\pi) \times \mathbb{R}^+ \\ U_j^+|_{x=0} = U_j^{*-}|_{z=0} \\ U_j^+|_{x=\pi} = 0 \\ U_j^+|_{t=0} = 0 \\ \partial_t U_0^+|_{t=0} = 0 \end{cases}$$

We can thus compute the profile U_{j+1}^{*-} since it is the unique solution the following well-posed equation:

$$\begin{cases} U_{j+1}^{*-} - c^2 \partial_{zz} U_{j+1}^{*-} = -\partial_{tt} U_{j-1}^{*-}, \quad \{z < 0\} \\ \partial_z U_{j+1}^{*-}|_{z=0} = \partial_x U_j^{+}|_{x=0} \\ \lim_{z \to -\infty} U_{j+1}^{*-} = 0 \end{cases}$$

Stability estimates in norm $L^\infty(]0, T[: L^2((-\infty, \pi)))$ can be obtained for the problem at hand by multiplication of the equation by $\partial_t U^\varepsilon$ and then integration by parts.

By linearity, these stability estimates can be applied to the error $W^\varepsilon := U_{\mathrm{app}}^\varepsilon - U^\varepsilon$.

Constructing the approximate solution at an order M large enough, we obtain that U^ε converges in $L^\infty(]0, T[: L^2((-\infty, \pi)))$ towards U, when $\varepsilon \to 0^+$ the same way as $U_{\mathrm{app}}^\varepsilon$. Finally, by construction of $U_{\mathrm{app}}^\varepsilon$, the convergence of U^ε towards U occurs as stated in Theorem 1.1.

3. Conclusion and perspectives

Let us answer the question asked in [3]: $U^{\varepsilon-}$ presents a boundary layer behavior in $\{x = 0^-\}$ since its approximate solution is composed *exclusively* of boundary layer profiles, which describes quick transitions at the boundary using a fast scale in ε. As a result of the loss in convergence induced by the boundary layers forming, we get the estimate stated in Theorem 1.1. In [3], the chosen small parameter is $\mu = \varepsilon^2$, hence, adopting the same notations as them, our estimate writes: $\|U^\mu - U\|_{L^\infty(]0,T[:\, L^2((-\infty,\pi)))} = \mathcal{O}(\sqrt{\mu})$, which is in agreement with the estimates given in [3]. As in the penalization approach proposed by Rauch in [5] and used by Bardos and Rauch in [4], as underlined by Droniou in [6], boundary layers form on one side of the boundary.

In order to sharpen penalization methods used in numerical applications, an interesting question would be, in the same line of mind as in [2], to see whether there is some alternative method of penalization preventing or minimizing the formation of boundary layers.

For the domain penalization method proposed in [3], the convergence of $U^\mu|_{x \in (0,\pi)}$ towards U is actually true in every Sobolev spaces but at the speed $\mathcal{O}(\sqrt{\mu})$. Due to the boundary layers forming, we do not get bounds establishing the convergence of $U^\mu|_{x \in (-\infty,\pi)}$ towards 0 for higher Sobolev norms, however the convergence in $L^\infty(]0, T[:\ L^2((-\infty, \pi)))$ occurs at the speed $\mathcal{O}(\mu^{3/4})$. For a penalization method not generating any boundary layers, the rate of convergence obtained theoretically would be in $\mathcal{O}(\mu)$, on each side of $\{x = 0\}$ and in every Sobolev norm.

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