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BiGlobal stability computations on curvilinear meshes

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Abstract

A numerical method is developed to perform BiGlobal stability computations on structured curvilinear meshes. The Linearized Euler Equations for an incompressible planar flow are considered. Perturbations are sought for in normal form, leading to a differential eigenvalue problem, which can be discretized on a Cartesian computational domain through a spectral collocation method based on Chebyshev polynomials. The Jacobian of the non-analytical coordinate change from the computational domain to the physical curvilinear domain is also calculated numerically using the same spectral method. This procedure is tested on several test cases with comparison to reference solutions obtained by 1D stability calculations. ***To cite this article: F. Longueteau, J.-P. Brazier, C. R. Mecanique 336 (2008).***

Résumé

Calculs de stabilité BiGlobale sur des maillages curvilignes. Une méthode numérique a été développée pour effectuer des calculs de stabilité BiGlobale sur des maillages curvilignes structurés. On considère les équations d'Euler linéarisées pour un écoulement incompressible bidimensionnel. La recherche des perturbations sous forme de modes normaux conduit à un problème différentiel aux valeurs propres, qui peut être discrétisé sur un maillage de calcul cartésien grâce à une méthode de collocation spectrale basée sur les polynômes de Tchebichev. La matrice jacobienne du changement de variables non analytique entre le domaine de calcul et le domaine physique curviligne est aussi calculée numériquement par la même méthode spectrale. Cette procédure est validée sur plusieurs cas tests par comparaison avec des solutions de référence obtenues par des calculs de stabilité 1D. ***Pour citer cet article : F. Longueteau, J.-P. Brazier, C. R. Mecanique 336 (2008).***

Keywords: Fluid mechanics; Linearized Euler Equations; BiGlobal stability; Spectral collocation

Mots-clés : Mécanique des fluides ; Équations d'Euler linéarisées ; Stabilité BiGlobale ; Collocation spectrale

Version française abrégée

Cette Note décrit une méthode numérique, appelée « transformation implicite », adaptée aux calculs de stabilité BiGlobale sur des maillages curvilignes structurés.

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Les équations de base sont les équations d'Euler pour un écoulement incompressible bidimensionnel plan, linéarisées autour d'un écoulement moyen (EEL). Dans le cadre de la théorie de la stabilité BiGlobale [2], les perturbations de vitesse et de pression sont recherchées sous la forme de modes normaux de pulsation ω et dont les amplitudes complexes dépendent des deux variables d'espace. L'introduction de cette forme dans les EEL produit un système différentiel aux valeurs propres généralisé (GEVP). Ce GEVP exprimé avec les variables physiques n'est en général pas directement discrétisable sur un maillage curviligne par les méthodes de collocation spectrale. L'idée est de définir sur le carré $[-1,1] \times [-1,1]$ un maillage de calcul cartésien, constitué des points de collocation et comportant le même nombre de points dans chaque direction que le maillage physique curviligne. On définit ensuite une application bijective $\mathcal{M}$ qui associe à chaque point (ξ_i, η_j) du maillage de calcul un point (x_{ij}, y_{ij}) du maillage physique (cf. Fig. 1). Le GEVP peut alors être réécrit en fonction des variables (ξ, η) à l'aide de la matrice jacobienne de $\mathcal{M}^{-1}$. Il peut ensuite être discrétisé dans chaque direction par une méthode de collocation spectrale basée sur les polynômes de Tchebichev [3]. Dans le cas général, $\mathcal{M}$ n'est pas définie par des formules analytiques mais seulement de manière implicite par l'association des points deux à deux. Mais comme les valeurs de x et y sont données par définition en chaque point de collocation, la matrice jacobienne de $\mathcal{M}$ peut être calculée elle aussi par la méthode de collocation spectrale. La matrice jacobienne de $\mathcal{M}^{-1}$ s'en déduit alors simplement par inversion. Le nouveau GEVP matriciel ainsi obtenu est ensuite résolu par un algorithme de calcul partiel du spectre autour d'une cible σ : l'algorithme d'Arnoldi [4].

Le code de calcul BIGSAM [5], pour BiGlobal Stability computations on an Arbitrary Mesh, résulte de l'implémentation de cette méthode numérique. On présente ensuite deux exemples de cas tests, pour lesquels l'écoulement moyen est une couche de mélange en tanh. La géométrie et l'écoulement moyen de ces cas tests sont académiques mais ils présentent une symétrie permettant d'effectuer une étude de stabilité 1D servant de référence (cf. Figs. 2 et 4) pour la comparaison avec les calculs BiGlobaux directs. Dans les deux cas, les pulsations et les structures spatiales bidimensionnelles des modes sont retrouvées avec une excellente précision (cf. Figs. 3 et 5).

Ces bons résultats laissent penser que la méthode de « transformation implicite » permettra d'étudier la stabilité BiGlobale d'écoulements complexes plus réalistes. L'extension aux fluides compressibles et la prise en compte d'une troisième composante de l'écoulement moyen pourront également être envisagées.

1. Introduction

Modal stability analysis can give significant insight on flow behaviour over cavities [1] but up to now, computations with spectral methods were mainly limited to rectangular geometries. The aim of this Note is to present a numerical method devoted to BiGlobal flow stability computations on curvilinear domains. As a first step, only two-dimensional incompressible flows are addressed. First, the set of equations to be solved is derived and the numerical method to solve the problem on structured curvilinear meshes is described. Then, the method is validated on two test cases with curvilinear geometries, with reference to quasi-1D solutions.

2. Development of the "implicit mapping" method

2.1. Linearized Euler Equations

A planar flow of incompressible fluid is considered, where the velocity $\mathbf{U} = (u, v)^t$, the pressure p and the constant density ρ satisfy the Euler equations. Following the classical stability approach, the flow variables are split into a mean part (subscript "0") and a small perturbation (subscript "1"), with $\varepsilon \ll 1$:

$$\mathbf{U} = \mathbf{U}_0 + \varepsilon \mathbf{U}_1 \quad \text{and} \quad p = p_0 + \varepsilon p_1 \tag{1}$$

Introducing the set of Eqs. (1) into Euler equations and selecting the first order terms in ε leads to the Linearized Euler Equations (LEE) which are the basic equations of our problem.

2.2. Harmonic perturbations

Next, according to the BiGlobal stability formulation [2], the perturbations are sought for in modal form of complex angular frequency ω with complex amplitudes depending upon the two Cartesian space coordinates x and y. Setting

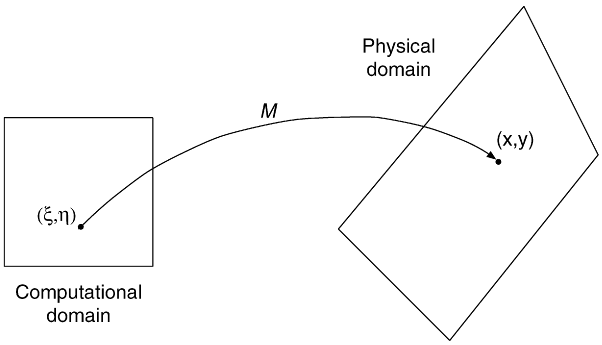

Fig. 1. The mapping between the computational and the physical domains.

$\mathbf{F} = (F, G)^t$, they read:

$$\mathbf{U}_1(x, y, t) = \mathbf{F}(x, y)\,\mathrm{e}^{-\mathrm{i}\omega t} \quad \text{and} \quad p_1(x, y, t) = P(x, y)\,\mathrm{e}^{-\mathrm{i}\omega t} \tag{2}$$

Introducing Eqs. (2) into the LEE, together with homogeneous boundary conditions, a generalized eigenvalue problem (GEVP) is obtained in which eigenelements are the vector $\mathbf{Z} = (F, G, P)^t$ and the angular frequency ω:

$$\begin{cases} \dfrac{\partial F}{\partial x} + \dfrac{\partial G}{\partial y} = 0 \\ \left(u_0\dfrac{\partial}{\partial x} + v_0\dfrac{\partial}{\partial y} + \dfrac{\partial u_0}{\partial x}\right)F + \dfrac{\partial u_0}{\partial y}G + \rho^{-1}\dfrac{\partial P}{\partial x} = \mathrm{i}\omega F \\ \dfrac{\partial v_0}{\partial x}F + \left(u_0\dfrac{\partial}{\partial x} + v_0\dfrac{\partial}{\partial y} + \dfrac{\partial v_0}{\partial y}\right)G + \rho^{-1}\dfrac{\partial P}{\partial y} = \mathrm{i}\omega G \end{cases} \tag{3}$$

This system can be written in matrix form:

$$\mathbf{A}(x, y, \partial_x, \partial_y)\mathbf{Z}(x, y) = \omega\mathbf{B}\mathbf{Z}(x, y) \tag{4}$$

2.3. Change of coordinates

In order to discretize the problem (4), the idea is to consider an application $\mathcal{M}$, bijective and differentiable, between a square computational mesh $\mathcal{D}_c$ and the mesh $\mathcal{D}_p$ in the physical space (Fig. 1). $\mathcal{M}$ defines an implicit mapping between two sets of coordinates: $\chi = (\xi, \eta)$ in $\mathcal{D}_c$ and $X = (x, y)$ in $\mathcal{D}_p$.

$$\mathcal{M} : \begin{pmatrix} \mathcal{D}_c \longmapsto \mathcal{D}_p \\ \chi \longrightarrow X \end{pmatrix}$$

Both $\mathcal{D}_c$ and $\mathcal{D}_p$ are assumed structured. They have the same number of points in each direction, so that each point (ξ_i, η_j) of $\mathcal{D}_c$ is implicitly associated by $\mathcal{M}$ to a unique point (x_{ij}, y_{ij}) in $\mathcal{D}_p$.

2.4. The GEVP in the computational domain

The goal now is to express the GEVP (4) completely in terms of χ instead of X. Let $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})^t$ and $\nabla' = (\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta})^t$. A direct application of the chain derivation rule gives $\nabla = \mathbf{L}^t \cdot \nabla'$, where $\mathbf{L} = \nabla\chi$. Using this relation, all the x and y derivatives of the flow variables can be replaced by ξ and η derivatives. Now the coefficients of the matrix $\mathbf{L}$ have to be evaluated. As $\mathcal{M}$ is differentiable, the Jacobian matrix $\mathbf{J}$ of $\mathcal{M}$ exists and it is defined by $\mathbf{J} = \nabla'\mathbf{X}$. Furthermore, since $\mathcal{M}$ is bijective, the matrix $\mathbf{J}$ is non-singular and $\mathbf{L} = \mathbf{J}^{-1}$. So it stems, by a simple 2×2 matrix inversion:

$$\mathbf{L} \equiv \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix} = \frac{1}{\det(\mathbf{J})}\begin{pmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} \\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi} \end{pmatrix} \tag{5}$$

Thereby, all the x and y derivatives of ξ and η can be expressed in terms of ξ and η derivatives of x and y. Hence the GEVP (4) can be completely expressed with the computational variables ξ and η:

$$\tilde{\mathbf{A}}(\xi, \eta, \partial_\xi, \partial_\eta)\tilde{\mathbf{Z}}(\xi, \eta) = \omega\tilde{\mathbf{B}}\tilde{\mathbf{Z}}(\xi, \eta) \tag{6}$$

2.5. Discretization

The problem (6) is discretized in each direction using a spectral collocation method based on Chebyshev polynomials [3]. The computational mesh is defined by the Gauss–Lobatto grid on $[-1, 1] \times [-1, 1]$, in which M collocation points are taken in ξ-direction and N in η-direction. The metric derivatives in the right-hand term of (5) can be also computed with the spectral derivation formulae, since the x and y coordinates of the physical point corresponding to each collocation point are known. The discretization of problem (6) leads to a matrix GEVP $\hat{\mathbf{A}}\hat{\mathbf{Z}} = \omega\hat{\mathbf{B}}\hat{\mathbf{Z}}$ with $3 \times M \times N$ coupled equations.

2.6. Numerical resolution

The matrix $\hat{\mathbf{A}}$ is a large non-sparse non-symmetric complex matrix and $\hat{\mathbf{B}}$ is singular. The GEVP is solved with a projection method suitable to this case, Arnoldi's algorithm [4], which computes only a part of the spectrum around a given target σ.

The "implicit mapping" and the resolution method explained above have been implemented in a code named BIGSAM [5] for BiGlobal Stability computations on an Arbitrary Mesh.

3. Validation of the "implicit mapping" method

3.1. Methodology

Many computations have been made to validate the method. Only the most striking results are presented here. We consider two-dimensional test-cases in which geometries and base flows have a periodicity in one spatial direction, say s. The perturbations are then sought in the form $Z(\tau)\,\mathrm{e}^{\mathrm{i}(\alpha s-\omega t)}$, where τ denotes the remaining spatial coordinate and α is the wave number in s direction. A temporal 1D stability computation is then performed. The reference 2D spatial solution is obtained by multiplying the 1D solution $Z(\tau)$ by $\mathrm{e}^{\mathrm{i}\alpha s}$.

Next, a BiGlobal stability computation is performed with BIGSAM using the reference eigenvalue ω_{1D} obtained before as a target for Arnoldi's algorithm. Finally, the eigenelements obtained with both methods are compared. For simplicity and conciseness, only results concerning the real part of the pressure P will be presented.

3.2. The planar channel

First a planar channel of length $L = 4$ and width $H = 2$ is considered, with Cartesian coordinates $x \in [0, L]$ and $y \in [-H/2, H/2]$ (Fig. 2, left). The base flow is a spatial mixing layer developing in x direction

$$u_0 = u_1 + \frac{1}{2}(u_2 - u_1)\left[1 + \tanh\left(\frac{y}{b}\right)\right], \quad v_0 = 0, \quad \rho = 1 \tag{7}$$

with upper velocity $u_2 = 1.3$, lower velocity $u_1 = 0.7$ and thickness $b = 0.4$. A slip condition is imposed at the walls and a periodicity condition is imposed on all the three unknowns F, G, P at the outflow boundary.

For the temporal 1D stability, the perturbation is sought in the form $Z(y)\,\mathrm{e}^{\mathrm{i}(\alpha x-\omega t)}$ with $\alpha = 2\pi/L$. Two Kelvin–Helmholtz hydrodynamic modes are obtained. We will focus on the amplified one, the angular frequency of which is $\omega_{1D} = 1.571 + 0.0913\mathrm{i}$. Fig. 2 (right) shows the real part of the pressure for this mode.

For the BiGlobal computation, we keep the same base flow and boundary conditions but the channel is rotated by an angle of 60 degrees, and the mesh is "swayed" with a sinus-like profile (Fig. 3, left). The mesh has 31 points in ξ and 25 points in η. The BiGlobal frequency found is $\omega_{2D} = 1.566 + 0.0981\mathrm{i}$. The relative error in modulus is lower than 1%. As it can be seen in Fig. 3 (right), the real part of the pressure associated with ω_{2D} is identical to the quasi-1D solution.

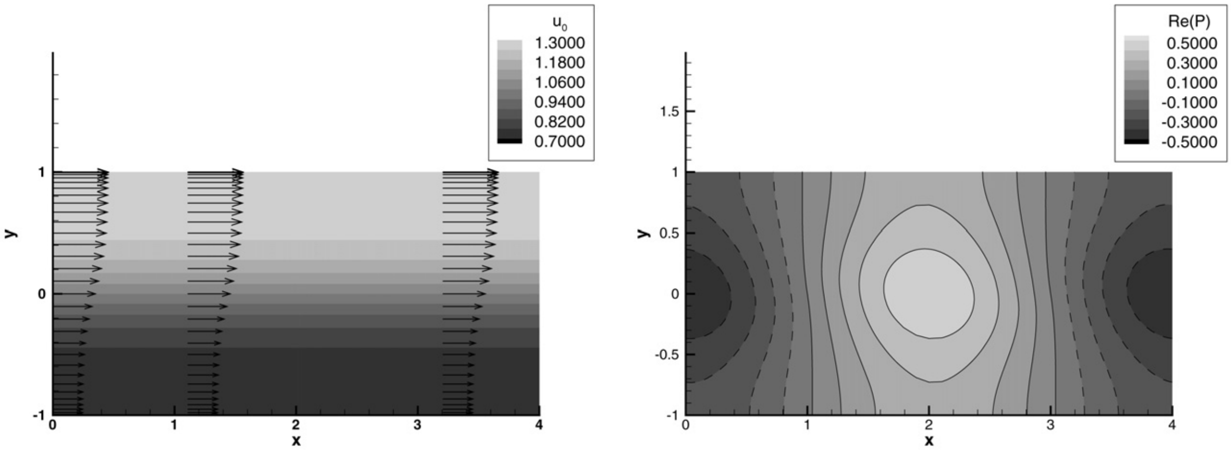

Fig. 2. Planar channel, reference solution: base flow (left), amplified eigenvector (right).

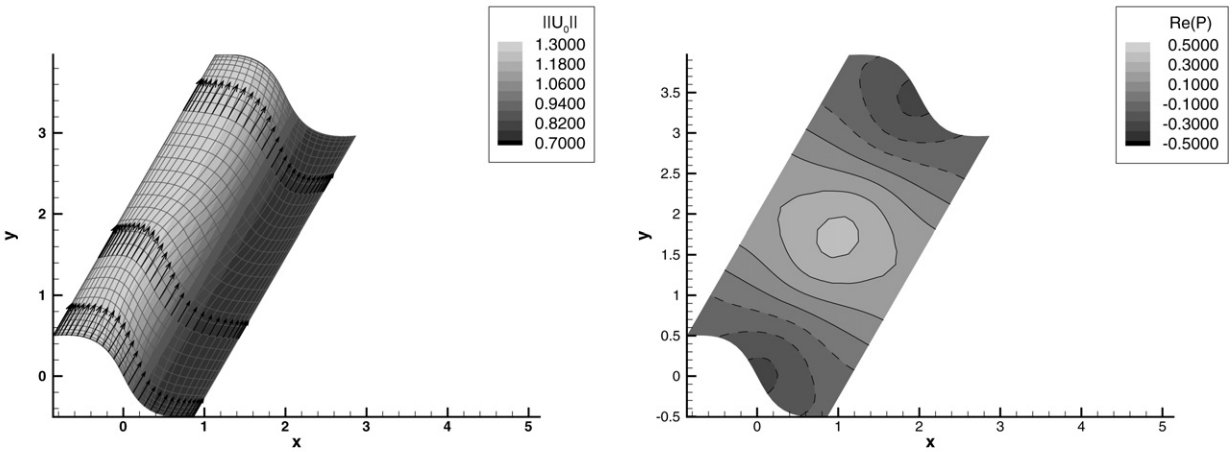

Fig. 3. Planar channel, BiGlobal computation: base flow (left), amplified eigenvector (right).

3.3. The planar disc

Next a circular cavity of radius $R = \sqrt{2}$ is considered with a cylindrical coordinate system (r, θ) (Fig. 4, left). Inside this cavity an arbitrary rotating flow is defined with a radial mixing layer profile:

$$u_\theta(r) = \frac{u_2}{2}\left(1 + \tanh\frac{r-h}{b}\right), \quad u_r = 0, \quad \rho = 1 \tag{8}$$

with upper velocity $u_2 = 0.6$, radius $h = R/2$ and thickness $b = 0.2$. A slip condition is imposed at the wall.

As the flow is invariant in θ, the 1D perturbations are sought in the form $Z(r)\,\mathrm{e}^{\mathrm{i}(n\theta-\omega t)}$, where n is the azimuthal wave number. The 1D stability computations have been performed with $n = 4$. Fig. 4 (right) shows the real part of the pressure fluctuation on one fourth of the disc associated with the computed frequency $\omega_{1D} = 1.742 + 0.119\mathrm{i}$.

For the BiGlobal computation, each vertex of the computational domain is mapped onto a vertex of the square centered in $(0, 0)$ and of side length equal to 2. Each boundary of the Gauss–Lobatto grid is mapped onto one fourth of the disc perimeter. Then the mesh is created with a grid generating software. (Fig. 5, left). Three mesh sizes have been used with respectively 60, 80 and 100 points in both ξ and η directions.

The frequencies corresponding to the azimuthal mode $n = 4$ are presented in Table 1. BiGlobal results are seen to converge toward 1D value, in spite of the mesh singularity in the corners. Furthermore, the associated eigenvector is very similar to the quasi-1D solution (Fig. 5, right).

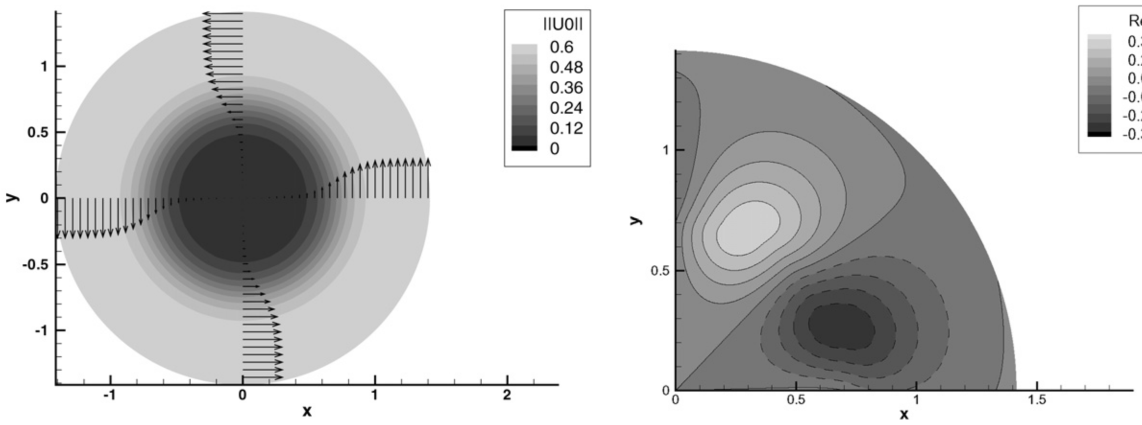

Fig. 4. Planar disc, reference solution: base flow (left), mode $n = 4$ (right).

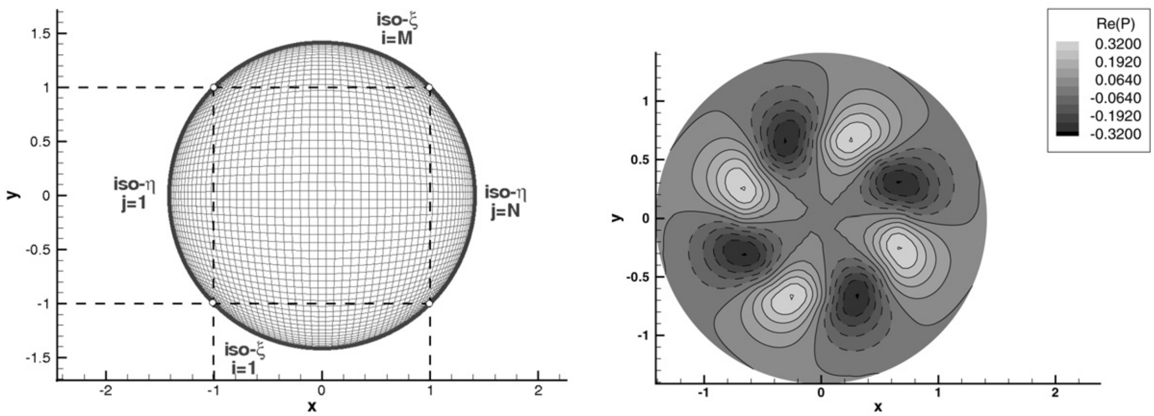

Fig. 5. Planar disc, BiGlobal computation: mesh (left), mode $n = 4$ (right).

Table 1
Planar disc: eigenvalue convergence for $n = 4$.

Mesh size	ω
60×60	$1.758 + 0.119i$
80×80	$1.746 + 0.129i$
100×100	$1.740 + 0.123i$
1D	$1.742 + 0.119i$

4. Conclusions

The “implicit mapping method” has been developed in order to perform BiGlobal stability computations on curvilinear domains. The resulting code has been tested for two test-cases, a mixing layer in a planar channel and another mixing layer in a plane circular cavity. In these cases, the BiGlobal results compare favourably with reference solutions coming from classical 1D stability analysis.

This new approach opens many perspectives for stability calculations on more complex 2D geometries that were not affordable in the past. Extensions to compressible fluids or to 3D flows with one invariant axis should also be possible.

References

[1] C.W. Rowley, T. Colonius, A.J. Basu, On self-sustained oscillations in two-dimensional compressible flow over rectangular cavities, Journal of Fluid Mechanics 455 (2002) 315–346.

[2] V. Theofilis, Advances in global linear instability analysis of non parallel and three-dimensional flows, Progress in Aerospace Sciences 39 (2003) 249–315.

[3] M.R. Khorrami, M.R. Malik, R.L. Ash, Application of spectral collocation techniques to the stability of swirling flows, Journal of Computational Physics 81 (1989) 206–229.

[4] W.E. Arnoldi, The principle of minimized iterations in the solution of the matrix eigenvalue problem, Quarterly Journal of Applied Mathematics 9 (1951) 17–29.

[5] Calculs d'instabilités absolues et BiGlobales autour d'un bec de bord d'attaque, Ph.D. Thesis, ISAE Toulouse, April 2008.