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C. R. Mecanique 336 (2008) 565–571
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Thermodynamical functions for a gas of point vortices

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Received 13 September 2006; accepted after revision 8 April 2008

Available online 2 June 2008

Presented by Évartiste Sanchez-Palencia

Abstract

We formulate nonlinear integro-differential equation for the averaged collective Hamiltonian of a gas of interacting two-dimensional vortices, derive its analytical solution, and discuss the equilibrium, axially-symmetrical, probability distributions that are possible for such a model. We also theoretically prove that the probability distribution for a system of 2D point vortices takes a form similar to the Gibbs distribution, but point out that the physical fundamentals of such a system differ from the standard theory of interacting particles. Furthermore, we find thermodynamical functions for positive and negative "temperature" of the system, and point out that the states with positive "temperature" correspond to stationary bell-shape vortex distributions, while the states with negative "temperature" correspond to distributions localized near container walls. ***To cite this article: E. Bécu et al., C. R. Mecanique 336 (2008).***

Résumé

Fonctions thermodynamiques du gaz des tourbillons ponctuels. Nous formulons l'équation non linéaire integro-différentielle pour l'Hamiltonien collectif moyenné d'un gas de tourbillons 2D intéragissant et trouvons sa solution analytique. Nous discutons aussi de la distribution d'équilibre de probabilité axisymétrique possible d'un tel modèle. Nous montrons également que la probabilité pour un système de tourbillons ponctuels doit prendre une forme similaire à la distribution de Gibbs. Notons que les fondaments physiques d'un tel systèm différent de la théorie standard des particules en intéraction. Nous trouvons les fonctions « thermodynamiques » pour les températures positives et négatives du système et discutons le fait que les états avec température positive correspondent à une distribution ayant le maximum central, tandis que les états avec température négative correspondent aux distributions localisées au voisinage des parois du containeur. ***Pour citer cet article : E. Bécu et al., C. R. Mecanique 336 (2008).***

Keywords: Dynamical systems; Hamiltonian description; Gas of localized vortices; Positive and negative "temperatures"

Mots-clés : Systèmes dynamiques ; Description Hamiltonienne ; Gaz des tourbillons localisés ; « Températures » positive et négative

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1631-0721/$ – see front matter
doi:10.1016/j.crme.2008.04.003

1. Introduction

Many studies have attempted to describe the final equilibrium states of quasi-inviscid, two-dimensional hydrodynamical systems. A stimulating observation arising from the field of research is that two-dimensional turbulent flows at very large Reynolds numbers frequently and spontaneously self-organize into large-scale (coherent) vortex structures. Many authors have examined statistical mechanics of a two-dimensional incompressible fluid in a vortex representation of Euler equations of motion (see, for example, [1–13] for the extensive bibliography). Alternatively, since the dynamics of vortex structures is highly nonlinear and is described by complex analytical equations of motion, there was a strong incentive to explore some variational principles: in fact, in a freely evolving system, if a steady state is to be reached, it can correspond to extremum of some functional. Despite different theoretical attempts to describe the equilibrium, the situation still remains controversial. For this reason, direct experimental studies play a crucial role. However, even on the experimental side, the situation is somewhat puzzling (see the review [14] and the extensive literature on the subject therein).

Intrigued by the theoretical and experimental disparity, in this Note we consider the simplest model of 2D turbulence as a system of N two-dimensional point vortices. The essence of the approach is to use the simplest model for a fluid motion possessing the continual number of degrees of freedom and to parameterize the field Hamiltonian using a finite number of parameters (generalized coordinates-moments) for which the Hamiltonian equations are deduced. It is well known that the equations of motion for vortices are look as those of Hamilton for particles and that such a model automatically respects all hydrodynamical laws of conservation. We therefore consider a two-dimensional "gas" of N 2D-point vortices where N is sufficiently large to reasonably approximate the turbulent "thermodynamical" functions of such a "gas". The properties of the "gas" will be described then by the "thermodynamical functions" corresponding to the quasi-limit with very large, but finite N. With these remarks in mind, henceforth we omit the quotation marks for "temperature", "gas", etc.

It is critical to highlight the differences between the dynamics of 2D-vortices and the dynamics of particles. First, Newton's first law in hydrodynamical systems is modified. An isolated vortex always remains in a stationary state, while a steady, straight-line motion takes place only either under the effect of another vortex with the same intensity but the opposite sign, or when interacting with a solid wall. The relativity principle of classical mechanics, according to which the states of rest and steady motion are equivalent, is invalid in the cases of vortices because the medium surrounding the vortex determines the selected frame of reference. The second law is altered even more dramatically. The derivative of the Hamiltonian of the system with respect to coordinate generates not force, but velocity multiplied on the intensity of the vortex, i.e., the external influence of another vortex causes not acceleration, but velocity [15].

Consequently, it is clear that the literally applying analogies from classical statistical mechanics for particles to vortices is not appropriate.

This Note focuses on formulating the nonlinear integro-differential equation for the averaged collective Hamiltonian of a gas of interacting bi-dimensional vortices, analytical solving the equation, and thoroughly deriving the thermodynamical functions for a gas of point vortices.

There are several key propositions that allow us to accomplish this. First, we assume that the point vortices do not approach each other too closely. In fact, as the distance between two vortices of the same sign diminishes below some characteristic distance, their velocities (relative to mutual rotation) start exceeding the sound speed in the medium, and the incompressibility assumption ($\operatorname{div}\mathbf{v} = 0$) breaks down. At such scales, one can no longer describe 2D turbulence as a system of point vortices. Second, we assume that the number of point vortices is finite. This assumption ensures that the full energy of the system is finite. If the energy expressed by the Hamiltonian H were infinite, due to presence in averaged functions of the exponential factor $\exp(-H/T)$ like Gibbs distribution, the convergence requirement for the statistical sum would imply that the temperature T must only be positive (see for classical particles [16,17]). On the other hand, when H (and the phase space volume) is finite, T can be either positive, or negative. Positive T means that only one stationary state corresponding to the bell-shaped distribution peak, exists. Negative T permits phase transitions and stationary distributions where multiple peaks (crystals) may form.

The final and key requirement is that one must consider the effect of all vortices on each other, not just the immediate neighbors. In classical hydrodynamics when the Hamiltonian $H_{ij} \sim -\ln r_{ij}$, the velocity fields created by the vortices are long ranged, $v \sim r_{ij}^{-1}$, and every vortex is influenced by the averaged collective field created by all the vortices in the system, even those very remote and itself. This poses two theoretical issues. First, in order to define such averaged collective functions, one needs to know the probability distribution for one vortex via the averaged collective Hamiltonian. However, as mentioned above, the dynamics of vortices differs from the dynamics of particles, and therefore, one cannot simply assume that the probability distribution takes the typical Gibbs function form. The form of the probability distribution needs to be derived from first principles. Second, to find the averaged collective Hamiltonian, an complex integro-differential equation needs to be solved.

2. Point vortices in the cylindrical container

Let us consider a system with a large but finite number of interacting 2D point vortices, $N \gg 1$. If the potential of their interaction were a short-acting one, the motion of each vortex were caused by the interaction with the nearest one. If the potential of interaction is a long-acting one, the motion of a vortex is caused by the collective action of all vortices. In this case, each vortex moves in the collective field created by the ensemble of other vortices. Therefore, the Hamiltonian of every vortex can be decomposed into two parts: the average (with respect to the motions of all other vortices) and the fluctuation, i.e. $H(x_1, \ldots, x_N) = \sum_{i=1}^{N} U_i(\mathbf{x}_i) + H'$. It is clear that fluctuations become essentially apparent only on the periphery of the vortex distribution. If we neglect the fluctuation component of the Hamiltonian, H', we simplify the problem. We consider a system with N identical and positive vortices localized in finite domain of space having a characteristic scale R. The evolution equations for the system can be made non-dimensional using the following characteristic scales: spatial, R, temporal, $\tau = 2\pi R^2/\Gamma$ and energy $H_0 = \Gamma^2/2\pi N$. The Hamiltonian, which usually represents the system's total energy expressed in terms of canonical variables x_i and y_i, then takes the following non-dimensional form:

$$H = \frac{1}{2N} \sum_{i \neq j;\, i,j=1}^{N} H_{ij} \tag{1}$$

Here, H_{ij} is the Hamiltonian of interaction between vortices i and j. The factor $1/2$ appears because the term describing interaction between vortices i and j is counted in the total sum twice as H_{ij} and H_{ji}. The motion of the vortices is described by the following dimensionless equations:

$$\partial_t x_i = \frac{\partial \overline{H}}{\partial y_i}, \qquad \partial_t y_i = -\frac{\partial \overline{H}}{\partial x_i} \tag{2}$$

In classical hydrodynamics, when $H_{ij} = \ln r_{ij}$, the field of velocities created by the vortices has a long range. To summarize, we came to the following conclusions: (a) each vortex moves virtually independently of its neighbors, and (b) each vortex moves as if in one field created by the ensemble of all other vortices, which we call the auto-accorded field averaged with respect to coordinates of all other vortices. After simple manipulations, we find the expression for the Hamiltonian

$$H \to H = \sum_i U_i, \quad \text{with } U_i \text{ defined by} \quad U_i = \int_D \mathrm{d}\mathbf{x}_j\, w_1(\mathbf{x}_j) H_{ij} \tag{3}$$

where we neglect the fluctuating part of the Hamiltonian. The problem then reduces to the problem of the motion of vortices, which do not interact among themselves, in the self-consistent field U_i. The word self-consistent means that the field U_i is an integral of the probability to find the configuration in a particular state where the probability itself is a function of the field, U_i: $w \sim \exp(-\beta U_i)$ for the canonical distribution. The free parameter β^{-1} is temperature of the gas of vortices.

The omission of the fluctuating term in the Hamiltonian describes the situation where the full energy of the fluid does not remain constant anymore, but instead experiences weak fluctuations. The system acts as if it is connected to a thermostat which controls the temperature of the system, but not the energy. The spatial distribution of the vortices is characterized then by the distribution similar to canonic Gibbs distribution.

In the domain, D, that we have been considering, vortices can move freely to any point in the domain, thus approaching each other, or concentrating near the border, etc. The distances between the vortices and their possible configurations are determined by the laws of conservation of the kinetic moment and energy. Obviously, the vortices cannot merge, or cross the boundary ∂D because such states have infinite energies, while the original vortices have finite energy, no matter how large is its absolute value. As the vortex approaches another vortex or the boundary (which serves as a mirror providing a reflected image of the vortex), the distance between the vortices decreases. However, the energy still has to remain finite, thus preventing the vortices from merging (or crossing the boundary).

A system configuration is characterized by an ensemble of canonical variables $X = \{X_i\} = \{x_i, y_i\}$ where, for point vortices, the space coordinate x_i plays a part of canonical coordinate, and the space coordinate y_i plays a part of canonical momentum. This configuration corresponds to point X in the phase space. Equality $H[X, R] = E$ "carves out" a hyper-surface in the phase space where the phase states are localized. Integral $\Gamma'(E, R) \equiv \partial_E \Gamma = \int_{D^N} \mathrm{d}X\, \delta(E - H[X, R])$ determines the energetic density of the phase states, i.e. the number of states of the system with energies between E and $E + \mathrm{d}E$. Here, $\delta(s)$ is the Dirac function. The total phase volume of the system is $\Gamma(E) \leqslant D^N$. Therefore, it is limited by the boundary ∂D which corresponds to the energy hyper-surface $\partial_E D$ characterized by value $E = +\infty$. Therefore, the energy hyper-surface with the final E, never intersects the boundary $\partial_E D$.

Following the classical considerations (see for example Landau and Lifshitz [16], Fermi [18]), the entropy of the system is defined via the density of phase states as:

$$S(E, R) = \ln\big[\Gamma'(E, R)\big] + Const_1 \tag{4}$$

This formulation is correct but differs from the largely used traditional definition via the full phase volume in spirit of L. Boltzmann. However, formulation (4) is more natural for several reasons. First, the full phase volume for the systems with bounded phase volume can be defined two ways: by $\Gamma(E, R) = \int_{D^N} \mathrm{d}X\, \theta(E - H[X, R])$ and by $\Gamma_1(E, R) = \int_{D^N} \mathrm{d}X\, \theta(H[X, R] - E)$. Here, X is a phase point. Obviously, $\Gamma_1(E, R) = D^N - \Gamma(E, R)$. On the other hand, it is natural that the theory does not have to depend on the choice of the specific form of Γ or Γ_1, or their value. Formulation (4) respects this requirement. Keeping in mind that the system has a finite number of identical elements, the phase volume of the system is even smaller: the same physical state corresponds to configuration states with $N!$ permutations of elements. So, the density of phase states is not $\Gamma'(E, R)$ but $(N!)^{-1}\Gamma'(E, R)$, so that $\Gamma'(E, R) = (N!)^{-1} \int_{D^N} \mathrm{d}X\, \delta(E - H[X, R])$. This modification is not essential, however, if the number N is fixed. If number N is a large but finite one, one can use the steepest descent method to find the asymptotical values of integrals which determine Γ'. The asymptotical value the integral is determined by the stationary points of function $G(\nu) = \nu E - \ln N! + N \ln \int_D X_1 \mathrm{e}^{-\nu U_1[X_1, R]}$ which is the logarithm of the subintegral expression. The stationary point is determined, thus, by the equation

$$G_\nu|_{\nu=\beta} \equiv \frac{\partial G}{\partial \nu}\bigg|_{\nu=\beta} = E - \frac{N}{Z_1(\beta)} \int_D \mathrm{d}X_1\, U_1(X_1, R) \exp\big(-\beta U_1(X_1, R)\big) = 0 \tag{5}$$

Here, we introduce notation (statistical sum) $Z_1(\beta) = \int_D \mathrm{d}X_1 \exp(-\beta U_1[X_1, R])$. The probability of states is introduced by the expression $w_1(X_1) = Z_1^{-1} \exp[-\beta U_1[X_1, R]]$ normalized on the unit. Let us now introduce function $F(\beta, R, N) = -N\beta^{-1} \ln Z_1$, where we accounted for the fact that the Hamiltonian, and consequently F, depend on the external parameter R. In this case,

$$w_1(X_1) \equiv \exp\big[\beta N^{-1} F - \beta U_1[X_1, R]\big] \tag{6}$$

This expression is called the canonical Gibbs distribution. Using (6), we obtain

$$\int_D \mathrm{d}X_1 \left[\beta \frac{\partial F}{\partial \beta} + F - N U_1[X_1, R]\right] \exp\big[\beta N^{-1} F - \beta U_1[X_1, R]\big] = 0 \tag{7}$$

which gives $E = N\langle U_1[X_1, R]\rangle_1 = \int_D \mathrm{d}X_1 [\beta\partial_\beta F + F] \exp[\beta N^{-1} F - \beta U_1[X_1, R]] = \beta\partial_\beta F + F = -N\partial_\beta \ln Z_1$. The expression for entropy can be obtained from definition [16], [18] as $S(E, R) = \ln \Gamma'(E, R) \simeq \beta E + N \ln \int_D \mathrm{d}X_1\, \mathrm{e}^{-\beta U_1[X_1, R]} - N \ln N + \cdots = \beta E - \beta F - N \ln N + \cdots = -N\beta\partial_\beta \ln Z_1 + N \ln Z_1 - N \ln N + \cdots = -N\beta^2 \partial_\beta(\beta^{-1} \ln Z_1) - N \ln N + \cdots$. After some mathematical rearrangement of the terms, the classical definition follows:

$$S(E, R) = -N \int_D \mathrm{d}X_1\, w_1(X_1) \ln w_1(X_1) - N \ln N \tag{8}$$

Thus, the entropy is proportional to the averaged value of the probability density logarithms, and is not an averaged value of some mechanical value. Consequently, both the temperature, β^{-1}, and the entropy are values characterizing in full the statistical ensemble, but not any particular mechanical microstate of the system.

Laboratory experiments are generally performed in containers with finite dimensions which are bounded by impermeable, solid walls. In this section, we describe how the presence of such physical boundaries influences the self-organization process in (quasi-)2D turbulent flows. What happens when the vortices collide with the wall? How does the geometry of the container influence the shapes of vortices that "survive" the self-organization?

The governing equations for dimensionless coordinates of vortex centers are given by Eq. (2), where the dimensionless Hamiltonian is defined by

$$\overline{H} = \sum_{i=1}^{N} \overline{U}_i, \quad \text{where } \overline{U}_i = \frac{1}{2N}\sum_{j=1}^{N} g(\mathbf{x}_i, \mathbf{x}_j) + \left[\frac{1}{N}\ln(1 - r_i^2)\right] \tag{9}$$

Here, $N \simeq N - 1 \gg 1$, $r_i^2 = x_i^2 + y_i^2$, $r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2$, $i \neq j$. Taking into account $N \gg 1$ and the fact that vortices keep clear too of boards, the term $[\cdots] = N^{-1}\ln(1 - r_i^2)$ will be omitted in the following calculations. The double sum with respect to i, j runs over all pairs, i.e., there are $N(N-1) \simeq N^2$ individual terms in this sum. The factor 2 reflects the symmetry condition $i \rightleftharpoons j$ in the expression for the Hamiltonian. Kernel $g(\mathbf{x}_i, \mathbf{x}_j)$ is given by the expression

$$g(\mathbf{x}_i, \mathbf{x}_j) = \frac{1}{2}\ln r_{ij}^2 - \frac{1}{2}\ln(1 - 2\mathbf{x}_i \cdot \mathbf{x}_j + r_i^2 r_j^2) = \frac{1}{2}\ln\frac{a - \cos\theta_{ij}}{b - \cos\theta_{ij}} \tag{10}$$

where $a = (r_i^2 + r_j^2)/2r_i r_j$ and $b = (1 + r_i^2 r_j^2)/2r_i r_j$. If $r_i = 1$ (or $r_j = 1$), we obtain $a = b$ and $g|_{r_i=1} = 0$. The asymmetry in the logarithmic terms arises from the boundary conditions. The sum in (9) with (10) involves (a) the usual free-space interactions which goes to $-E_{\min}$ (the value of the minimal energy $-E_{\min}$ is defined by the conservation of angular momentum) when $r_{ij} \to r_{\min}$ (as in the case of same-signed vortices that we considered above) and (b) the interaction of an individual vortex with the images of all *other* vortices. These terms become singular if $r_i, r_j \to R$, i.e., the vortex crosses the boundary. We neglect the terms describing interaction vortex–image for each vortex when $N \gg 1$. Because various combinations of terms with $r_i \to R$ or $r_{ij} \to 0$ are possible, we can confidently conclude that H can have either negative value, $-E_{\min}$ (when $r_{ij} \to 0$), or positive value, $+\infty$ (when $r_s \to 1$).

To obtain the averaging distribution, we replace $(2N)^{-1}\sum_j \cdots \to \int_D \mathrm{d}\mathbf{x}\, w_1(\mathbf{x}) \cdots$ and neglect small term of order $N^{-1/2} \to 0$. We obtain (for the dimensionless values) that

$$\overline{U}_1(\mathbf{x}_1) = \int_D' \mathrm{d}\mathbf{x}\, w_1(\mathbf{x}) g(\mathbf{x}_1, \mathbf{x}) \to \overline{U}_1(\mathbf{x}_1) = \mathrm{e}^{\beta F_1}\int_D' \mathrm{d}\mathbf{x}\, g(\mathbf{x}_1, \mathbf{x})\mathrm{e}^{-\beta \overline{U}_1(\mathbf{x})} \tag{11}$$

We can write the equations to find $Y(\mathbf{x}_1) = \exp(-\beta\overline{U}_1(\mathbf{x}))$ or $w(\mathbf{x}_1)$

$$-\beta^{-1}\ln Y(\mathbf{x}_1) = \mathrm{e}^{\beta F_1}\int_D' \mathrm{d}\mathbf{x}\, g(\mathbf{x}_1, \mathbf{x}) Y(\mathbf{x}) \rightleftarrows -\beta^{-1}\ln w_1(\mathbf{x}_1) - \int_D' \mathrm{d}\mathbf{x}\, g(\mathbf{x}_1, \mathbf{x}) w_1(\mathbf{x}) = -F_1 \tag{12}$$

Consider an axially symmetrical distribution $w_1(\mathbf{x}_1) = w_1^{(0)}(r_1)$. In this case, from Eq. (12) it follows:

$$-\frac{1}{\beta}\ln w_1^{(0)}(r_1) - \int_0^1 \mathrm{d}r\, r w_1^{(0)}(r)\left[\frac{1}{2}\int_{-\pi}^{\pi}\mathrm{d}\theta\,\ln\frac{b - \cos(\theta - \theta_1)}{a - \cos(\theta - \theta_1)}\right] = -F_1 \tag{13}$$

After integrating on angular variables, we can rewrite Eq. (13) as

$$-\frac{1}{\beta}\ln w_1^{(0)}(r_1) - \pi\left[\int_0^{r_1}\mathrm{d}r\, r w_1^{(0)}(r)\right]\ln r_1 - -\pi\int_{r_1}^{1}\mathrm{d}r\, r w_1^{(0)}(r)\ln r = -F_1 \tag{14}$$

The solution to this equation is normalized by the condition $2\pi\int_0^1 \mathrm{d}r\, r w_1^{(0)}[\beta, F_1; r] = 1$, which gives the equation of state, i.e. the dependence $F_1 = F_1(\beta)$.

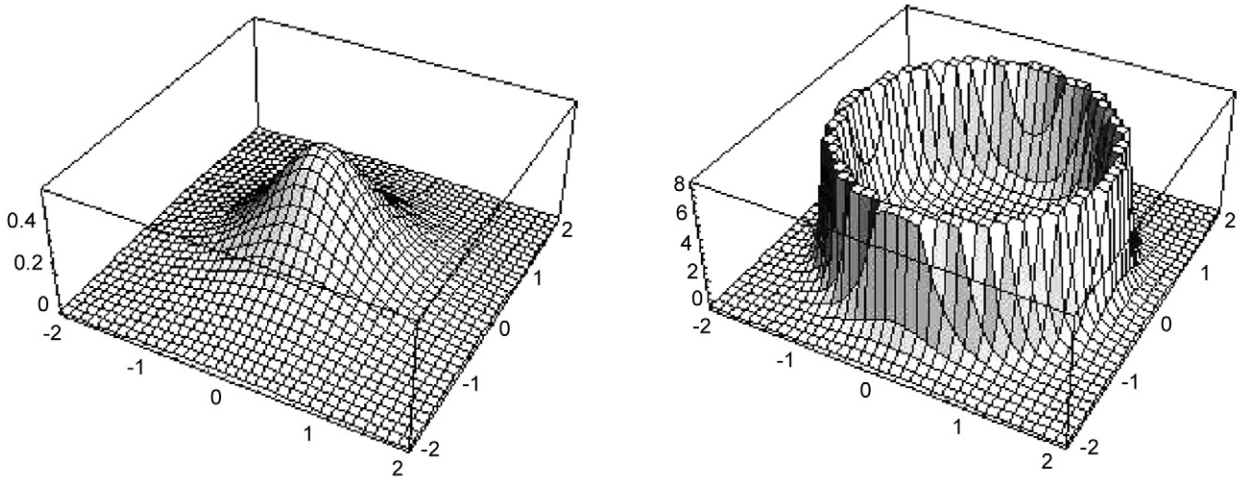

Fig. 1. Distribution of the vorticity for the magnitude factor $a = \pi$ (positive temperature) on the left and $a = \pi^{-1}$ (negative temperature) on the right.

Fig. 1. Répartition de la vorticité pour le facteur de magnitude $a = \pi$ (température positive) à gauche et $a = \pi^{-1}$ (température négative) à droite.

Let us take on an analytical solution for $Y(r;\mu)$ defined from the relationship $w_1^{(0)}(r) = Y(r;\mu)\exp(\beta F_1)$. Here, parameter μ is defined by the expression $\mu = \pi\beta\exp(\beta F_1)$, i.e. $F_1 = \ln(\mu/\pi\beta)$. The equation yields

$$\ln Y(r) + \mu\left[\left(\int_0^r \mathrm{d}s\, sY(s)\right)\ln r + \int_r^1 \mathrm{d}s\, sY(s)\ln s\right] = 0 \tag{15}$$

The nonlinear equation has solution $Y(r) = a(1+\frac{1}{8}\mu ar^2)^{-2}$, where $a = Y(0)$ is an amplitude factor. Now, we use the normalization condition which gives $2\mu\int_0^1 \mathrm{d}s\, sY(s) = (\mu a)/(1+\frac{1}{8}\mu a) = \beta$. We find from here $\mu = 8a^{-1}(\pm\sqrt{a}-1)$, $\beta = \pm 8a^{-1/2}(\pm\sqrt{a}-1)$. Finally, the free energy calculated from $F_1 = \ln(\mu/\pi\beta)$ is given by

$$F_1 = \ln\left[\pm\frac{8}{a}(\pm\sqrt{a}-1)\frac{\sqrt{a}}{8\pi(\pm\sqrt{a}-1)}\right] = \ln\left[\pm\frac{1}{\pi\sqrt{a}}\right] \tag{16}$$

Only the solution with $0 < a < \infty$ has physical meaning. Therefore,

$$\mu = \frac{8}{a}(\sqrt{a}-1), \quad Y(r) = \frac{a}{(1+(\sqrt{a}-1)r^2)^2}, \quad \beta = \frac{8(\sqrt{a}-1)}{\sqrt{a}}, \quad F_1 = \ln\left[\frac{1}{\pi\sqrt{a}}\right] \tag{17}$$

Eq. (17) presents the key result of the work. All thermodynamical functions can now be found. The full entropy is calculated from $S(\beta) = NS_1(\beta) = N\beta^2\partial_\beta F_1$. The full energy and the temperature are given by $E(\beta) = N[\ln[1/\pi\sqrt{a}\,] - (\sqrt{a}-1)]$ and by $\theta \equiv \beta^{-1} = \sqrt{a}/8(\sqrt{a}-1) \to \sqrt{a} = 8\theta/(8\theta-1) \equiv 8/(8-\beta) > 0$, i.e. the temperature of the vortex gas can be either positive, $\theta > 1/8$, and negative, $\theta < 0$. The free energy and the internal energy as functions of temperature, are $F(\theta) = NF_1 = N\ln[(8\theta-1)/8\pi\theta]$, $E(\theta) = NE_1 = N[\ln[(8\theta-1)/8\pi\theta] - (8\theta-1)^{-1}]$. The analytically calculated picture of vorticity distribution for both positive and negative temperatures is shown in Fig. 1.

3. Conclusion

We have shown that the probability distribution for a system of point vortices takes a form similar to the Gibbs distribution, but the physical fundamentals of such a system differ from the standard theory of classical particles. We also analytically solved the nonlinear integro-differential equation for the averaged collective Hamiltonian and derived all necessary thermodynamical functions.

The statistical mechanics and thermodynamics of point vortices in a closed domain are described by a canonical ensemble with Gibbs-like probability distribution $w_c(X) = Z_c^{-1}\exp[-\beta H[X]]$ $(A1)$. The energy of only the fluid flow is not necessarily conserved in this case, because there exists an energy transfer in and out of the thermostat (see [9], p. 4450). The partition function is given by the integral $Z(\beta) = \int_D \mathrm{d}X \exp(-\beta H[X])$ $(A2)$. Parameter β can be both positive and negative. The domain of possible values of β is defined by the convergence of integral $(A2)$ and the condition that $w_c(X) \geqslant 0$. The temperature β^{-1} is related to the averaged energy $\langle E \rangle$ defined by $\langle E \rangle = \int \mathrm{d}X\, H[X] Z^{-1} \exp[-\beta H[X]]$. The value of the energy E is calculated from the initial distribution of vortices $E = (2N)^{-1} \sum'_{i,j} \ln[(b_{ij} - \cos\theta_{ij})/(a_{ij} - \cos\theta_{ij})]$ $(A3)$, where $a_{ij} = (r_i^2 + r_j^2)/2r_i r_j$ and $b_{ij} = (1 + r_i^2 r_j^2)/2r_i r_j$. If $r_i = 1$ (or $r_j = 1$), we obtain $a = b$ and $g|_{r_i=1} = 0$. Equation $E = \langle E \rangle = N \partial_\beta(\beta F_1(\beta))$ $(A4)$ fixes the value of β corresponding to the given initial configuration. If $\beta > 0$, the system evolves into a configuration with a central peak. If $\beta \sim 0$, a quasi-uniform spot is formed. If $\beta < 0$, an axially-symmetrical distribution of vortices appears with the maximum near the border.

Classical turbulence theories, while providing important guidance, do not explicitly address geometrical issues. Such issues require new tools that complement information from analysis of turbulence. Two turbulent-flow cases confirm the need for such extensions. In particular, the analysis of irregular level sets makes it a better register of small-scaled information, which is difficult to infer from spectral data alone.

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