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ScienceDirect
C. R. Mecanique 336 (2008) 656–663
Comptes Rendus Mecanique
http://france.elsevier.com/direct/CRAS2B/

Analytical development of disturbed matrix eigenvalue problem applied to mixed convection stability analysis in Darcy media

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Received 19 February 2008; accepted after revision 30 May 2008

Presented by Évariste Sanchez-Palencia

Abstract

This work consists in evaluating algebraically and numerically the influence of a disturbance on the spectral values of a diagonalizable matrix. Thus, two approaches will be possible; to use the theorem of disturbances of a matrix depending on a parameter, due to Lidskii and primarily based on the structure of Jordan of the no disturbed matrix. The second approach consists in factorizing the matrix system, and then carrying out a numerical calculation of the roots of the disturbances matrix characteristic polynomial. This problem can be a standard model in the equations of the continuous media mechanics. During this work, we chose to use the second approach and in order to illustrate the application, we choose the Rayleigh–Bénard problem in Darcy media, disturbed by a filtering through flow. The matrix form of the problem is calculated starting from a linear stability analysis by a finite elements method. We show that it is possible to break up the general phenomenon into other elementary ones described respectively by a disturbed matrix and a disturbance. A good agreement between the two methods was seen. ***To cite this article: H.B. Hamed, R. Bennacer, C. R. Mecanique 336 (2008).***

Résumé

Développement analytique d'un problème à valeurs propres d'une matrice perturbée appliqué à l'analyse de stabilité de la convection mixte en milieu de Darcy. Ce travail consiste à évaluer algébriquement et numériquement l'influence d'une perturbation sur les valeurs spectrales d'une matrice diagonalisable. Ainsi, deux approches seront possibles ; utiliser le théorème de perturbations d'une matrice dépendant d'un paramètre, dû à Lidskii et essentiellement basé sur la structure de Jordan de la matrice perturbée. La seconde approche consiste à factoriser le système matriciel puis procéder à un calcul numérique des racines du polynôme caractéristique de la matrice des perturbations. Ce problème peut être un modèle type dans les équations de la mécanique des milieux continus. Au cours de ce travail, nous avons choisi d'utiliser la seconde approche et d'utiliser comme application illustrative, le problème de la convection de Rayleigh–Bénard en milieu de Darcy, perturbée par un débit filtrant. La forme matricielle du problème est calculée à partir d'une analyse de stabilité linéaire par une méthode d'éléments finis. Nous démontrons qu'il est possible de décomposer le phénomène général en d'autres élémentaires décrits respectivement par une matrice perturbée et une perturbation. Un bon accord entre les deux méthodes a été relevé. ***Pour citer cet article : H.B. Hamed, R. Bennacer, C. R. Mecanique 336 (2008).***

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1631-0721/$ – see front matter © 2008 Published by Elsevier Masson SAS on behalf of Académie des sciences.
doi:10.1016/j.crme.2008.06.002

Keywords: Computational fluid mechanics; Eigenvalue problem; Perturbed matrices; Computing Fluid Dynamics: CFD; Mixed convection; Linear stability; Algebraic development

Mots-clés : Mécanique des fluides numérique ; Problème à valeurs propres ; Matrice perturbée ; Convection mixte ; Stabilité linéaire ; Développement algébrique

1. Introduction

The general situation is to reduce into diagonal form a linear operator describing the mixed convection in Darcy media. This operator is obtained via linear stability analysis performed with the finite elements method. At the threshold of the instabilities, the mixed convection in the porous media can be described as filtering flow destabilized by a vertical temperature gradient (or constant heat flux). It can be also considered as a Rayleigh–Bénard problem disturbed by the filtering flow. In particular, we study the pattern of formation of progressive waves in the considered porous medium heated from below in the presence of a horizontal flow. The system has an asymptotic response ad infinitum. When the system exhibits a supercritical bifurcation, a linear stability analysis is performed in order to locate the transition from the rest state towards transverse 2D rolls. The presence of a through flow breaks the rotational symmetry of the system at the supercritical instability threshold and selects transversal rolls among the infinity of unstable modes.

This Note gives the main algebraic transformations to describe the problem by a disturbed matrix, which is the Bénard convection matrix, with the perturbation matrix containing the influence of the added boundary condition. A large number of continuous media mechanics can use the same approach. This subject has at the same time a fundamental, academic and practical interest. Indeed, the mixed convection makes it possible to explain certain weather phenomena and it intervenes in industrial applications having strong economic stakes such as cooling or manufacture of electronic components. The work presented here constitutes a new pattern of determination of the threshold of instabilities and has application practises, for example, in CVD (Chemical Vapour Deposits) and other crystal growth technics. This subject is currently open and very few publications are available in the literature.

2. Solution method

Starting from the general canonical form of eigenvalues problems (cf. Eq. (1)) we can immediately see that this system has n solutions, where n is the dimension of the matrix $[E]$. We suppose that $[E]$ is invertible.

$$\big([E]-\lambda I_d\big)\{F\}=0 \tag{1}$$

I_d is the identity application. A trivial solution is possible only if the determinant of $[E-\lambda I_d]$ is null. In this way, Eq. (1) yields a set of eigenvalues λ_i (where $1\leqslant i\leqslant n$) with their corresponding eigenvectors $\{F\}_i$, $1\leqslant i\leqslant n$, which are the solutions of the system. The eigenvalues obtained are not necessary simple and can be arranged obeying the following relation:

$$|\lambda_j|\leqslant|\lambda_p|\leqslant\cdots\leqslant|\lambda_k|\leqslant\cdots\leqslant|\lambda_m|,\quad 1\leqslant j,p,k,m,\ldots\leqslant n \tag{2}$$

Now we suppose that the matrix $[E]$ can be factorized as follows:

$$[E]=[K]^{-1}[B][K_\theta]^{-1}[B],\quad \text{where} \tag{3a}$$

$$[K_\theta]=[K]-\varepsilon[B] \tag{3b}$$

The matrices $[K]$ and $[B]$ are two invertible matrices, and α is a small real number called the perturbation factor. It is then easy to remark that the matrix $[E(\varepsilon=0)]=([K]^{-1}[B])^2$ has a particular form. We will name it $[E^0]$ and consider it as a reference perturbed matrix (supposed studied previously). Such a form is classically obtained in fluid dynamics as in the Rayleigh–Bénard problem.

Now we propose to start an algebraic development in order to give a general relation between the controlling parameters and the stability of the problem considered. The goal will be to get two distinct eigenvalue problems where the first is related to the reference problem and the second focusing on the perturbation (α). We substitute $[K_\theta]$ by its expression (Eq. (3b)) in the global matrix $[E(\alpha)]$ (Eq. (3a)), and, after factorizing, get:

$$
\begin{aligned}
[E(\varepsilon)] &= [K]^{-1}[B]\big[[K]-\varepsilon[B]\big]^{-1}[B] \\
&= [K]^{-1}[B][K]^{-1}\big[I_d-\varepsilon[B][K]^{-1}\big]^{-1}[B]
\end{aligned}
$$

It is shown by linear algebra, when M is an invertible square matrix, that

$$
[I_d+M]^{-1}=\sum_{k=0}^{\infty}(-1)^k[M]^k \tag{4}
$$

This series is called the Von Neumann series and is converging only if any norm of $[M]$ is smaller than unity ($\|M\|<1$). In our case we have to respect the condition:

$$
\big\|\varepsilon[B][K]^{-1}\big\|<1 \tag{5}
$$

Using (4) and bringing out the first term of the series, then we factorize the matrix $[E(\alpha)]$

$$
\begin{aligned}
\big[E(\varepsilon)\big] &= [K]^{-1}[B][K]^{-1}\big[I_d\underbrace{-\varepsilon[B][K]^{-1}}_{[M]}\big]^{-1}[B] \\
&= \underbrace{[K]^{-1}[B][K]^{-1}[B]}_{[E^0]}[B]^{-1}\left(\sum_{k=0}^{\infty}(-1)^k\big(-\varepsilon[B][K]^{-1}\big)^k\right)[B] \\
&= \big[E^0\big]\left[I_d+[B]^{-1}\sum_{k=1}^{\infty}\big(\varepsilon[B][K]^{-1}\big)^k[B]\right]
\end{aligned}
$$

Then we can write

$$
\begin{aligned}
&\big[E(\varepsilon)\big]-\lambda I_d=0 \\
&\quad\Rightarrow \big[E^0\big]\left[I_d+[B]^{-1}\left(\sum_{k=1}^{\infty}\big(\varepsilon[B][K]^{-1}\big)^k\right)[B]\right]-\lambda I_d=0 \\
&\quad\Rightarrow \big[E^0\big]+\left[\big[E^0\big][B]^{-1}\left(\sum_{k=1}^{\infty}\big(\varepsilon[B][K]^{-1}\big)^k\right)[B]\right]-\lambda I_d=0 \\
&\quad\Rightarrow \big[E^0\big]+\big[E^{\text{perturb}}(\varepsilon)\big]-\lambda I_d=0
\end{aligned}
$$

The obtained perturbation matrix is defined as follows:

$$
\big[E^{\text{perturb}}(\varepsilon)\big]=\big[E^0\big][B]^{-1}\left(\sum_{k=1}^{\infty}\big(\varepsilon[B][K]^{-1}\big)^k\right)[B]
$$

Finally we obtain:

$$
\big(\big[E^0\big]+\big[E^{\text{perturb}}(\varepsilon)\big]\big)-\lambda I_d=0 \tag{6}
$$

Property. *If we have the following properties* $\lambda_M=\mathrm{eig}(M)$; $\lambda_X=\mathrm{eig}(X)$ *where* M *and* X *are two invertible matrices, then*:

$$
\begin{cases} M-\lambda_M I_d=0 \\ X-\lambda_X I_d=0 \end{cases} \xrightarrow{\sum} (M+X)-(\lambda_M+\lambda_X)I_d=0
$$

On the basis of the eigenvalues equation (6) *and using the above property, we have*

$$
(\lambda_{E^0})_i+(\lambda_{E^{\text{perturb}}})_i=\lambda_i \tag{7}
$$

The consequent equation (7) *will be true for all* $1\leqslant i\leqslant n$, *where* n *is the* $[E^0]$ *and* $[E^{\text{perturb}}]$ *dimension.*

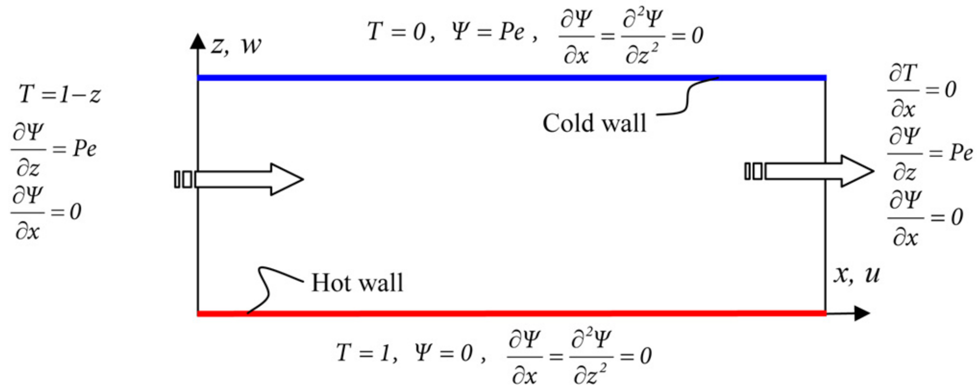

Fig. 1. The physical model and dimensionless boundary conditions.

Fig. 1. Modèle physique et conditions aux frontières adimensionnelles.

If the reference classical problem is well known ($(\lambda_{E^0})_i$), it remains to determine $(\lambda_{E^{\text{perturb}}})_m$

$$(\lambda_{E^{\text{perturb}}})_m = m\text{th eigenvalue of } \left([E^0][B]^{-1}\left(\sum_{k=1}^{\infty}\left(\varepsilon[B][K]^{-1}\right)^k\right)[B]\right) \tag{8}$$

3. Explicit solution to the Rayleigh–Bénard problem

3.1. Physical problem definition

The system to be studied consists of a horizontal rectangular cavity with an aspect ratio $A = L/H$, where L' and H' are respectively the length and the height of the cavity, filled with an incompressible Newtonian fluid. The enclosure, sketched in Fig. 1, is heated from the bottom and cooled from the top. The set of equations is given in non-dimensional form in Eq. (9), under stream function formulation. Here, $R_T = Ra/Da$ is the Darcy–Rayleigh number which expresses the balance between buoyancy and viscous forces, $Da = \mu + eK/\mu_f L'^2$ is the Darcy number, which usually takes very small values compared with unity in porous media, T is the temperature and Pr is the Prandtl number. The operator J is defined by: $J(f,g) = \frac{\partial f}{\partial z}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial z}$

$$\begin{cases} \nabla^2\psi = Pr\,R_T\dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial t} + J(\psi, T) = \nabla^2 T \end{cases} \tag{9}$$

The general solution of the system of Eqs. (9) can be decomposed, regardless of how high the R_T is imposed, into a sum of pure diffusive (ψ_C and T_C) and convective solutions and ($\tilde{\psi}$ and $\tilde{\theta}$). Thus, we introduce Peclet number such as $Pe = Re \times Pr$ where Re is the Reynolds number, and we introduce the following transformations:

$$\begin{cases} \psi = \psi_C + \tilde{\psi}(t,x,z) \\ T = T_C + \tilde{\theta}(t,x,z) \end{cases} \quad \text{where} \quad \begin{cases} \psi_C = Pe \times z \\ T_C = C_C = 1 - z \end{cases} \tag{10}$$

Substituting expressions (10) into the system (9) and making some algebraic calculus yields the following system of governing equations:

$$\begin{cases} \nabla^2\tilde{\psi} = Pr\,R_T\dfrac{\partial\tilde{\theta}}{\partial x} \\ \dfrac{\partial\tilde{\theta}}{\partial t} + \dfrac{\partial\tilde{\psi}}{\partial z}\dfrac{\partial\tilde{\theta}}{\partial x} + Pe\dfrac{\partial\tilde{\theta}}{\partial x} - \dfrac{\partial\tilde{\psi}}{\partial x}\left(\dfrac{\partial\tilde{\theta}}{\partial z} - 1\right) = \nabla^2\tilde{\theta} \end{cases} \tag{11}$$

The system (11) is the perturbation equations. The boundary conditions for the perturbations are:

$$\nabla\tilde{\theta}\cdot\vec{n}|_{x=0,A}=0,\quad \tilde{\theta}|_{z=0,1},\quad \tilde{\psi}|_{\Gamma}=0,\quad \left.\frac{\partial\tilde{\psi}}{\partial x}\right|_{\Gamma}=0\quad\text{and}\quad \left.\frac{\partial^2\tilde{\psi}}{\partial z^2}\right|_{\Gamma}=0 \tag{12}$$

The linear stability of the rest state is now investigated in terms of the governing parameters of the problem. The perturbed stream function, and temperature fields can be expressed as:

$$\begin{cases}\tilde{\psi}(x,z,t)=\psi_0 F(x,z)e^{pt}\\ \tilde{\theta}(x,z,t)=\theta_0 G(x,z)e^{pt}\end{cases} \tag{13}$$

where $F(x,z)$ and $G(x,z)$, are space functions describing the fields $\tilde{\psi}(x,z,t)$ and $\tilde{\theta}(x,z,t)$ at the onset of convection, p is a complex number which expresses the perturbation growth rate and ψ_0 and θ_0 are constants. Considering the marginal stability ($p=0$) and substituting Eqs. (13) into (11) yields (neglecting the second order terms)

$$\begin{cases}\psi_0\dfrac{1}{Pr}\nabla^2 F=R_T\theta_0\dfrac{\partial G}{\partial x}\\ Pe\,\theta_0\dfrac{\partial G}{\partial x}+\psi_0\dfrac{\partial F}{\partial x}=\theta_0\nabla^0 G\end{cases} \tag{14}$$

The numerical method used to solve system (15) and (14), is the same as described in details by [1] and only the main steps are given here. The calculus domain is discretized into four-node rectangular Hermit-cubic-elements. Using the Bubnov–Galerkin procedure, the resulting space-discretized equations are expressed as follows:

$$\begin{cases}\psi_0[K]\{F\}=R_T\theta_0[B]\{G\}\\ \psi_0[B]\{F\}=\theta_0[K_\theta]\{G\}\end{cases} \tag{15}$$

where $[B]$, $[K]$ and $[K_\theta]$ are $m\times m$ matrices, describing, respectively, the transport-term-integral, the stream function and the temperature coefficients. $\{F\}$ and $\{G\}$ are unknown vectors of size m. Here $m=4n$, where n is the total nodes number. The corresponding elementary matrices are now given:

$$\begin{aligned}&(K_{i,j})^e_{1\leqslant i,j\leqslant 4}=\int_\Omega \frac{1}{Pr}\nabla\mathcal{N}_i\nabla\mathcal{N}_j\,d\Omega;\qquad (K_{\theta_{i,j}})^e_{1\leqslant i,j\leqslant 4}=\int_\Omega\left(\nabla\mathcal{N}_i\nabla\mathcal{N}_j-Pe\frac{\partial\mathcal{N}_i}{\partial x}\mathcal{N}_j\right)d\Omega\\ &(K_{i,j})^e_{1\leqslant i,j\leqslant 4}=\int_\Omega \nabla\mathcal{N}_i\nabla\mathcal{N}_j\,d\Omega;\qquad (B_{i,j})^e_{1\leqslant i,j\leqslant 4}=\int_\Omega\frac{\partial\mathcal{N}_i}{\partial x}\mathcal{N}_j\,d\Omega\end{aligned} \tag{16}$$

The functions $\mathcal{N}_i(x,z)$ defined on the calculus domain Ω as the basis functions, are the Hermit's shape interpolation. The above integrals are evaluated using the Gauss integration algorithm.

$$\left([E]-\frac{1}{R_T}I_d\right)\{F\}=0 \tag{17}$$

The matrix $[E]=[K]^{-1}[B][K_\theta]^{-1}[B]$ is an $m\times m$ invertible square matrix composed of products of the global matrices, and I_d is the identity application. Eq. (17) can be rearranged to the classical canonical form:

$$[E-\lambda I_d]\{F\}=0 \tag{18}$$

where, λ represents the eigenvalue $\lambda=1/R_T$ and $\{F\}$ is the eigenvector related to λ. Considering that the values of the critical Rayleigh number, R_T^C, are inversely proportional to the eigenvalues, the smallest value which generates the loss of flow stability will be obtained starting from the spectral ray of the matrix.

$$\lambda_m=\frac{1}{R_T^C} \tag{19}$$

4. Application to convective flows

We first illustrate RB solution as a reference case followed by the application to mixed convection. In this case there is no inlet flow, which means that $Pe=0$, the $[K_\theta]$ is reduced to the $[K]$ matrix, and Eq. (12) is reduced to the following eigenvalue problem:

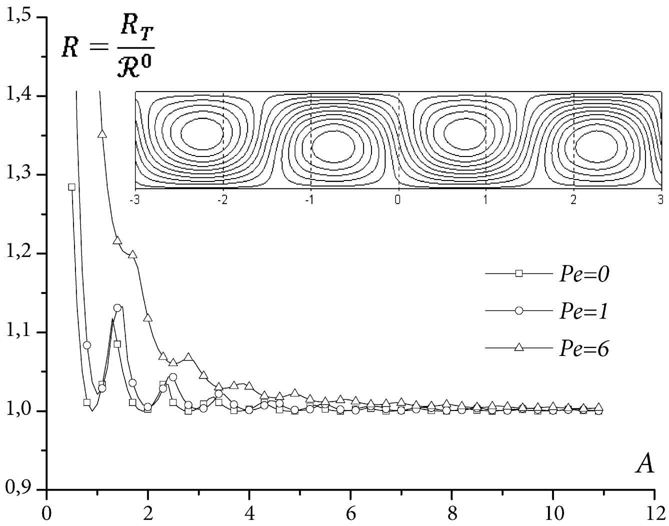

Fig. 2. Normalized Darcy–Rayleigh number in the onset of mixed convection according to the aspect ratio.

Fig. 2. Darcy–Rayleigh normalisé en fonction du rapport de forme.

$$[K]^{-1}[B][K]^{-1}[B] - \frac{1}{R_T} I_d = 0 \tag{20}$$

Remember that: $[E^0] = [K]^{-1}[B][K]^{-1}[B]$.

The critical Rayleigh number R_T^C is given according to the spectral ray of the global matrix $[E^0]$, and is only depending on the cavity aspect ratio, and converges asymptotically to a constant $\mathcal{R}^0$ for high A value (see Fig. 2, $Pe = 0$).

In the case of mixed convection, a through flow from one side of the duct is now made; in the other side, we extract the same flow in order to maintain the equation of mass continuity. It means that $Pe \neq 0$ on vertical walls. We remember that the operator describing the mixed convection is:

$$\big[E(Pe)\big] = [K]^{-1}[B][K_\theta]^{-1}[B] \tag{21}$$

The critical Rayleigh number $R_T^C = 1/\lambda_m$ is given according to the spectral ray of the global matrix $[E(Pe)]$. It also depends on the aspect ratio of the cavity, Reynolds and Prandtl numbers. A series of curves for different flow parameters is provided in Fig. 2, which illustrates the influence of the Peclet numbers, $Pe = Re \times Pr$, on the normalized critical Rayleigh number $R_T/\mathcal{R}^0$.

It is now determined that the problem considered has the same form and conditions as that of Eq. (1). Then it remains to verify condition (5) in order to transform the mixed convection on reference natural convection problem coupled to a perturbation one. The 2D transverse rolls birth for vanishing Re numbers i.e. small $Pe = Re \times Pr$, we numerically verify that:

$$\big\| Pe[B][K]^{-1} \big\| < 1 \tag{22}$$

So the problem (Eq. (20)) can be written as the sum of Rayleigh–Bénard and hydrodynamic perturbation matrices (see Eq. (6)):

$$\big[[E^0] + \big[E_{\text{hydr}}^{\text{perturb}}\big]\big] - \frac{1}{R_T} I_d = 0 \tag{23}$$

There we can conclude (based on Eq. (7)) that the critical Rayleigh related to the spectral ray of the $[E]$ matrix is:

$$R_T^C(Pe) = \frac{\mathcal{R}^0}{1 + \mathcal{R}^0 \sum_{k=1}^{+\infty} (-Pe)^k |(\lambda_k)_m|} \tag{24}$$

Now we will illustrate the formula (24) by an application using a development of the Neumann serial for the case $n = 2$. These eigenvalues are calculated with the same manner we have calculated $\mathcal{R}^0$. Fig. 3 is showing the range

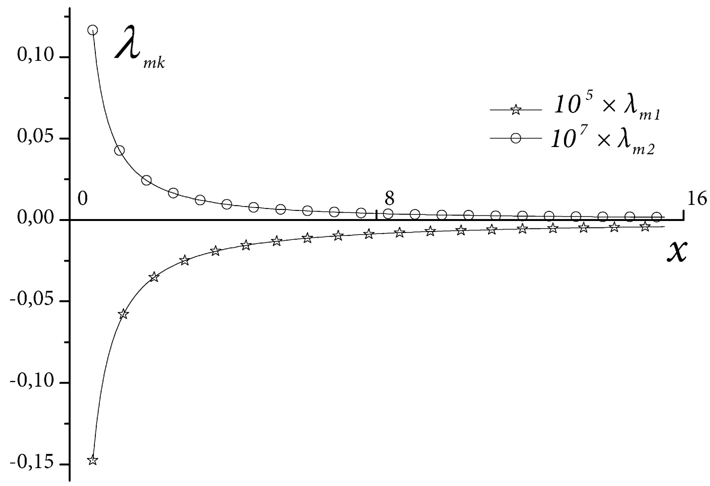

Fig. 3. Eigenvector corresponding to the most destabilizing mode.

Fig. 3. Vecteur propre correspondant au mode le plus déstabilisant.

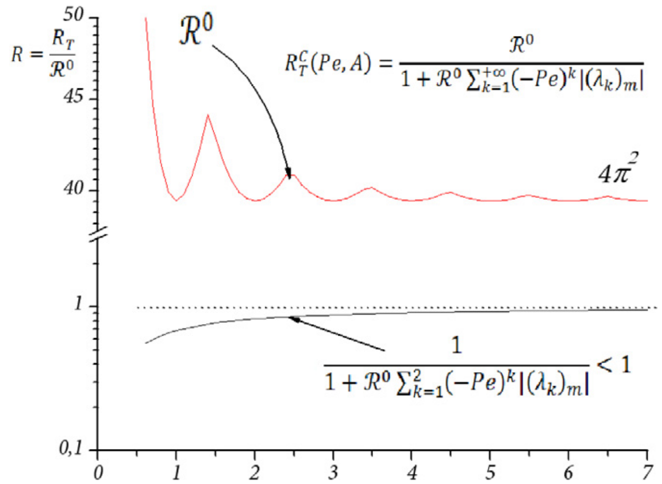

Fig. 4. Calculated perturbation coefficient. The critical value of Darcy–Rayleigh number is given by $\mathcal{R}^0$ × perturbation coefficient given by $1/(1+\mathcal{R}^0\sum_{k=1}^{+\infty}(-Pe)^k|(\lambda_k)_m|)$.

Fig. 4. Représentation du coefficient de perturbation calculé à l'ordre 2. La valeur critique du Darcy–Rayleigh est donnée par $\mathcal{R}^0$ × coefficient de perturbation, donné par $1/(1+\mathcal{R}^0\sum_{k=1}^{+\infty}(-Pe)^k|(\lambda_k)_m|)$.

of the calculated eigenvalues. These values are very small, and are decreasingly tending toward the null value, which means, firstly that the Von Neumann term is quantifying the perturbation induced by the travelling flow, and secondly that the effect of this perturbation vanish ad infinitum. The expression of critical Rayleigh is the ratio of the classical natural convection $\mathcal{R}^0$ value by and the added perturbation. In Fig. 4 we evaluate this added quantity, and show that it is close strictly lower than one for the different Reynolds values. Thus, we find that the travelling flow has stabilizing effect on the global problem. For important values of A the critical R_T^C tends to $4\pi^2$ corresponding to the classical Rayleigh–Bénard problem in Darcy Porous media [2]. The behaviour of this tendency allows us to answer the question underlined in previous works ([3] or [4]).

Finally, in Fig. 5 we show the very good agreement between the obtained critical Darcy–Rayleigh by the direct resolution of the problem (Eq. (20) and by the two separate parts of the problem using (24).

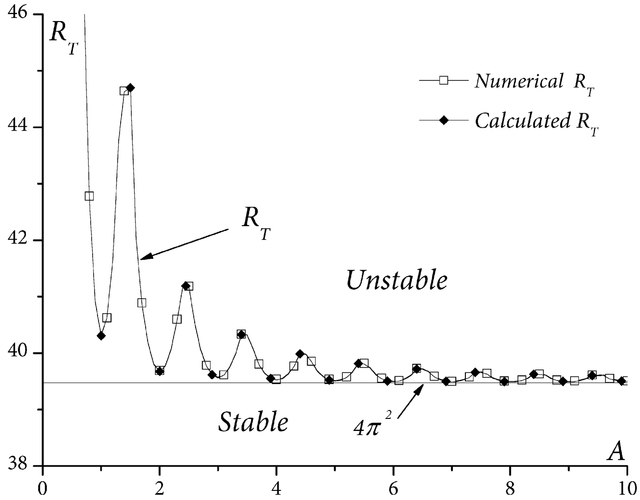

Fig. 5. Graphic comparison between calculated and numerical critical Darcy–Rayleigh numbers. $Re = 2$, $Pr = 0.71$.

Fig. 5. Comparaison graphique entre le Darcy–Rayleigh critique calculé et numérique. $Re = 2$, $Pr = 0.71$.

5. Conclusions

Theoretical development of the equation of the eigenvalues was undertaken in order to write the problem in the form of a state of reference subjected to a disturbance. The established approach is illustrated based on the effect of a through flow (due to the average flow induced from the side of the cell) on stability and bifurcations of a fluid under Rayleigh–Bénard convection (confined natural convection) in a porous medium.

An explicit general formula of critical Rayleigh is given. We numerically check the validity of the formula found.

The basic governing equations were discretized using the finite element method, which led us to the determination of the critical Rayleigh number by the reduction to the diagonal form of a global matrix containing all the parameters of the system. The effect of the aspect ratio was taken into account. It is shown that, in natural convection, the loss of stability has a close relationship with the tightening of the wavelength. In mixed convection, this effect is less and less important that the number of *Pe* is large. The critical Rayleigh number tends to the asymptotical reference value, $4\pi^2$, with the aspect ratio increase. We pursue this study on the double diffusive problem where multiple solutions are possible (see [5–11]).

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