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C. R. Mecanique 336 (2008) 493–499

COMPTES RENDUS MECANIQUE

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Effects of strong convection on the cooling process for a long or thin pipe

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Received 1 April 2007; accepted 11 March 2008

Available online 28 April 2008

Presented by Évariste Sanchez-Palencia

Abstract

In this Note a heat flow through a thin pipe filled with fluid is studied. The pipe is cooled by the exterior medium. Depending on the ratio between the pipe's thickness ε and the Reynolds number Re^{ε}, we obtain three different macroscopic models via rigorous asymptotic analysis. For small Re^{ε} the fluid in the pipe is perfectly cooled, i.e. it assumes the temperature of the surrounding medium. For large Re^{ε}, the fluid is not cooled at all, i.e. it maintains the same temperature as it had when it entered the pipe. Between those two cases there is a critical value of Re^{ε} when the macroscopic model is described by an ODE keeping the effects of the surrounding medium as well as the entering temperature. ***To cite this article: S. Marušić et al., C. R. Mecanique 336 (2008).***

Résumé

Effets d'une convection forte sur le processus de refroidissement d'un tube long ou mince. Dans cette Note, nous étudions la conduction thermique dans un tube mince rempli par un fluide. Le tube est refroidi par le milieu extérieur. Suivant le rapport entre l'épaisseur du tube ε et le nombre de Reynolds Re^{ε}, on obtient, via une analyse asymptotique rigoureuse, trois modèles différents. Pour Re^{ε} assez petit, le fluide est parfaitement refroidi, c'est-à-dire qu'il prend la température du milieu extérieur au tube. Pour Re^{ε} grand, le fluide n'est pas du tout refroidi, c'est-à-dire qu'il reste à la température qu'il avait à l'entrée du tube. Entre ce deux cas existe un cas critique où le modèle macroscopique est donné par une EDO où cohabitent les effets du milieu extérieur au tube ainsi que ceux de la température à l'entrée de celui-ci. ***Pour citer cet article : S. Marušić et al., C. R. Mecanique 336 (2008).***

Keywords: Heat transfer; Heat conduction; Asymptotic analysis; High convection

Mots-clés : Transferts thermiques ; Conduction thermique ; Analyse asymptotique ; Convection forte

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1631-0721/$ – see front matter
doi:10.1016/j.crme.2008.03.003

Version française abrégée

On considère la conduction thermique dans un tube de longueur L et d'épaisseur d. Le paramètre géométrique $\varepsilon = d/L$ est supposé petit, c'est-à-dire que le tube est soit mince soit long. Le processus est gouverné par une équation de convection–diffusion linéaire. Le tube est entouré par un milieu ayant une température donnée G. Le fluide dans le tube est refroidi (ou chauffé) par le milieu environnant et le processus est régi par une loi linéaire de type Newton. Par soucis de simplicité, on suppose que la partie hydrodynamique est connue et que la vitesse est unidirectionnelle (par exemple, du type de celle d'un écoulement de Poiseuille). Notre but est de trouver une loi macroscopique gouvernant le processus par une analyse asymptotique rigoureuse par rapport au paramètre ε. Pour cela, on écrit d'abord l'équation sous une forme adimensionnelle :

$$\frac{\partial \varphi_\varepsilon}{\partial t} - \Delta\varphi_\varepsilon + Re^\varepsilon Pr\, w^\varepsilon \frac{\partial \varphi_\varepsilon}{\partial x_1} = 0 \quad \text{dans } \Omega_\varepsilon^T = \Omega_\varepsilon \times]0, T[, \qquad \varphi_\varepsilon = h_k \quad \text{pour } x_1 = k \text{ et } k = 0, 1 \tag{1}$$

$$\frac{\partial \varphi_\varepsilon}{\partial \mathbf{n}} = Nu(G - \varphi_\varepsilon) \quad \text{sur } \Gamma_\varepsilon^T = \Gamma_\varepsilon \times]0, T[, \qquad \varphi_\varepsilon(x, 0) = \varphi_0(x), \quad x \in \Omega_\varepsilon \tag{2}$$

où Re^ε est le nombre de Reynolds, Pr est le nombre de Prandtl, Nu est le nombre de Nusselt et $\Omega_\varepsilon =]0, 1[\times \varepsilon S$, $S \subset \mathbf{R}^2$ est un domaine borné, et $\Gamma_\varepsilon =]0, 1[\times \varepsilon\partial S$. Les conditions précises imposées sur G et h_k sont décrites dans la version anglaise. Suivant le rapport entre l'épaisseur du tube ε et le nombre de Reynolds Re^ε, nous trouvons les trois cas suivants :

- $\varepsilon Re^\varepsilon \ll 1$ – la température extérieure G domine le processus. A la limite, la température du fluide dans le tube est égale à G. Les effets de la température h_0 du fluide entrant dans le tube ne sont présents que dans une couche limite au voisinage de l'entrée du tube.
- $\varepsilon Re^\varepsilon = \mathrm{O}(1)$ – C'est le cas critique où les effets de la température environnante et ceux de la vitesse de convection sont du même ordre et ils sont tous les deux préservés à la limite. On obtient une EDO du premier ordre gouvernant le processus macroscopique. Seule la condition aux limites entrante, c'est-à-dire en $x = 0$ est conservée alors que la condition sortante, en $x_1 = 1$ disparait. Ce qui a physiquement un sens, car la température du fluide entrant le tube peut être imposée, mais la température du fluide sortante est déterminée par le processus (même si dans le problème microscopique (1) il a fallu imposer des conditions des deux côtés pour que le problème soit bien posé). Le problème macroscopique possède une solution explicite donnée par (22).
- $\varepsilon Re^\varepsilon \gg 1$ – la convection est dominante. Les effets de la température extérieure G sont négligeables et la température limite est égale à h_0. En bref, le fluide s'écoule trop vite pour pouvoir être refroidi (ou réchauffé).

Pour mener notre analyse, on dilate le domaine Ω_ε afin de définir $\Omega =]0, 1[\times S$, $\Gamma =]0, 1[\times \partial S$. Pour $(x_1, y) \in \Omega$, on définit $\phi^\varepsilon(x_1, y, t) = \varphi_\varepsilon(x_1, \varepsilon y, t)$. Pour un tel ϕ^ε, on obtient alors les équations (5), (6). On a le théorème suivant :

Théorème 0.1. *Soit ϕ_ε une solution de* (5), (6).

(a) *Si* $\lim_{\varepsilon\to 0} \varepsilon Re^\varepsilon = 0$ *alors $\phi_\varepsilon \rightharpoonup G$ faible dans $L^2(\Omega^T)$.*
(b) *Si* $\lim_{\varepsilon\to 0} \varepsilon Re^\varepsilon = Re^0 > 0$ *alors $\phi_\varepsilon \to \phi_0$ dans $L^2(\Omega^T)$, où ϕ_0 est la solution unique du problème* (11).
(c) *Si* $\lim_{\varepsilon\to 0} \varepsilon Re^\varepsilon = +\infty$ *alors $\varphi_\varepsilon \to h_0$. La convergence est forte dans $L^2(\Omega^T)$ si* $\lim_{\varepsilon\to 0} \varepsilon^2 Re^\varepsilon = 0$ *et forte dans $L^2_{\sqrt{w}}(\Omega^T)$ sinon. Par $L^2_{\sqrt{w}}$ on dénote l'espace des fonctions L^2 par rapport à la mesure avec un poids* $\sqrt{w(y,t)}\,\mathrm{d}x_1\,\mathrm{d}y\,\mathrm{d}t$*, où $w = |\mathbf{v}|$ est la norme de la vitesse de convection.*

La méthode classique pour étudier ce genre de problèmes (voir par exemple [5] ou [3]) utilise des développements asymptotiques raccordés. Nous évitons cette approche lourde en faisant une analyse d'erreur directe. Pour cela, nous profitons du non-changement du signe de la vitesse de convection. On prouve en même temps que le passage entre ces trois modèles est continu.

1. Introduction

We consider the heat conduction through a thin pipe with length L and cross-section diameter d. The ratio between geometric parameters of the pipe $\varepsilon = d/L$ is supposed to be small, i.e. the pipe is either long or thin. The process is described by a linear convection–diffusion equation. The pipe is surrounded by some medium with given temperature G. The fluid in the pipe is cooled (or heated) by the surrounding medium and the process is described simply by the Newton's cooling law. We study only the thermic part of the system assuming that the hydrodynamic part is known. As shown in [1], in case of liquid, it is reasonable to take that the fluid velocity has the Poiseuille form (i.e. it is unidirectional with a parabolic profile) since the buoyancy forces resulting from the thermic expansion of the fluid are negligible compared to the main Poiseuille flow. Our goal is to find the macroscopic law describing the behavior of the fluid, via rigorous asymptotic analysis with respect to the parameter ε.

First of all we want to work with an adimensionalised equation and with appropriate space and time scales that capture the most important phenomena governing the process. We denote by $\bar{x} \in \mathbf{R}^3$ the space variable and τ the (physical) time. Next we introduce the new (rescaled) variable $x = \frac{\bar{x}}{L}$. Now our, rescaled, domain can be described as $\Omega_\varepsilon =]0,1[\times \varepsilon S$, $S \subset \mathbf{R}^2$. The pipe is filled by fluid. Let μ be the viscosity and ρ the density of the fluid. Let κ be it's thermal conductivity and c_p the specific heat capacity at constant pressure. Finally, V_ε is the characteristic velocity of the fluid. We denote by $Re^\varepsilon = \mu^{-1} V_\varepsilon \rho L$ the Reynolds number and by $Pr = \kappa^{-1}\mu c_p$, the Prandtl number. It is convenient to introduce the characteristic time of the process as $\overline{T} = L^2 c_p \rho \kappa^{-1}$ and a new, rescaled, time $t = \tau/\overline{T}$. Since the pipe is submersed in an ambient with a temperature different from the fluid temperature, we impose the Robin's boundary condition on the side of the pipe resulting from Newton's cooling law. By β we denote the heat transfer coefficient and by $Nu = \beta L \kappa^{-1}$ the Nusselt number. Assuming that the flow through the pipe has a given Poiseuille profile and denoting its adimensionalised velocity $\mathbf{v}^\varepsilon = w(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}, t)\mathbf{e}_1$, the problem in non-dimensional form reads

$$\frac{\partial \varphi_\varepsilon}{\partial t} - \Delta\varphi_\varepsilon + Re^\varepsilon\, Pr\, w^\varepsilon \frac{\partial \varphi_\varepsilon}{\partial x_1} = 0 \quad \text{in } \Omega_\varepsilon^T = \Omega_\varepsilon \times]0,T[, \qquad \varphi_\varepsilon = h_k \quad \text{for } x_1 = k \text{ and } k = 0,1 \tag{3}$$

$$\frac{\partial \varphi_\varepsilon}{\partial \mathbf{n}} = Nu\,(G - \varphi_\varepsilon) \quad \text{on } \Gamma_\varepsilon^T = \Gamma_\varepsilon \times]0,T[, \qquad \varphi_\varepsilon(x,0) = \varphi_0(x), \quad x \in \Omega_\varepsilon \tag{4}$$

with $w^\varepsilon(x_2,x_3,t) = w(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}, t)$. Here $\Gamma_\varepsilon =]0,1[\times \varepsilon\partial S$. Due to the small thickness of the domain it is reasonable to suppose that $h_k = h_k(t)$, i.e. that the boundary temperatures are constant on the pipe's cross-sections. In addition we assume that h_k are of class C^1 on $[0,T]$. Again, due to the domain thickness, we can assume that $G = G(x_1,t) \in C^1([0,1]\times[0,T])$. As for the Poiseuille velocity, we assume that $w(\cdot/\varepsilon,\cdot) \in C^1([0,T]; C^2(\overline{\Omega_\varepsilon}))$, $w \geqslant 0$. Furthermore, we suppose that the mean velocity on the cross-section of the pipe is strictly positive, in the sense that there exists constant $w_0 > 0$ such that $\langle w\rangle \geqslant w_0 > 0$. The initial temperature φ_0 is assumed to be in $L^2(\Omega_\varepsilon)$, independent on ε.

Several papers where heat conduction (stationary or not) through a thin domain was considered, with different boundary conditions, can be found. Let us just mention a few of them. A similar problem was studied by A. Sili in [2], but with no convection. In fact, Sili studies heat conduction through an elastic bar instead of a pipe filled by a fluid, so in his case convection does not take place. His main goal is to find the effects of the heat transfer coefficient (assumed to depend on ε) on the macroscopic law governing the process. Contrary to his work, we keep the heat transfer coefficient constant (independent on ε) and concentrate mainly on effects caused by convection. The asymptotic expansion and boundary layer effects for a similar problem were studied in [3]. In that paper the convection term is taken into account, but the Reynolds number is assumed to be low, so that convection effects are not present in the macroscopic model. A method for modelling conduction through a network of pipes, as well as some other techniques of asymptotic analysis, are described in the book [4]. Classical method of matched expansions can be learned from [5].

2. Rescaling and a priori estimates

For our analysis we rescale the domain Ω_ε to define $\Omega =]0,1[\times S$, $\Gamma =]0,1[\times\partial S$. For $(x_1,y) \in \Omega$ we put $\phi^\varepsilon(x_1,y,t) = \varphi_\varepsilon(x_1,\varepsilon y,t)$. For such ϕ^ε we have the equations in $\Omega^T = \Omega\times]0,T[$

$$\frac{\partial \phi_\varepsilon}{\partial t} - \varepsilon^{-2}\Delta_y\phi_\varepsilon - \frac{\partial^2\phi_\varepsilon}{\partial x_1^2} + Pr\,Re^\varepsilon w(y,t)\frac{\partial \phi_\varepsilon}{\partial x_1} = 0 \quad \text{in } \Omega^T, \qquad \phi_\varepsilon = h_k \quad \text{for } x_1 = k \text{ and } k = 0,1 \tag{5}$$

$$\frac{\partial \phi_\varepsilon}{\partial \mathbf{n}_y} = \varepsilon\, Nu\, (G - \phi_\varepsilon) \quad \text{on } \Gamma^T = \Gamma \times]0, T[, \qquad \phi_\varepsilon(x_1, y, 0) = \varphi_0(x_1, \varepsilon y), \quad (x_1, y) \in \Omega \tag{6}$$

Using the standard approach we derive the estimates:

Theorem 2.1. *There exists a constant $C > 0$ independent from ε, such that*

$$|\phi_\varepsilon|_{L^\infty(0,T;L^2(\Omega))} + \left|\frac{\partial \phi_\varepsilon}{\partial x_1}\right|_{L^2(0,T;L^2(\Omega))} \leqslant \frac{C}{\sqrt{\varepsilon}}(\sqrt{\varepsilon} + \varepsilon\, Re^\varepsilon) \tag{7}$$

$$|\nabla_y \phi_\varepsilon|_{L^2(0,T;L^2(\Omega))} \leqslant C\varepsilon(1 + \varepsilon\, Re^\varepsilon) \tag{8}$$

$$|\phi_\varepsilon|_{L^2(0,T;L^2(\Gamma))} \leqslant C(\sqrt{\varepsilon} + 1 + \varepsilon\, Re^\varepsilon) \tag{9}$$

Proof. Let $z^\varepsilon(x_1, t) = \mathrm{e}^{-\frac{1}{\varepsilon}}[h_0(t)(\mathrm{e}^{\frac{1-x_1}{\varepsilon}} - x_1) + h_1(t)(\mathrm{e}^{\frac{x_1}{\varepsilon}} + x_1 - 1)]$. We use $\varphi_\varepsilon - z^\varepsilon$ as a test function in (3) to prove the claim using standard techniques. □

If $\varepsilon\, Re^\varepsilon \leqslant C$, from that estimates we deduce that we can extract a subsequence (denoted again by the same symbol) and a function ϕ_0 such that

$$\phi_\varepsilon \rightharpoonup \phi^0 \quad \text{weakly in } L^2(\Omega^T) \text{ and } L^2(\Gamma^T). \tag{10}$$

Unfortunately, for $\lim_{\varepsilon\to 0} \varepsilon\, Re^\varepsilon = +\infty$ our a priori estimates are insufficient to get compactness.

3. Convergence

Depending on Re^ε, we have three characteristic cases:

- $\varepsilon\, Re^\varepsilon \ll 1$ – the exterior temperature G dominates the process. In the limit, the temperature of the fluid in the pipe equals G. The effects of the temperature of the entering fluid h_0 can be seen only in some small boundary layer near the pipe's end.
- $\varepsilon\, Re^\varepsilon = \mathrm{O}(1)$ – the critical case when the influences of the ambiental temperature G and the convection velocity are of the same order and they both remain in the limit. We obtain an ODE of the first order that governs the macroscopic process. Only the upstream boundary condition at $x_1 = 0$ is kept, while the downstream condition at $x_1 = 1$ disappears. The problem can be solved explicitly.
- $\varepsilon\, Re^\varepsilon \gg 1$ – the convection is dominant. The effects of the exterior temperature G are negligible and the limit temperature is equal to h_0. In short, fluid runs through the pipe too fast to be cooled (or heated) by the exterior medium.

The result can be formulated precisely as follows:

Theorem 3.1. *Let ϕ_ε be the solution of* (5), (6).

(a) *If* $\lim_{\varepsilon\to 0} \varepsilon\, Re^\varepsilon = 0$ *then $\phi_\varepsilon \rightharpoonup G$ weakly in $L^2(\Omega^T)$.*[1]

(b) *If* $\lim_{\varepsilon\to 0} \varepsilon\, Re^\varepsilon = Re^0 > 0$. *Then $\phi_\varepsilon \to \phi_0$, strongly in $L^2(\Omega^T)$, where ϕ_0 is the unique solution to the problem*

$$Pr\, Re^0 \langle w\rangle \frac{\partial \phi^0}{\partial x_1} + Nu\, |\partial S|(\phi^0 - G) = 0, \quad 0 < x_1 < 1, \qquad \phi^0(0) = h_0, \quad \langle w\rangle = \int_S w(y,t)\, \mathrm{d}y \tag{11}$$

(c) *If* $\lim_{\varepsilon\to 0} \varepsilon\, Re^\varepsilon = +\infty$ *then $\phi_\varepsilon \to h_0$. The convergence is strong in $L^2(\Omega^T)$ if* $\lim_{\varepsilon\to 0} \varepsilon^2 Re^\varepsilon = 0$ *and strong in $L^2_{\sqrt{w}}(\Omega^T)$ otherwise. Here $L^2_{\sqrt{w}}$ denotes the space of square integrable functions with respect to the weighted measure $\sqrt{w(y)}\,\mathrm{d}x_1\,\mathrm{d}y\,\mathrm{d}t$, and $w = |\mathbf{v}|$ stands for the norm of the convection velocity.*

[1] The convergence can be proved to be strong in $L^2(\Omega)$ but it requires different technique.

Proof of assertion (a). Let $\psi \in H_0^1(]0,1[\times]0,T[)$. We plug $\psi(x_1,t)$ as a test function in our problem and we get

$$-\varepsilon \int_{\Omega^T} \phi_\varepsilon \frac{\partial \psi}{\partial t} + \varepsilon \int_{\Omega^T} \frac{\partial \psi}{\partial x_1}\frac{\partial \phi_\varepsilon}{\partial x_1} - \varepsilon\, Pr\, Re^\varepsilon \int_{\Omega^T} w\phi_\varepsilon \frac{\partial \psi}{\partial x_1} = Nu \int_{\Gamma^T} (G-\phi_\varepsilon)\psi \tag{12}$$

If $\lim_{\varepsilon\to 0} \varepsilon\, Re^\varepsilon = 0$, we arrive at $\int_0^T\int_0^1 \psi(G-\phi^0) = 0$ implying that $\phi^0 = G$ and proving (a). To prove (b) we pass to the limit in (12) and we obtain

$$|\partial S|\, Nu \int_0^1 \psi(G-\phi^0) = -Pr\, Re^0 \int_S w \int_0^1 \phi^0 \psi'$$

This justifies Eq. (11), but not the initial condition. Since the equation without an initial condition has a continuum of solutions we do not have a uniqueness of the solution, i.e. the sequence $\{\phi_\varepsilon\}_{\varepsilon>0}$ can have more than one cluster point. Thus we must justify the initial condition in order to have a convergence and, to do so, we need a different approach. Our weak L^2 convergence does not imply the convergence of traces. It is important to note that we cannot hope for H^1 convergence, or convergence in any topology in which the trace operator is continuous. For example, in case of the linear parabolic equation, as we have here, the weak $L^2(0,T;H^1(\Omega))$ convergence is natural. However, it would imply the convergence of both traces, on the left end $x_1=0$ as well as on the right end $x_1=1$. But, we know that only one of two boundary conditions can be kept (if any), since our limit equation is of first order. The standard approach for finding out which boundary condition should be kept and which one has to be neglected, is to derive a more precise asymptotic approximation. In particular, one needs to study boundary layers on pipe's ends. We use a simpler approach avoiding the computation of tedious asymptotic expansion. It turns out that we keep only the left (upstream) boundary condition, which is natural from the physical point of view. Indeed, the temperature of the fluid exiting the pipe should be determined by the process and not prescribed in advance. In the initial problem (3), (4), we had to impose some boundary condition on the right end of the pipe to close up the problem, but its influence is only present in some small boundary layer near the right end of the pipe and it does not affect the macroscopic behavior of the process. The same analysis will imply the result in the third case with dominant convection. □

4. Proof of the convergence theorem

It is well known from the classical boundary layer theory (see e.g. [5]) that the choice of the boundary condition that will be kept at the limit depends on the sign of the coefficient multiplying the first order term. In our case that coefficient is $w\, Pr\, \varepsilon\, Re^\varepsilon \geqslant 0$, so that the condition on the right end $\phi_\varepsilon(0,y,t) = h_0$ should be kept. Thus our approximation reads

$$Pr\, \varepsilon\, Re^\varepsilon\, M \frac{\partial \phi_\varepsilon^0}{\partial x_1} = \frac{|\partial S|}{|S|} Nu(G-\phi_\varepsilon^0), \quad \phi_\varepsilon^0(0,t) = h_0, \quad \text{with } M = \begin{cases} \langle w\rangle & \text{if } \lim_{\varepsilon\to 0} \varepsilon^2 Re^\varepsilon = 0 \\ w & \text{otherwise} \end{cases} \tag{13}$$

Our goal is to justify our approximation proving that the choice of boundary condition was correct. **In the case** $\lim_{\varepsilon\to 0} \varepsilon^2 Re^\varepsilon$ **we proceed as follows:** We start the justification by defining two auxiliary functions A and B

$$-\Delta_y A = \frac{|\partial S|}{|S|} \quad \text{in } S, \qquad -\frac{\partial A}{\partial \mathbf{n}_y} = 1 \quad \text{on } \partial S \tag{14}$$

$$-\Delta_y B = w - \langle w\rangle \quad \text{in } S, \quad -\frac{\partial B}{\partial \mathbf{n}_y} = 0 \quad \text{on } \partial Sq, \quad \langle w\rangle = \int_S w \tag{15}$$

Now, let $H^\varepsilon = \phi_\varepsilon - \phi_\varepsilon^0$. Then

$$\begin{aligned} &\varepsilon\left(\frac{\partial H^\varepsilon}{\partial t} - \frac{\partial^2 H^\varepsilon}{\partial x_1^2}\right) - \varepsilon^{-1}\Delta_y H^\varepsilon + Pr\, w \varepsilon\, Re^\varepsilon \frac{\partial H^\varepsilon}{\partial x_1} + Nu \frac{|\partial S|}{|S|}(G-\phi_\varepsilon^0) \\ &\quad = \varepsilon\left(-\frac{\partial \phi_\varepsilon^0}{\partial t} + \frac{\partial^2 \phi_\varepsilon^0}{\partial x_1^2}\right) - Pr\, \varepsilon\, Re^\varepsilon \big(w - \langle w\rangle\big)\frac{\partial \phi_\varepsilon^0}{\partial x_1} \\ &H^\varepsilon(0,y,t) = 0, \quad \frac{\partial H^\varepsilon}{\partial \mathbf{n}} + \varepsilon\, Nu\, (\phi_\varepsilon - G) = 0 \quad \text{on } \Gamma^T \end{aligned} \tag{16}$$

Since ϕ_ε^0 can be computed explicitly, it is easy to see that $-\frac{\partial\phi_\varepsilon^0}{\partial t}+\frac{\partial^2\phi_\varepsilon^0}{\partial x_1^2}=\mathrm{O}((\varepsilon\,Re^\varepsilon)^{-1}+(\varepsilon\,Re^\varepsilon)^{-2})$ for each $(x,t)\in\,]0,1[\times]0,T[$. Multiplying the above equation by H^ε and integrating over $\Omega(p)=]0,p[\times S$ with respect to (x_1,y) and over $]0,T[$, with respect to t and denoting

$$\gamma(p)=\{p\}\times S,\qquad \Gamma(p)=]0,p[\times\partial S$$

we obtain

$$\begin{aligned}&\frac{\varepsilon}{2}\left|H^\varepsilon(T)\right|^2_{L^2(\Omega(p))}+\varepsilon^{-1}|\nabla_y H^\varepsilon|^2_{L^2(0,T;L^2(\Omega(p)))}+\varepsilon\left|\frac{\partial H^\varepsilon}{\partial x_1}\right|^2_{L^2(0,T;L^2(\Omega(p)))}+Nu\,|H^\varepsilon|^2_{L^2(0,T;L^2(\Gamma(p)))}\\&\quad+\frac{Pr}{2}\,\varepsilon\,Re^\varepsilon\int_0^T\int_{\gamma(p)}w|H^\varepsilon|^2=Nu\int_0^T\int_0^p\left(\int_S\frac{|\partial S|}{|S|}(\phi_\varepsilon^0-G)H^\varepsilon-\int_{\partial S}(\phi_\varepsilon^0-G)H^\varepsilon\right)+\frac{\varepsilon}{2}\int_0^T\int_{\gamma(p)}\frac{\partial}{\partial x_1}H^\varepsilon\\&\quad+\frac{\varepsilon}{2}\left|\varphi_0-\phi_\varepsilon^0(\cdot,0)\right|^2_{L^2(\Omega(p))}-Pr\,\varepsilon\,Re^\varepsilon\int_0^T\int_{\Omega(p)}\big(w-\langle w\rangle\big)\frac{\partial\phi_\varepsilon^0}{\partial x_1}|H^\varepsilon|^2-\varepsilon\int_0^T\int_{\Omega(p)}\left(\frac{\partial\phi_\varepsilon^0}{\partial t}-\frac{\partial^2\phi_\varepsilon^0}{\partial x_1^2}\right)H^\varepsilon\end{aligned}\tag{17}$$

To get the above relation we have used simple computation

$$\int_0^p\int_S\frac{\partial^2 H^\varepsilon}{\partial x_1^2}H^\varepsilon=-\left|\frac{\partial H^\varepsilon}{\partial x_1}\right|^2_{L^2(\Omega(p))}+\frac{1}{2}\int_S\frac{\partial}{\partial x_1}\left|H^\varepsilon(p,y,t)\right|^2\mathrm{d}y$$

We also used that, since $H^\varepsilon(0,y,t)=0$ we have

$$\int_{\Omega(p)}w\frac{\partial}{\partial x_1}|H^\varepsilon|^2=\int_{\gamma(p)}w(y)|H^\varepsilon|^2\,\mathrm{d}y$$

Now is the time to use the auxiliary functions A, B. We have

$$\begin{aligned}&\int_0^p\left(\int_S\frac{|\partial S|}{|S|}(\phi_\varepsilon^0-G)H^\varepsilon-\int_{\partial S}(\phi_\varepsilon^0-G)H^\varepsilon\right)=\int_0^p\left(-\int_S\Delta_y A(\phi_\varepsilon^0-G)H^\varepsilon-\int_{\partial S}(\phi_\varepsilon^0-G)H^\varepsilon\right)\\&\quad=\int_0^p(\phi_\varepsilon^0-G)\int_S\nabla_y A\nabla_y H^\varepsilon\leqslant C\varepsilon+\frac{1}{4\varepsilon}|\nabla_y H^\varepsilon|^2_{L^2(\Omega(p))}\end{aligned}\tag{18}$$

$$\begin{aligned}&\int_{\Omega(p)}\big(w-\langle w\rangle\big)\frac{\partial\phi_\varepsilon^0}{\partial x_1}H^\varepsilon=-\int_0^p\frac{\partial\phi_\varepsilon^0}{\partial x_1}\int_{\gamma(x_1)}\Delta_y B H^\varepsilon=\int_0^p\frac{\partial\phi_\varepsilon^0}{\partial x_1}\int_{\gamma(x_1)}\nabla_y B\nabla_y H^\varepsilon\\&\quad\leqslant C\varepsilon^2\,Re^\varepsilon+\frac{1}{4\varepsilon^2\,Pr\,Re^\varepsilon}|\nabla_y H^\varepsilon|^2_{L^2(\Omega(p))}\end{aligned}\tag{19}$$

We apply (18) and (19) on (17) leading to

$$\frac{Pr}{2}\int_0^T\int_{\gamma(p)}w|H^\varepsilon|^2\leqslant C\left(\varepsilon+\frac{1}{Re^\varepsilon}+\frac{1}{\varepsilon(Re^\varepsilon)^2}+\frac{1}{\varepsilon^2(Re^\varepsilon)^3}+\varepsilon^2\,Re^\varepsilon\right)+\frac{\varepsilon}{2Re^\varepsilon}\int_0^T\int_{\gamma(p)}\frac{\partial}{\partial x_1}|H^\varepsilon|^2\tag{20}$$

Next we notice that

$$\int_0^1 dp\int_{\gamma(p)}\frac{\partial}{\partial x_1}|H^\varepsilon|^2=\int_\Omega\frac{\partial}{\partial x_1}|H^\varepsilon|^2=|H^\varepsilon|^2_{L^2(\gamma(1))}=|S\|h_1|^2-|\phi_\varepsilon^0|^2_{L^2(\gamma(1))}$$

Finally, we integrate (20) with respect to p over [0, 1] and we conclude that

$$\int_0^T\int_\Omega w|H^\varepsilon|^2 \leqslant C\left(\varepsilon + \frac{1}{Re^\varepsilon} + \frac{1}{\varepsilon^2 (Re^\varepsilon)^3} + \varepsilon^2 Re^\varepsilon\right) \tag{21}$$

To get rid of w we again use the auxiliary function B. We obtain

$$\int_\Omega w|H^\varepsilon|^2 = \int_\Omega \big(w - \langle w\rangle\big)|H^\varepsilon|^2 + \langle w\rangle \int_\Omega |H^\varepsilon|^2 = 2\int_\Omega \nabla_y B \nabla_y H^\varepsilon H^\varepsilon + \langle w\rangle \int_\Omega |H^\varepsilon|^2$$

Thus, using the a priori estimates (7)–(9), we get

$$\begin{aligned}|H^\varepsilon|^2_{L^2(\Omega^T)} &\leqslant \int_{\Omega^T} \frac{w}{\langle w\rangle}|H^\varepsilon|^2 + C|\nabla_y H^\varepsilon|_{L^2(\Omega^T)}|H^\varepsilon|_{L^2(\Omega^T)} \\ &\leqslant C\left(\varepsilon + \frac{1}{Re^\varepsilon} + \varepsilon^2 Re^\varepsilon + \frac{1}{\varepsilon^2(Re^\varepsilon)^3}\right) + \frac{1}{2}|H^\varepsilon|^2_{L^2(\Omega^T)}\end{aligned}$$

In the case $\lim_{\varepsilon\to 0}\varepsilon^2 Re^\varepsilon \neq 0$ **we take** $M = w$. Then we have no use of B since the term $Pr\,\varepsilon\,Re^\varepsilon (w - \langle w\rangle)\frac{\partial\phi^0_\varepsilon}{\partial x_1}$ no longer appears in (16). On the other hand we get an additional term $\varepsilon^{-1}\Delta_y\phi^0_\varepsilon = \mathrm{O}(\varepsilon^{-3}(Re^\varepsilon)^{-2} + \varepsilon^{-4}(Re^\varepsilon)^{-3})$. Thus in (21) we get $C(\varepsilon^{-3}(Re^\varepsilon)^{-2} + \varepsilon^{-4}(Re^\varepsilon)^{-3})$. As, in our case, that term tends to 0 when $\varepsilon \to 0$ we prove the claim as in the previous case.

We have derived a general asymptotic approximation (13). That ODE can be easily solved and we have

$$\phi^0_\varepsilon(x_1, t) = \mathrm{e}^{-x_1/K^\varepsilon}\left(h_0(t) + \frac{1}{K^\varepsilon}\int_0^{x_1} \mathrm{e}^{\xi/K^\varepsilon} G(\xi, t)\,\mathrm{d}\xi\right), \tag{22}$$

with

$$K^\varepsilon = Nu^{-1}|\partial S|^{-1} Pr\,\varepsilon\,Re^\varepsilon\,|S| M = \omega\,\varepsilon\,Re^\varepsilon$$

$\omega = Nu^{-1}|\partial S|^{-1}\,Pr\,M$, independent from ε.

We have proved that, as $\varepsilon \to 0$, $\phi_\varepsilon - \phi^0_\varepsilon \to 0$, provided that $\lim_{\varepsilon\to 0}\varepsilon^2(Re^\varepsilon)^3 = +\infty$. That covers all three cases except the case of small Re^ε which was handled before by the weak convergence. □

One can now directly compute the following pointwise limits:

$$\lim_{\varepsilon Re^\varepsilon\to 0}\phi^0_\varepsilon(x_1, t) = G(x_1, t), \qquad \lim_{\varepsilon Re^\varepsilon\to +\infty}\phi^0_\varepsilon(x_1, t) = h_0(t),$$

$$\lim_{\varepsilon Re^\varepsilon\to Re^0}\phi^0_\varepsilon(x_1, t) = \mathrm{e}^{-x_1/K}\left(h_0(t) + \frac{1}{K}\int_0^{x_1}\mathrm{e}^{\xi/K} G(\xi, t)\,\mathrm{d}\xi\right)$$

That finishes the proof of all the claims from Theorem 3.1 . It also shows that the passage from one model to another is in some sense continuous.

Acknowledgement

This research was supported by the Ministry of Science, Education and Sports, Republic of Croatia, Grant 037-0372787-2797.

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