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A gap in the continuous spectrum of an elastic waveguide ☆

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Abstract

A periodic elastic waveguide is found out such that the continuous spectrum of the elasticity problem operator contains a gap. This effect can be used for constructing elastic wave filters. ***To cite this article: S.A. Nazarov, C. R. Mecanique 336 (2008).***

Résumé

Un gap dans le spectre continu d'un guide d'onde élastique. On exhibe un guide périodique d'onde élastique tel que le spectre continu de l'opérateur du problème élastique contienne un gap. Cet effet peut être utilisé pour construire des filtres d'ondes elastiques. ***Pour citer cet article : S.A. Nazarov, C. R. Mecanique 336 (2008).***

Keywords: Elastic periodic waveguide; Gap in continuous spectrum

Mots-clés : Guide périodique d'onde élastique ; Gap dans un spectre continu

1. Statement of the elasticity problem in a waveguide

Let Π be a periodic solid, anisotropic and inhomogeneous. Π is the interior of the set

$$\overline{\Pi} = \bigcup_{j\in\mathbb{Z}} \overline{\varpi_j}, \tag{1}$$

where $\mathbb{Z} = \{0, \pm 1, \ldots\}$, $\varpi_j = \{x = (y, z)\colon\ (y, z - j) \in \varpi\}$ and ϖ is the periodicity cell, i.e., a domain with a Lipschitz boundary and a compact closure in the layer $\{x\colon\ y = (y_1, y_2) \in \mathbb{R}^2, z \in [0, 1]\}$. We assume that $\Pi \subset \mathbb{R}^3$ is a domain with a Lipschitz boundary, in particular, a connected set. The lateral side $\partial\varpi^{\#} = \{x \in \partial\varpi\colon\ z \in (0, 1)\}$ is divided into two sets γ and $\partial\varpi^{\#} \setminus \overline{\gamma}$. The union $\Gamma = \bigcup \gamma_j$ denotes the clamped surface and $\partial\Pi \setminus \overline{\Gamma}$ is free of traction. We suppose that $\mathrm{mes}_2\, \gamma > 0$.

We present the boundary value problem on the propagation and the diffraction of elastic waves in the periodic waveguide Π by means of the matrix notation (see, e.g. [1])

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$$D(-\nabla_x)^\top A(x)D(\nabla_x)u(x)=\lambda\rho(x)u(x),\quad x\in\Pi,$$
$$D\big(n(x)\big)^\top A(x)D(\nabla_x)u(x)=0,\quad x\in\partial\Pi\setminus\overline{\Gamma},\qquad u(x)=0,\quad x\in\Gamma \tag{2}$$

Here $n(x)$ is the unit outward normal and λ is a spectral parameter, the square of the frequency. Furthermore, the displacement vector $u=(u_1,u_2,u_3)^\top$ is interpreted in the fixed Cartesian coordinate system x as a column in $\mathbb{R}^3$ and $\top$ stands for transposition. The strain column of height 6

$$\varepsilon=(\varepsilon_{11},\varepsilon_{22},\varepsilon_{33},\sqrt{2}\varepsilon_{23},\sqrt{2}\varepsilon_{31},\sqrt{2}\varepsilon_{12})^\top \tag{3}$$

and the stress column σ of the same structure are in the relationship

$$\sigma(u;x)=A(x)\varepsilon(u;x),\quad \varepsilon(u;x)=D(\nabla_x)u(x), \tag{4}$$

$$D(\nabla_x)^\top=\begin{pmatrix}\partial_1 & 0 & 0 & 0 & 2^{-1/2}\partial_3 & 2^{-1/2}\partial_2\\ 0 & \partial_2 & 0 & 2^{-1/2}\partial_3 & 0 & 2^{-1/2}\partial_1\\ 0 & 0 & \partial_3 & 2^{-1/2}\partial_2 & 2^{-1/2}\partial_1 & 0\end{pmatrix},\quad \nabla_x=\begin{pmatrix}\partial_1\\ \partial_2\\ \partial_3\end{pmatrix},\ \partial_j=\frac{\partial}{\partial x_j} \tag{5}$$

The symmetric stiffness matrix A of size 6×6 in the Hooke's Law from (4) and the material density ρ are measurable functions and satisfy the boundless and positivity properties

$$c_A|a|^2\leqslant a^\top A(x)a\leqslant C_A|a|^2,\quad a\in\mathbb{R}^6,\qquad c_\rho\leqslant\rho(x)\leqslant C_\rho\quad\text{for a.a. } x\in\Pi \tag{6}$$

where c_A, C_A and c_ρ, C_ρ are positive constants. Moreover, they imply exponential perturbations of A^0 and ρ^0 which are 1-periodic in z, namely, $A^0(y,z\pm 1)=A^0(y,z)$, $\rho^0(y,z\pm 1)=\rho^0(y,z)$ and

$$\big|A(x)-A^0(x)\big|+\big|\rho(x)-\rho^0(x)\big|\leqslant c_0\exp\big(-\delta_0|z|\big),\quad \delta_0>0 \tag{7}$$

For A^0 and ρ^0, inequalities (6) with the positive constants c_{A^0}, C_{A^0} and c_{ρ^0}, C_{ρ^0} are valid as well.

Owing to possible irregularities of the boundary and coefficients of differential operators, we deal with the variational formulation of the inhomogeneous problem (2)

$$\big(AD(\nabla_x)u,D(\nabla_x)v\big)_\Pi-\lambda(\rho u,v)_\Pi=f(v),\quad v\in H_0^1(\Pi;\Gamma) \tag{8}$$

where $(\ ,\ )$ is the inner product in the Lebesgue space $L^2(\Pi)$, $H_0^1(\Pi;\Gamma)$ is the Sobolev space of functions vanishing at Γ and $f\in H_0^1(\Pi;\Gamma)^*$ is a continuous functional. If $f=0$, (8) becomes a spectral problem.

The wave phenomenon is intimately related to the continuous spectrum of problem (8)

$$\Sigma_c=\bigcup_{p\in\mathbb{N}}\big[\Lambda_p^-,\Lambda_p^+\big]\subset\mathbb{R}_+=(0,+\infty) \tag{9}$$

where $\mathbb{N}=\{1,2,\ldots\}$ and $0<\Lambda_p^-<\Lambda_p^+$, $\Lambda_p^-\to\infty$ as $p\to\infty$ (see Section 2). Namely, for $\lambda\in\Sigma_c$ the model problem, obtained from (8) by setting $A=A^0$, $\rho=\rho^0$ and $f=0$, has an oscillating solution in the form of Floquet's wave

$$\exp(i\eta z)U(y,z) \tag{10}$$

where $\eta\in[0,2\pi)$ and $U\in H^1_{\mathrm{per}}(\varpi)$ is 1-periodic in z. In the case $\lambda\notin\Sigma_c$ problem (8) enjoys the Fredholm alternative, i.e. it admits a solution $u\in H_0^1(\Pi;\Gamma)$ if and only if $f\in H_0^1(\Pi;\Gamma)^*$ satisfies the orthogonality condition $f(v)=0$ for $v\in K$ where K is the subspace of solutions to the homogeneous problem (8), $\dim K<\infty$.

The main goal of this Note is to demonstrate that the continuous spectrum (9) may possess a gap. More precisely, we construct a family of waveguides (1), for which

$$\max\big\{\Lambda_1^+,\ldots,\Lambda_5^+\big\}<\Lambda_6^- \tag{11}$$

The numbers in (11) are but the endpoints of the desired gap. We emphasize that the existence of the gap is guarantied under a restriction on the shape of ϖ and the constants c_{A^0}, C_{A^0} and c_{ρ^0}, C_{ρ^0} only.

In the literature the existence results for gaps in the continuous spectrum are mainly related to scalar problems in the mathematic physics (see [2,3] and others). The author does not have met a similar result in elasticity. The gaps under discussion can be used for constructing elastic wave filters and dampers.

2. The spectral pencil on the periodicity cell

The partial Gel'fand transform

$$u(y,z)\mapsto \mathbf{u}(y,z,\eta)=\frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\exp\bigl(-\mathrm{i}\eta(z+k)\bigr)v(y,z+k) \tag{12}$$

(see [4] and, e.g., [5,6]) realizes the isometric isomorphism $L^2(\Pi)\cong L^2(0,2\pi;L^2(\varpi))$. Here $L^2(0,2\pi;L^2(\varpi))$ is the Lebesgue space of abstract functions with the norm

$$\bigl\|\mathbf{u};L^2\bigl(0,2\pi;L^2(\varpi)\bigr)\bigr\|=\Biggl(\int_0^{2\pi}\bigl\|\mathbf{u}(\cdot,\eta);L^2(\varpi)\bigr\|^2\,\mathrm{d}\eta\Biggr)^{1/2}$$

Note that $(y,z)\in\Pi$ on the left of (12), but $(y,z)\in\varpi$ on the right. Furthermore, transform (12) establishes the isomorphisms $H^1(\Pi)\approx L^2(0,2\pi;H^1_{\mathrm{per}}(\varpi))$ and $H^1(\Pi)^*\approx L^2(0,2\pi;H^1_{\mathrm{per}}(\varpi)^*)$. The Gel'fand transform and the corresponding Perceval theorem correlate the family of problems in the periodicity cell

$$\langle\mathbf{u},\mathbf{v}\rangle_\eta:=\bigl(A^0D_\eta(\nabla_x)\mathbf{u},D_{\bar\eta}(\nabla_x)\mathbf{v}\bigr)_\varpi=\lambda\bigl(\rho^0\mathbf{u},\mathbf{v}\bigr)_\varpi,\quad \mathbf{v}\in H^1_{0,\mathrm{per}}(\varpi;\gamma) \tag{13}$$

with the model problem in the waveguide Π. Here $D_\eta(\nabla_x)=D(\nabla_y,i\eta+\partial/\partial z)$ and $\eta\in[0,2\pi)$.

Since the problem involves the square of the spectral parameter η, the quadratic pencil $\eta\mapsto\mathfrak{A}(\eta;\lambda)$ is associated with (13). According to a general result in [7] on holomorphic pencils, eigenvalues of $\mathfrak{A}$ are normal on the complex plane $\mathbb{C}$ and have no finite accumulation point. Furthermore, the spectrum of $\mathfrak{A}$ is invariant with respect to the shifts $\pm2\pi$ along the real axis $\mathbb{R}$.

For a real η, the form on the left-hand side of (13) is Hermitian. Let $\eta\in[0,2\pi)$ be fixed and let $\mathcal{H}_\eta$ be the Sobolev space $H^1_{0,\mathrm{per}}(\varpi;\gamma)\subset H^1_0(\varpi;\gamma)$ of 1-periodic in z functions with the specific inner product $\langle\,,\rangle_\eta$. The operator T_η, defined by

$$\langle T_\eta\mathbf{u},\mathbf{v}\rangle_\eta=\bigl(\rho^0\mathbf{u},\mathbf{v}\bigr)_\varpi,\quad \mathbf{u},\mathbf{v}\in\mathcal{H}_\eta \tag{14}$$

is positive, compact, and symmetric, therefore, self-adjoint. Thus, the essential spectrum of T_η coincides with $M^\infty=0$ and the discrete spectrum forms the infinitesimal positive sequence

$$M_1(\eta)\geqslant M_2(\eta)\geqslant\cdots\geqslant M_p(\eta)\geqslant\cdots\to+0 \tag{15}$$

where eigenvalues are listed according to multiplicity. By the definition in (13) and (14), $\Lambda_p(\eta)=M_p(\eta)^{-1}$ with $p\in\mathbb{N}$ become eigenvalues of problem (13) with the parameter $\eta\in\mathbb{R}$ fixed. They depend on η continuously and, by an obvious argument, 2π-periodically. The following assertion is proved in [8] (see also [5,6]).

Theorem 2.1. *The operator of problem* (8)

$$H^1_0(\Pi;\Gamma)\ni u\mapsto f\in H^1_0(\Pi;\Gamma)^* \tag{16}$$

is Fredholm if and only if the segment $[0,2\pi)\subset\mathbb{R}\subset\mathbb{C}$ *is free of the spectrum of the pencil* $\eta\mapsto\mathfrak{A}(\eta;\lambda)$.

We now are in position to conclude that the essential spectrum of problem (8) is but the set (9) where

$$\pm\Lambda_p^\pm=\max\bigl\{\pm\Lambda_p(\eta)\bigm|\eta\in[0,2\pi)\bigr\} \tag{17}$$

To make this conclusion precise, we endow $\mathcal{H}:=H^1_0(\Pi;\Gamma)$ with the specific inner product

$$\langle u,v\rangle=\bigl(AD(\nabla_x)u,D(\nabla_x)v\bigr)_\Pi \tag{18}$$

and introduce the positive, continuous, and self-adjoint operator T by the formula

$$\langle Tu,v\rangle=(\rho u,v)_\Omega,\quad u,v\in H^1_0(\Pi;\Gamma) \tag{19}$$

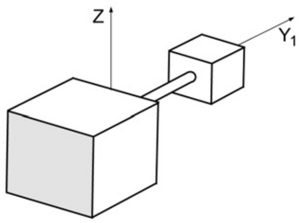

Fig. 1. The periodicity cell.

Since Π is not bounded, T cannot be compact. We emphasize especially the relationship $\mu = \lambda^{-1}$ between eigenvalues of T and problem (8), respectively. The essential spectrum of T coincides with

$$\{0\} \cup \{\mu \in \mathbb{R}_+ \colon \mu^{-1} \in \Sigma_c\}. \tag{20}$$

Note that, by a result in [8], the kernel of mapping (16) is finite-dimensional even in the case when the segment $[0, 2\pi)$ contains a point of the spectrum of the pencil $\eta \mapsto \mathfrak{A}(\eta; \lambda)$. This means that the second set in (20) is covered with the continuous spectrum of T.

3. Korn's inequalities

The formula

$$\|u; H^1(\varpi)\|^2 \leqslant c(\varpi)\|D(\nabla_x)u; L^2(\varpi)\|^2, \quad u \in H_0^1(\varpi; \gamma) \tag{21}$$

is known as the Korn inequality (see [9,10] and others). The change $u(x) \mapsto \mathbf{u}(x) = \exp(-i\eta z)u(x)$, $\eta \in [0, 2\pi)$, turns (21) into the relation

$$\|\mathbf{u}; H^1(\varpi)\|^2 \leqslant C'(\varpi)\|D_\eta(\nabla_x)\mathbf{u}; L^2(\varpi)\|^2, \quad \mathbf{u} \in H_0^1(\varpi; \gamma)$$

which, together with the positivity condition (6), ensures $\langle\, , \,\rangle_\eta$ to be an inner product in the Hilbert space $\mathcal{H}_\eta$. Furthermore, summing inequalities (21) written for the cells ϖ_j in (1), we derive the Korn inequality for $u \in H_0^1(\Pi; \Gamma)$ which proves (18) to be an inner product in $\mathcal{H}$.

Let the periodicity cell ϖ be composed from the two cubes

$$\varpi^- = (-1, 0) \times (-1/2, 1/2) \times (0, 1), \qquad \varpi^+ = (1, 3/2) \times (-1/4, 1/4) \times (1/4, 3/4)$$

connected by the thin cylinder $\varpi^h = \{x\colon\ y_1 \in [0, 1), |y_2|^2 + |z - 1/2|^2 < h^2\}$ of radius $h \in (0, 1/4]$ (see Fig. 1). Let also the face $\gamma = \{-1\} \times (-1/2, 1/2) \times (0, 1)$ of the bigger cube, shown shaded on Fig. 1, be clamped. If $h \to 0$, the periodicity cell implies a junction of two massive bodies and a thin rod. A method developed in [11], permits one to make Korn's inequality on the junction asymptotically sharp with respect to the small geometrical parameter h by distributing weights in Sobolev norms of the displacements u_1, u_2 and u_3. In the sequel we use only the following consequence of this inequality:

Theorem 3.1. *Let the field $u \in H_0^1(\varpi; \gamma)$ on the periodicity cell described above, verify the orthogonality conditions*

$$\int_{\varpi^+} u_p(x)\,\mathrm{d}x = 0, \quad p = 1, 2, \qquad \int_{\varpi^+} (x - x^+) \times u(x)\,\mathrm{d}x = 0 \in \mathbb{R}^3 \tag{22}$$

where $\times$ stands for the vector product in the Euclidean space $\mathbb{R}^3$ and x^+ for the mass center of ϖ^+. Then the estimate

$$\|u; H^1(\varpi^-)\|^2 + h^2\|u; H^1(\varpi^h \cup \varpi^+)\|^2 \leqslant C_\varpi\|D(\nabla_x)u; L^2(\varpi)\|^2 \tag{23}$$

is valid with a constant C_ϖ which depends on neither $h \in (0, 1/4]$, nor u.

Let us demonstrate that, without the orthogonality conditions (22), the constant C_ϖ in (23) cannot be independent of h. We set

$$u^p(x) = e_{1+p}\chi(x_1) - x_{1+p}e_1\frac{\partial\chi}{\partial x_1}(x_1), \qquad u^{2+p}(x) = e_{1+p}x_1\chi(x_1) - x_{1+p}e_1\frac{\partial}{\partial x_1}\big(x_1\chi(x_1)\big)$$
$$u^5(x) = (x_3e_2 - x_2e_3)\chi(x_1) \tag{24}$$

where e_q and e_3 are the unit vectors of the axes $x_q = y_q$ and $x_3 = z$, respectively, $p, q = 1, 2$, and $\chi \in C^\infty(\mathbb{R})$ is a cut-off function which equals 1 for $x_1 \geqslant 1$ and 0 for $x_1 \leqslant 0$. The restriction of u^q on ϖ^+ becomes a rigid motion which violates at least one of the orthogonality conditions in (22). Furthermore,

$$\big\|u^q; L^2(\varpi)\big\|^2 \geqslant \big\|u^q; L^2\big(\varpi^+\big)\big\|^2 \geqslant \mathbf{c} > 0, \quad q = 1, \ldots, 5 \tag{25}$$

At the same time, all non-trivial strains $\varepsilon_{jk}(u^q)$ are presented in the list

$$\varepsilon_{11}\big(u^p; x\big) = -x_{1+p}\frac{\partial^2\chi}{\partial x_1^2}(x_1), \qquad \varepsilon_{11}\big(u^{2+p}; x\big) = -x_{1+p}\frac{\partial^2}{\partial x_1^2}\big(x_1\chi(x_1)\big)$$
$$\varepsilon_{1+p1}\big(u^5; x\big) = \frac{1}{2}(-1)^p x_{1+p}\frac{\partial\chi}{\partial x_1}(x_1)$$

and do not vanish on the ligament ϖ^h only. Hence,

$$\big\|D(\nabla_x)u^q; L^2(\varpi)\big\|^2 \leqslant \mathbf{C}h^4, \quad q = 1, \ldots, 5 \tag{26}$$

and inequality (23) cannot hold true for the displacement fields (24).

4. A gap in the continuous spectrum

We employ the max–min principle (see, e.g., [12, Theorem 10.2.2]) for the operator $-T_\eta$ in (14):

$$-M_q(\eta) = \max_{\mathcal{E}_q\subset\mathcal{H}_\eta}\ \inf_{\mathbf{u}\in\mathcal{E}_q\setminus\{0\}} \frac{\langle -T_\eta\mathbf{u}, \mathbf{u}\rangle_\eta}{\langle\mathbf{u}, \mathbf{u}\rangle_\eta} \tag{27}$$

Here $\mathcal{E}_q$ is a subspace in $\mathcal{H}_\eta$ of co-dimension $q-1$, i.e., $\dim(\mathcal{H}_\eta \ominus \mathcal{E}_q) = q-1$. For $q = 5$, any $\mathcal{E}_5$ contains a non-trivial linear combination $\mathbf{u}(x) = \exp(-\mathrm{i}\eta z)(a_1u^1(x) + \cdots + a_5u^5(x))$ and, therefore, by (25) and (26), we get

$$-M_5(\eta) \leqslant -\frac{(\rho^0u, u)_\varpi}{(\mathcal{A}^0D(\nabla_x)u, D(\nabla_x)u)_\varpi} \leqslant -\frac{c_{\rho^0}\mathbf{c}}{C_{A^0}\mathbf{C}h^4} \quad\Longleftrightarrow\quad \Lambda_5(\eta) \leqslant c_\Lambda h^4 \tag{28}$$

On the other hand, let $\mathcal{E}_6$ consists of vector functions $\mathbf{u} \in H^1_{0,per}(\varpi; \gamma)$ such that $u = \exp(\mathrm{i}\eta z)\mathbf{u}$ verifies (22). By virtue of (27) and (23), we have

$$-M_6(\eta) \geqslant \inf_u \frac{-(\rho^0u, u)_\varpi}{(A^0D(\nabla_x)u, D(\nabla_x)u)_\varpi} \geqslant -\frac{C_{\rho^0}C_\varpi}{c_{A^0}h^2} \quad\Longleftrightarrow\quad \Lambda_6(\eta) \geqslant C_\Lambda h^2,\ C_\Lambda > 0 \tag{29}$$

We take $h_0 > 0$ such that $c_\Lambda h_0^4 \leqslant C_\Lambda h_0^2$. Then, for $h \in (0, h_0)$ we find out that, in view of (28), (29) and (17), relation (11) is valid that ensures the desired gap in the continuous spectrum (9).

Assuming a single small cube in Π to be sufficiently heavy (see figure and cf. [13]), one readily gets an eigenvalue of problem (2) on the interval $(0, \Lambda_1^-)$, i.e., in the discrete spectrum below the threshold. An example of an eigenvalue inside the gap is not known yet.

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