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Asymptotics of the solution of a Dirichlet spectral problem in a junction with highly oscillating boundary

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Abstract

We study the asymptotic behavior of the eigenelements of the Dirichlet problem for the Laplacian in a bounded domain, a part of whose boundary, depending on a small parameter ε, is highly oscillating; the frequency of oscillations of the boundary is of order ε and the amplitude is fixed. We present second-order asymptotic approximations, as $\varepsilon \to 0$, of the eigenelements in the case of simple eigenvalues of the limit problem. ***To cite this article: Y. Amirat et al., C. R. Mecanique 336 (2008).***

Résumé

Approximation asymptotique des éléments propres du problème de Dirichlet pour le Laplacien dans un domaine à frontière fortement oscillante. Nous étudions le comportement asymptotique des éléments propres du problème de Dirichlet pour le Laplacien dans un domaine borné dont une partie de la frontière, dépendant d'un petit paramètre ε, est fortement oscillante ; la fréquence des oscillations est d'ordre ε et leur amplitude est fixe. Nous présentons des approximations asymptotiques d'ordre deux des éléments propres dans le cas de valeurs propres simples du problème limite. ***Pour citer cet article : Y. Amirat et al., C. R. Mecanique 336 (2008).***

Keywords: Asymptotic expansion; Spectral problem; Oscillating boundary

Mots-clés : Développement asymptotique ; Problème spectral ; Frontière oscillante

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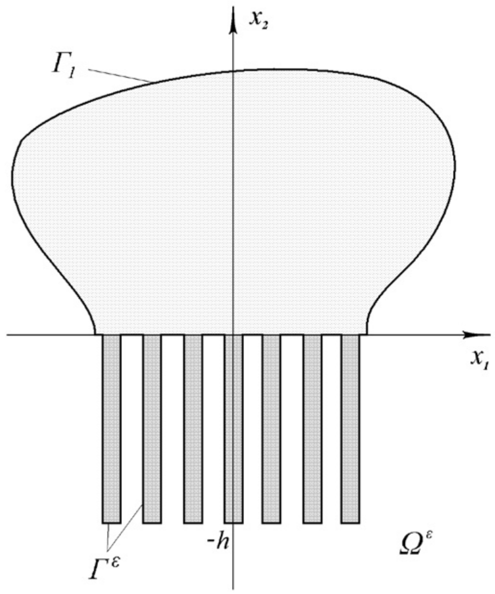

Fig. 1. Membrane with oscillating boundary.

1. Introduction and setting of the problem

Boundary-value problems involving rapidly oscillating boundaries or interfaces frequently arise when modeling problems of physics and engineering sciences, such as the scattering of acoustic and electromagnetic waves on small periodic obstacles, the free vibrations of strongly inhomogeneous elastic bodies, electric current through rough interfaces, fluids over rough walls, and coupled fluid–solid periodic structures. The mathematical analysis of these problems consists in studying the large scale behavior of the solution.

In this Note we consider a two dimensional Dirichlet spectral problem in a bounded domain, a part of whose boundary is highly oscillating. Our aim is to construct accurate asymptotic approximations of the eigenvalues and corresponding eigenfunctions. Let us mention that the case where the frequency and the amplitude of oscillations of the boundary are of the same order ε is considered in [1,2]. Other boundary conditions where considered in [3–5]. Other aspects of these problems were studied in [6–8].

Let Ω^+ be a bounded domain in $\mathbb{R}^2$, located in the upper half space. We assume the boundary $\partial\Omega^+$ to be piecewise smooth, consisting of the parts: $\partial\Omega^+ = \Gamma_0 \cup \Gamma_1$, where Γ_0 is the segment $(-\frac{1}{2}, \frac{1}{2})$ on the abscissa axis, Γ_1 coincides with lines $x_1 = -\frac{1}{2}$ and $x_1 = \frac{1}{2}$ at neighborhood of the abscissa axis. Let $\varepsilon = \frac{1}{2\mathcal{N}+1}$ be a small parameter, where $\mathcal{N} \gg 1$. Assume that $0 < a < \frac{1}{2}$, $h > 0$ and define (see Fig. 1)

$$\Omega^-_{j,\varepsilon} = \left\{x \in \mathbb{R}^2\colon\ -\varepsilon a < x_1 - \varepsilon j < \varepsilon a, -h < x_2 \leqslant 0\right\}, \qquad \Omega^-_\varepsilon = \bigcup_{j=-\mathcal{N}}^{\mathcal{N}} \Omega^-_{j,\varepsilon}$$

$$\Omega^\varepsilon = \Omega^+ \cup \Omega^-_\varepsilon, \qquad \Gamma^\varepsilon = \partial\Omega^\varepsilon \backslash \overline{\Gamma_1}$$

We consider the spectral problem

$$-\Delta u_\varepsilon = \lambda_\varepsilon u_\varepsilon \quad \text{in } \Omega^\varepsilon, \qquad u_\varepsilon = 0 \quad \text{on } \partial\Omega^\varepsilon \tag{1}$$

and study the asymptotic behavior, as $\varepsilon \to 0$, of the eigenvalue λ_ε and the corresponding eigenfunction u_ε.

2. Uniform bounds and convergence results

The following theorems deal with uniform estimates, convergence of solutions to nonhomogeneous boundary-value problems associated with (1), and convergence of eigenelements:

Theorem 2.1. *Let $F_\varepsilon \in L_2(\Omega^\varepsilon)$, let Q be an arbitrary compact in the complex plane which does not contain eigenvalues of the boundary-value problem*

$$-\Delta u_0^+ = \lambda_0 u_0^+ \quad \text{in } \Omega^+, \qquad u_0^+ = 0 \quad \text{on } \partial\Omega^+ \tag{2}$$

and let $\lambda \in Q$. Then:

(i) *the boundary-value problem*

$$-\Delta U_\varepsilon = \lambda U_\varepsilon + F_\varepsilon \quad \text{in } \Omega^\varepsilon, \qquad U_\varepsilon = 0 \quad \text{on } \partial\Omega^\varepsilon$$

has, for ε small enough, a unique solution satisfying the estimate

$$\|U_\varepsilon\|_{H^1(\Omega^\varepsilon)} \leqslant C_1 \|F_\varepsilon\|_{L_2(\Omega^\varepsilon)}$$

uniformly with respect to ε and λ;

(ii) *assume that there is $F_0 \in L_2(\Omega^+)$ such that, as $\varepsilon \to 0$,*

$$\|F_\varepsilon - F_0\|_{L_2(\Omega^+)} + \|F_\varepsilon\|_{L_2(\Omega^\varepsilon \setminus \overline{\Omega^+})} \to 0$$

and let U_0 be the solution of the boundary–value problem

$$-\Delta U_0 = \lambda U_0 + F_0 \quad \text{in } \Omega^+, \qquad U_0 = 0 \quad \text{on } \partial\Omega^+$$

Then

$$\|U_\varepsilon - U_0\|_{H^1(\Omega^+)} + \|U_\varepsilon\|_{H^1(\Omega^\varepsilon \setminus \overline{\Omega^+})} \to 0$$

uniformly with respect to λ.

Theorem 2.2. *Assume that the multiplicity of the eigenvalue λ_0 of problem* (2) *is equal to p. Then*:

(i) *there are p eigenvalues of problem* (1) (*with multiplicities taken into account*) *converging to λ_0, as $\varepsilon \to 0$;*

(ii) *if $\lambda_\varepsilon^1, \ldots, \lambda_\varepsilon^p$ are the eigenvalues of problem* (1)*, which converge to λ_0 and $u_\varepsilon^1, \ldots, u_\varepsilon^p$ are the corresponding eigenfunctions, orthonormal in the space $L_2(\Omega^\varepsilon)$, then for any sequence $\varepsilon_k \underset{k\to\infty}{\longrightarrow} 0$ there exists a subsequence $\varepsilon_{k'} \to 0$ such that*

$$\left\|u_\varepsilon^j - u_0^{+,j}\right\|_{H^1(\Omega^+)} + \|u_\varepsilon^j\|_{H^1(\Omega^\varepsilon \setminus \overline{\Omega^+})} \to 0$$

as $\varepsilon = \varepsilon_{k'} \to 0$. Here, $u_0^{+,1}, \ldots, u_0^{+,p}$ denote the eigenfunctions of problem (2)*, corresponding λ_0 and orthonormal in $L_2(\Omega^+)$.*

The proof of these results is analogous to the proof from [9].

3. Asymptotic approximation of the eigenelements

In this section we construct the asymptotics, as $\varepsilon \to 0$, of the eigenelements λ_ε, u_ε to the problem (1) by means of the method of matching asymptotic expansions [10]. We consider the case where the limit λ_0 of λ_ε, as $\varepsilon \to 0$, is a simple eigenvalue of problem (2) and we denote by u_0 the corresponding eigenfunction, normalized in $L_2(\Omega^+)$.

We first construct an asymptotic approximation of the solution of (1) in Ω^+. For this, we define a real number λ_1 and a function u_1^+ in Ω^+ satisfying the boundary-value problem

$$-\Delta u_1^+ = \lambda_0 u_1^+ + \lambda_1 u_0^+ \quad \text{in } \Omega^+, \qquad u_1^+ = q(a)\frac{\partial u_0^+}{\partial x_2} \quad \text{on } \Gamma_0, \qquad u_1^+ = 0 \quad \text{on } \Gamma_1 \tag{3}$$

where $q(a)$ is an arbitrary constant which will be specified below. By the solvability condition of problem (3), we have

$$\lambda_1 = -q(a) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\partial u_0^+}{\partial x_2}\right)^2 (x_1)\, dx_1 \tag{4}$$

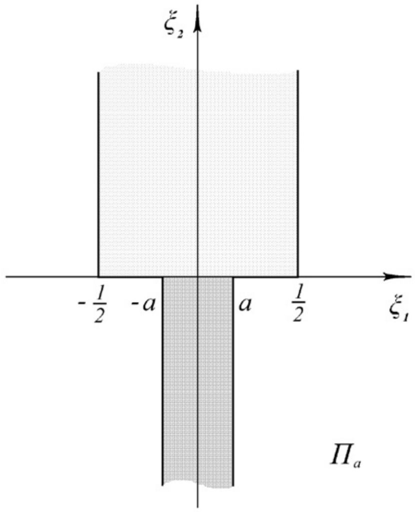

Fig. 2. Cell of periodicity.

To determine uniquely the solution of (3), we assume in addition that $\int_{\Omega^+} u_1^+(x)u_0^+(x)\,\mathrm{d}x = 0$. Denote

$$\tilde{\lambda}_\varepsilon = \lambda_0 + \varepsilon\lambda_1, \qquad \tilde{u}_\varepsilon^+ = u_0^+ + \varepsilon u_1^+ \tag{5}$$

It follows from (2) and (3) that $\tilde{u}_\varepsilon^+$ belongs to $C^\infty(\overline{\Omega^+})$ and satisfies the boundary-value problem

$$-\Delta\tilde{u}_\varepsilon^+ = \tilde{\lambda}_\varepsilon\tilde{u}_\varepsilon^+ + \tilde{f}_\varepsilon^+ \quad \text{in } \Omega^+, \qquad \tilde{u}_\varepsilon^+ = 0 \quad \text{on } \Gamma_1$$

where $\tilde{f}_\varepsilon^+ = -\varepsilon^2\lambda_1 u_1^+$. Obviously, $\|\tilde{f}_\varepsilon^+\|_{L_2(\Omega^+)} = \mathrm{O}(\varepsilon^2)$ and $\|\tilde{u}_\varepsilon^+\|_{L_2(\Omega^+)} = 1 + \mathrm{o}(1)$. The pair $(\tilde{\lambda}_\varepsilon, \tilde{u}_\varepsilon^+)$ given by (5) is defined to be an asymptotic approximation of the solution to problem (1) in Ω^+.

Let us now consider the domain Ω_ε^-. Introduce the notations (see Fig. 2):

$$\Pi^+ = \left(-\frac{1}{2}, \frac{1}{2}\right) \times (0, +\infty), \quad \Pi_a^- = (-a, a) \times (-\infty, 0), \quad \gamma(a) = (-a, a) \times \{0\}$$

$$\Pi_a = \Pi^+ \cup \Pi_a^- \cup \gamma(a), \quad \Gamma^+ = \left(\left\{-\frac{1}{2}\right\} \times (0, +\infty)\right) \cup \left(\left\{\frac{1}{2}\right\} \times (0, +\infty)\right), \quad \Gamma_a^- = \partial\Pi_a \backslash \overline{\Gamma^+}$$

$$\tilde{\Pi}_a = \overline{\Pi_a} \backslash \big(\{(-a, 0)\} \cup \{(a, 0)\}\big), \quad \Pi_a(R) = \{\xi \in \Pi_a\colon\ \xi_2 < R\}, \quad \tilde{\Pi}_a(R) = \{\xi \in \tilde{\Pi}_a\colon\ \xi_2 < R\}$$

Consider the boundary-value problem

$$\Delta_\xi X = 0 \quad \text{in } \Pi_a, \qquad X = 0 \quad \text{on } \Gamma_a^-, \qquad \frac{\partial X}{\partial \xi_1} = 0 \quad \text{on } \Gamma^+$$

One can show that problem (3) has a solution X belonging to $C^\infty(\tilde{\Pi}_a(R)) \cap H^1(\Pi_a(R))$ for any $R > 0$, even with respect to ξ_1 and having the differentiable asymptotics

$$\partial_\xi^\beta X(\xi) = \mathrm{O}\big(\mathrm{e}^{\frac{\pi}{a}\xi_2}\big), \quad \text{as } \xi_2 \to -\infty, \qquad \partial_\xi^\beta\big(X(\xi) - \xi_2 - q(a)\big) = \mathrm{O}\big(\mathrm{e}^{-2\pi\xi_2}\big), \quad \text{as } \xi_2 \to \infty$$

where now the constant $q(a)$ is

$$q(a) = \frac{a}{\pi}\big(4\ln 2 - \big[(1-2a)\ln(1-2a) + (1+2a)\ln(1+2a)\big]\big) \tag{6}$$

Then, consider the boundary-value problems:

$$\Delta_\xi \tilde{X} = \frac{\partial X}{\partial \xi_1} \quad \text{in } \Pi_a, \qquad \tilde{X} = 0 \quad \text{on } \partial\Pi_a \tag{7}$$

$$\Delta_\xi X_1 = \frac{\partial \tilde{X}}{\partial \xi_1} \quad \text{in } \Pi_a, \qquad X_1 = 0 \quad \text{on } \Gamma_a^-, \qquad \frac{\partial X_1}{\partial \xi_1} = 0 \quad \text{on } \Gamma^+ \tag{8}$$

$$\Delta_\xi X_2 = X \quad \text{in } \Pi_a, \qquad X_2 = 0 \quad \text{on } \Gamma_a^-, \qquad \frac{\partial X_2}{\partial \xi_1} = 0 \quad \text{on } \Gamma^+ \tag{9}$$

Arguing as in [1], we can prove that there exists a constant $0 < c < \frac{\pi}{a}$ such that each of problems (7)–(9) has a solution in $C^\infty(\tilde{\Pi}_a(R)) \cap H^1(\Pi_a(R))$ for any $R > 0$, with the differentiable asymptotics

$$\partial_\xi^\beta \tilde{X}(\xi) = \mathrm{O}\big(\mathrm{e}^{\mp c\xi_2}\big) \quad \text{as } \xi_2 \to \pm\infty, \qquad \partial_\xi^\beta X_j(\xi) = \mathrm{O}\big(\mathrm{e}^{c\xi_2}\big), \quad \text{as } \xi_2 \to -\infty$$

$$\partial_\xi^\beta\big(X_1(\xi) - q_1(a)\big) + \partial_\xi^\beta\left(X_2(\xi) - \frac{1}{6}\xi_2^3 - \frac{1}{2}q(a)\xi_2 - q_2(a)\right) = \mathrm{O}\big(\mathrm{e}^{-c\xi_2}\big), \quad \text{as } \xi_2 \to \infty$$

where $q_j(a)$ denote some constants. Due to the evenness of the function X, $\tilde{X}$ is odd in ξ_1, X_j is even in ξ_1 and thus $\tilde{X}$ and X_j have 1-periodic extensions in ξ_1 for which we keep the same notations $\tilde{X}$, X_j.

Consider now the function defined by

$$\tilde{v}_\varepsilon(\xi; x_1) = \varepsilon v_1(\xi; x_1) + \varepsilon^2 v_2(\xi; x_1) + \varepsilon^3 v_3(\xi; x_1) \tag{10}$$

where

$$v_1(\xi; x_1) = \alpha_0(x_1)X(\xi), \qquad v_2(\xi; x_1) = \alpha_1(x_1)X(\xi) - 2\alpha_0'(x_1)\tilde{X}(\xi)$$
$$v_3(\xi; x_1) = \alpha_2(x_1)X_2(\xi) + 4\alpha_0''(x_1)X_1(\xi) - 2\alpha_1'(x_1)\tilde{X}(\xi)$$

and $\alpha_0(x_1) = \frac{\partial u_0^+}{\partial x_2}(x_1, 0)$, $\alpha_1(x_1) = \frac{\partial u_1^+}{\partial x_2}(x_1, 0)$, $\alpha_2(x_1) = -(\alpha_0''(x_1) + \lambda_0\alpha_0(x_1))$. Denote

$$\tilde{\Omega}_\varepsilon = \overline{\Omega_\varepsilon}\Bigg\backslash\left(\bigcup_{j=-\mathcal{N}}^{\mathcal{N}}\big(\{-\varepsilon a + \varepsilon j\} \times \{0\}\big) \bigcup_{j=-\mathcal{N}}^{\mathcal{N}}\big(\{\varepsilon a + \varepsilon j\} \times \{0\}\big)\right)$$

We easily verify that the function $\tilde{v}_\varepsilon(\frac{x}{\varepsilon}; x_1)$ belongs to $C^\infty(\tilde{\Omega}^\varepsilon) \cap H^1(\Omega^\varepsilon)$ and satisfies $\tilde{v}_\varepsilon(\frac{x}{\varepsilon}; x_1) = 0$, as $x_1 = \pm\frac{1}{2}$. Moreover, for fixed $r > 0$ and sufficiently small such that Γ_1 coincides with the straight lines $x_1 = \pm\frac{1}{2}$ as $0 < x_2 < r$, the function $\tilde{v}_\varepsilon(\frac{x}{\varepsilon}; x_1)$ satisfies the boundary-value-problem

$$-\Delta\tilde{v}_\varepsilon = \tilde{\lambda}_\varepsilon\tilde{v}_\varepsilon + \tilde{f}_\varepsilon^- \quad \text{in } \Omega^\varepsilon, \qquad \tilde{v}_\varepsilon = 0 \quad \text{on } \partial\Omega^\varepsilon \cap \big((-\infty, \infty) \times (-h, r)\big) \tag{11}$$

where

$$\tilde{f}_\varepsilon^-(x) = -\varepsilon^2\left(\left(\frac{\partial^2}{\partial x_1^2} + \tilde{\lambda}_\varepsilon\right)\big(v_2(\xi; x_1) + \varepsilon v_3(\xi; x_1)\big) + 2\frac{\partial^2}{\partial x_1\partial\xi_1}v_3(\xi; x_1)\right)\Bigg|_{\xi = \frac{x}{\varepsilon}}$$

Moreover, we show that $\|\tilde{f}_\varepsilon^-\|_{L_2(\Omega_\varepsilon^-)} = \mathrm{O}(\varepsilon^{\frac{5}{2}})$. The pair $(\tilde{\lambda}_\varepsilon, \tilde{v}_\varepsilon(\frac{x}{\varepsilon}; x_1))$ given by (5) and (10) is then defined to be an asymptotic approximation of the solution of problem (1) in Ω_ε^-.

Let us introduce the functions defined, for $\xi_2 > 0$, by

$$X^+(\xi) = X(\xi) - \xi_2 - q(a), \quad X_1^+(\xi) = X_1(\xi) - q_1(a), \quad X_2^+ = X_2(\xi) - \frac{1}{6}\xi_2^3 - \frac{1}{2}q(a)\xi_2 - q_2(a)$$

and

$$\tilde{v}_\varepsilon^+(\xi; x_1) = \varepsilon v_1^+(\xi; x_1) + \varepsilon^2 v_2^+(\xi; x_1) + \varepsilon^3 v_3^+(\xi; x_1)$$

where

$$v_1^+(\xi; x_1) = \alpha_0(x_1)X^+(\xi), \qquad v_2^+(\xi; x_1) = \alpha_1(x_1)X^+(\xi) - 2\alpha_0'(x_1)\tilde{X}(\xi)$$
$$v_3^+(\xi; x_1) = \alpha_2(x_1)X_2^+(\xi) + 4\alpha_0''(x_1)X_1^+(\xi) - 2\alpha_1'(x_1)\tilde{X}(\xi)$$

Denote

$$\tilde{u}_{\varepsilon,0}(x) = \begin{cases} u_0^+(x) + \varepsilon u_1^+ + \varepsilon v_1^+\left(\dfrac{x}{\varepsilon}; x_1\right) \text{in } \Omega^+ \\ \varepsilon v_1\left(\dfrac{x}{\varepsilon}; x_1\right) \text{in } \Omega_\varepsilon^- \end{cases} \tag{12}$$

and

$$\tilde{u}_{\varepsilon,1}(x)=\begin{cases} u_0^+(x)+\varepsilon u_1^+ +\varepsilon v_1^+\left(\dfrac{x}{\varepsilon};x_1\right)+\varepsilon^2 v_2^+\left(\dfrac{x}{\varepsilon};x_1\right)\text{in } \Omega^+ \\ \varepsilon v_1\left(\dfrac{x}{\varepsilon};x_1\right)+\varepsilon^2 v_2\left(\dfrac{x}{\varepsilon};x_1\right)\ \text{in } \Omega_\varepsilon^- \end{cases} \tag{13}$$

We observe that $\tilde{u}_{\varepsilon,0}\in H^1(\Omega^\varepsilon)$ and $\tilde{u}_{\varepsilon,1}\notin H^1(\Omega^\varepsilon)$ since it has a jump as $x_2=0$. We also verify that $\|\tilde{u}_{\varepsilon,j}\|_{L_2(\Omega^\varepsilon)}\to 1$, as $\varepsilon\to 1$. Set

$$u_{\varepsilon,j}=\frac{\tilde{u}_{\varepsilon,j}}{\|\tilde{u}_{\varepsilon,j}\|_{L_2(\Omega^\varepsilon)}},\quad j=0,1 \tag{14}$$

Finally we have the following result:

Theorem 3.1. *Let u_ε be an eigenfunction, normalized in $L_2(\Omega^\varepsilon)$ and corresponding to the eigenvalue λ_ε, let $u_{\varepsilon,j}$ $(j=0,1)$ be the normalized functions defined by* (12)–(14)*, and let λ_1 and $q(a)$ be defined by* (4) *and* (6)*, respectively. We have*

$$\lambda_\varepsilon=\lambda_0+\varepsilon\lambda_1+\mathrm{O}(\varepsilon^2)$$

and

$$\|u_\varepsilon-u_{\varepsilon,0}\|_{L_2(\Omega^\varepsilon)}+\|u_\varepsilon-u_{\varepsilon,1}\|_{H^1(\Omega^+)}+\|u_\varepsilon-u_{\varepsilon,1}\|_{H^1(\Omega_\varepsilon^-)}=\mathrm{O}(\varepsilon^2).$$

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