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Asymptotic modeling of a Coulomb frictional Signorini problem for the von Kármán plates

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Abstract

We study in this Note the asymptotic modeling of Coulomb frictional unilateral contact problem between an elastic non-linear von Kármán plate and a rigid obstacle. To this end we use a formal asymptotic expansions method in terms of the half-thickness of the plate as the parameter. The leading term of the asymptotic expansion is characterized by two-dimensional von Kármán plate problem with Signorini conditions but without friction. ***To cite this article: D.A. Chacha, A. Bensayah, C. R. Mecanique 336 (2008).***

Résumé

Modélisation asymptotique du problème de Signorini avec frottement de Coulomb pour les plaques de von Kármán. Nous étudions dans cette Note la modélisation asymptotique d'un problème de contact unilatéral avec frottement de Coulomb entre une plaque élastique non linéaire de type von Kármán et un obstacle rigide. A cet effet nous employons la méthode des développements asymptotique formelle en termes de la demi-épaisseur de la plaque comme paramètre. Le terme du premier ordre significatif du développement asymptotique est caractérisé par un problème bidimensionnel de plaque de von Kármán avec les conditions de Signorini mais sans frottement. ***Pour citer cet article : D.A. Chacha, A. Bensayah, C. R. Mecanique 336 (2008).***

Keywords: Friction; Signorini problem; Coulomb friction; von Kármán plate; Asymptotic expansion; Unilateral contact

Mots-clés : Frottement ; Problème de Signorini ; Frottement de Coulomb ; Plaque de von Kármán ; Développement asymptotique ; Contact unilatéral

Version française abrégée

On considère dans le cadre de l'élasticité non linéaire une plaque élastique, homogène et isotrope de type von Kármán, de surface moyenne ω et d'épaisseur 2ε, occupant un domaine borné connexe Ω^ε de $\mathbb{R}^3$, dont les forces appliquées sont de type volumique de densité $f^\varepsilon \in (L^2(\Omega^\varepsilon))^3$ et surfacique de densité $g^\varepsilon \in (L^2(\Gamma_-^\varepsilon))^3$ sur la face inférieure, sur le bord latéral Γ_0^ε les forces sont de pression de type von Kármán de densité $(\tilde{F}_1^\varepsilon, \tilde{F}_2^\varepsilon) \in (L^2(\gamma))^2$.

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La plaque en question entre en contact unilatéral avec frottement de Coulomb avec une fondation rigide, la zone de contact est Γ_+^ε la face supérieure de la plaque. Ainsi le problème considéré est formulé par (CP^ε). L'objectif de la présente note est d'étudier le comportement asymptotique du problème précédent lorsque $\varepsilon \downarrow 0$. Afin d'atteindre cet objectif on utilise la méthode des développements asymptotiques d'une façon formelle. Dans une première étape on donne la formulation variationnelle (VP^ε) du problème classique (CP^ε). Dans la deuxième étape , on commence par une mise à l'échelle (the scalling) du domaine Ω^ε vers un domaine Ω indépendant de ε et des données (1)–(4), ensuite on formule le problème (VP^ε) dans le domaine de référence Ω en tenant compte des mises à l'échelle considérées, on obtient le problème variationnel ($\mathrm{SVP}(\varepsilon)$). Dans la troisième étape, on effectue un développement asymptotique formel du champs des déplacements-contraintes $(u(\varepsilon), \sigma(\varepsilon))$ mise à l'échelle, ensuite on substitue le développement précédent dans le problème ($\mathrm{SVP}(\varepsilon)$). On obtient au premier ordre significatif (ε^0) le problème (SVP(0)) qui est un problème de contact unilatéral mais sans frottement. Ce travail vient suite aux travaux de J.C. Paumier [1,2] sur la modélisation asymptotique d'un problème de Signorini avec frottement de Coulomb pour une plaque linéaire de type Kirchhoff–Love totalement encastrée sur le bord latéral. La méthode employée par J.C. Paumier pour réaliser l'étude de la modélisation du problème précédent est la méthode de convergence. Il a obtenu le résultat suivant : la solution du problème tridimensionnel mise à l'échelle converge fortement dans l'espace des déplacements cinématiquement admissibles vers la solution d'un problème de Signorini sans frottement pour le modèle de plaque bidimensionnel. On signale que récemment A. Léger et B. Miara [3] ont généralisé le travail de J.C. Paumier [2] sans frottement au cas des coques peu profondes dans le cadre de l'élasticité linéaire.

1. Introduction

The contact problem is an important problem in computational mechanics, it is characterized by unilateral inequalities, describing the physical impossibility of tensile contact tractions (except under special circumstances) and of material interpenetration. Additional inequalities and/or non-linearities are introduced when friction laws are taken into account. These complex boundary conditions can lead to problems with existence and uniqueness of quasi-static solution and to lack of convergence of numerical algorithms. In frictional problems, there can also be lack of stability, leading to stick-slip motion and frictional vibrations. In 2002–2003, J.C. Paumier [1,2] studied the Signorini problem with friction in the Kirchhoff–Love theory of plates. During his research task he raised some open questions:

– how is it possible to get a lower-dimensional model including friction?
– is the study of the quasi-static case possible?
– is it possible to replace a part of the clamped condition by an unilateral one?
– is this approach valid for shells and rods?
– what happens in the non-linear case (von Kármán equations)?
– what happens for other constitutive laws?

It is announced that A. Léger and B. Miara [3] generalized the work of J.C. Paumier to the case of linearized shallow shell but without friction, which gives a partial answer to the fourth open question. The study carried out by J.C. Paumier is the modeling of a Coulomb frictional unilateral contact problem between an elastic thin plate and rigid foundation, within the framework of linear elasticity, by a two-dimensional Signorini model without friction by using the method of convergence. One can find the same results by using the method of the asymptotic expansions, one obtains that with the first significant order the same model obtained by the method of convergence. A major advantage in the modeling of thin structure in linear elasticity is the possibility of the justification of the convergence of the 3D model towards the 2D model, which is not the case in general within the framework of non-linear elasticity. Our objective in this work is to answer the fifth open question by a formal asymptotic analysis. So we generalized the study of J.C. Paumier to the non-linear plate of von Kármán but by using a formal asymptotic expansions method.

2. Setting of the problem

In this Note, we use the following conventions and notations: Greek indices (except ε) belong to the set $\{1, 2\}$, Latin indices belong to the set $\{1, 2, 3\}$, the symbols of differentiation $\partial_j^\varepsilon = \partial/\partial x_j^\varepsilon$, $\partial_j = \partial/\partial x_j$, δ_{ij} the Kronecker symbols, and the summation convention with respect to the repeated indices is systematically used.

Let $\Omega^\varepsilon = \omega \times]-\varepsilon, +\varepsilon[$, where ε is a small parameter, be an open bounded set from $\mathbb{R}^3$, such that ω is an open subset from $\mathbb{R}^2$ with Lipschitz boundary γ. We denote the lateral boundary of Ω^ε by $\Gamma_0^\varepsilon = \gamma \times [-\varepsilon, \varepsilon]$, the upper and the lower faces are denoted, respectively, by Γ_+^ε and Γ_-^ε. We suppose that Ω^ε is occupied by a non-linear, elastic, homogeneous, isotropic body. In its natural configuration: a plate of thickness 2ε whose Lamé's constants are denoted $\lambda > 0, \mu > 0$ and assumed to be independent of ε. The plate is supposed to be subjected to a body force of density $f^\varepsilon \in (L^2(\Omega^\varepsilon))^3$, its lower face subjected to a surface force of density $g^\varepsilon \in (L^2(\Gamma_-^\varepsilon))^3$ and submitted, on Γ_0^ε to applied surface forces of the "von Kármán's type" which are horizontal, and only their resultant $(\tilde{F}_1^\varepsilon, \tilde{F}_2^\varepsilon) \in (L^2(\gamma))^2$ after integration across the thickness is given along the boundary γ. Therefore, the displacements u^ε derived from this situation verify u_α^ε independent of x_3^ε and $u_3^\varepsilon = 0$ on Γ_0^ε which mean that the only horizontal displacements of equal direction and magnitude are allowed along each vertical segment of the lateral face Γ_0^ε. For more details on the von Kármán equations we return to [4] and [5]. We suppose that the upper face Γ_+^ε of the plate is in unilateral Coulomb frictional contact with a rigid foundation. Let Λ denote the frictional coefficient, $\Theta^\varepsilon = \{x^\varepsilon \in \mathbb{R}^3 \mid (x_1^\varepsilon, x_2^\varepsilon) \in \omega,\ x_3^\varepsilon \geqslant \varepsilon d\}$ the foundation domain, where d ($\geqslant 0$) is the gap function defined on Γ_+^ε which describes the distance between the upper face and the rigid foundation measured in the normal direction, $\bar{v}$ the trace of v on Γ_+^ε and $\underline{v}$ the trace of v on Γ_-^ε.

Our aim is to find the asymptotic behavior of the equilibrium state of the plate Ω^ε which is characterized by a displacement vector u^ε solution of the classical problem:

$$
(\mathrm{CP}^\varepsilon)\quad \begin{cases} -\partial_j^\varepsilon \hat{\sigma}_{ij}^\varepsilon = f_i^\varepsilon \quad \text{in } \Omega^\varepsilon \\ u_\alpha^\varepsilon \text{ independent of } x_3^\varepsilon \text{ and } u_3^\varepsilon = 0 \quad \text{on } \Gamma_0^\varepsilon \\ \frac{1}{2\varepsilon}\int_{-\varepsilon}^{+\varepsilon} \hat{\sigma}_{\alpha\beta}^\varepsilon \cdot \nu_\beta^\varepsilon \,\mathrm{d}x_3^\varepsilon = \tilde{F}_\alpha^\varepsilon \quad \text{on } \gamma \\ \hat{\sigma}_{ij}^\varepsilon n_j^\varepsilon = g_i^\varepsilon \quad \text{on } \Gamma_-^\varepsilon \\ \bar{u}_3^\varepsilon \leqslant \varepsilon d, \quad \hat{\sigma}_{33}^\varepsilon \leqslant 0, \quad \hat{\sigma}_{33}^\varepsilon(\bar{u}_3^\varepsilon - \varepsilon d) = 0 \quad \text{on } \Gamma_+^\varepsilon \text{ (the Signorini conditions)} \\ \left.\begin{array}{l} |\hat{\sigma}_T^\varepsilon| < \Lambda|\hat{\sigma}_{33}^\varepsilon| \Rightarrow u_T^\varepsilon = 0 \quad \text{on } \Gamma_+^\varepsilon \\ |\hat{\sigma}_T^\varepsilon| = \Lambda|\hat{\sigma}_{33}^\varepsilon| \Rightarrow \exists\delta > 0, \quad u_T^\varepsilon = -\delta\hat{\sigma}_T^\varepsilon, \quad \hat{\sigma}_T^\varepsilon = (\hat{\sigma}_{\alpha 3}^\varepsilon) \quad \text{on } \Gamma_+^\varepsilon \end{array}\right\} \text{(the Coulomb friction conditions)} \end{cases}
$$

where: $\hat{\sigma}_{ij}^\varepsilon = \sigma_{ij}^\varepsilon + \sigma_{kj}^\varepsilon \partial_k^\varepsilon u_i^\varepsilon$, $\sigma_{ij}^\varepsilon = \lambda E_{pp}^\varepsilon(u^\varepsilon)\delta_{ij} + 2\mu E_{ij}^\varepsilon(u^\varepsilon)$ the components of stress tensor, $E_{ij}^\varepsilon(u^\varepsilon) = \frac{1}{2}(\partial_i^\varepsilon u_j^\varepsilon + \partial_j^\varepsilon u_i^\varepsilon + \partial_i^\varepsilon u_k^\varepsilon \partial_j^\varepsilon u_k^\varepsilon)$ the components of the non-linear train tensor, $n^\varepsilon = (n_i^\varepsilon)$ is the unit outer normal vector along the boundary of the plate Ω^ε, $\nu^\varepsilon = (\nu_\alpha^\varepsilon)$ is the unit outer normal vector along the boundary of the set ω and the subscripts T to the tangential components. To give a weak formulation of our problem we introduce some notations. Let

$$
\begin{aligned}
&V(\Omega^\varepsilon) = \{v \in W^{1,4}(\Omega^\varepsilon) \mid v \text{ independent of } x_3^\varepsilon \text{ on } \Gamma_0^\varepsilon\}, \qquad V_0(\Omega^\varepsilon) = \{v \in W^{1,4}(\Omega^\varepsilon) \mid v = 0 \text{ on } \Gamma_0^\varepsilon\} \\
&\vec{V}(\Omega^\varepsilon) = V(\Omega^\varepsilon) \times V(\Omega^\varepsilon) \times V_0(\Omega^\varepsilon), \qquad K(\Omega^\varepsilon) = \{v \in V_0(\Omega^\varepsilon) \mid \bar{v} \leqslant \varepsilon d \text{ on } \Gamma_+^\varepsilon\} \\
&\vec{K}(\Omega^\varepsilon) = V(\Omega^\varepsilon) \times V(\Omega^\varepsilon) \times K(\Omega^\varepsilon)
\end{aligned}
$$

Multiplying the system of equilibrium equations in $(\mathrm{CP}^\varepsilon)$ by functions v_i^ε and integrating over the set Ω^ε, using the Green formulas and the boundary conditions we obtain:

The variational formulation of the classical problem $(\mathrm{CP}^\varepsilon)$ is:

$$
(\mathrm{VP}^\varepsilon)\quad \begin{cases} \text{Find } u^\varepsilon \in \vec{K}(\Omega^\varepsilon) \text{ such that:} \\ \int_{\Omega^\varepsilon} \hat{\sigma}_{ij}^\varepsilon \partial_j^\varepsilon v_i^\varepsilon \,\mathrm{d}x^\varepsilon = L^\varepsilon(v^\varepsilon) + 2\varepsilon\int_\gamma \tilde{F}_\alpha^\varepsilon \tilde{v}_\alpha^\varepsilon \,\mathrm{d}\gamma + \langle \hat{\sigma}_{i3}^\varepsilon, \bar{v}_i^\varepsilon\rangle, \quad \forall v^\varepsilon \in \vec{V}(\Omega^\varepsilon) \\ \langle \hat{\sigma}_{33}^\varepsilon, \bar{v}_3^\varepsilon - \bar{u}_3^\varepsilon\rangle \geqslant 0, \quad \forall v_3^\varepsilon \in K(\Omega^\varepsilon) \\ \langle \hat{\sigma}_{\alpha 3}^\varepsilon, \bar{v}_\alpha^\varepsilon - \bar{u}_\alpha^\varepsilon\rangle + \langle \Lambda|\hat{\sigma}_{33}^\varepsilon|, |\bar{v}_T^\varepsilon| - |\bar{u}_T^\varepsilon|\rangle \geqslant 0, \quad \forall v_\alpha^\varepsilon \in V(\Omega^\varepsilon) \end{cases}
$$

where: $L^\varepsilon(v^\varepsilon) = \int_{\Omega^\varepsilon} f_i^\varepsilon v_i^\varepsilon \,\mathrm{d}x^\varepsilon + \int_{\Gamma_-^\varepsilon} g_i^\varepsilon \underline{v}_i^\varepsilon \,\mathrm{d}\Gamma^\varepsilon$ and $\langle \hat{\sigma}_{i3}^\varepsilon, \bar{\phi}_i^\varepsilon\rangle = \int_{\Gamma_+^\varepsilon} \hat{\sigma}_{i3}^\varepsilon \cdot \bar{\phi}_i^\varepsilon \,\mathrm{d}\Gamma^\varepsilon$.

3. Asymptotic study

3.1. The scaled problem

Let $\Omega = \omega \times]-1, +1[$, $\Gamma_- = \omega \times \{-1\}$, $\Gamma_+ = \omega \times \{+1\}$, $\Gamma_0 = \partial\omega \times [-1, +1]$. Let $x = (x_i) \in \bar{\Omega}$ denote a generic point in the set $\bar{\Omega}$.

We now transform the domain Ω^ε having the thickness 2ε into fixed domain Ω independent of ε via the simple mapping:

$$\chi^\varepsilon : x^\varepsilon \in \Omega^\varepsilon \to x \in \Omega, \quad \text{where } x_\alpha = x_\alpha^\varepsilon,\ x_3 = x_3^\varepsilon/\varepsilon. \tag{1}$$

Hence:

$$\chi^\varepsilon(\Omega^\varepsilon) = \Omega, \quad \chi^\varepsilon(\Gamma_\pm^\varepsilon) = \Gamma_-, \quad \chi^\varepsilon(\Gamma_+^\varepsilon) = \Gamma_+, \quad \chi^\varepsilon(\Gamma_0^\varepsilon) = \Gamma_0, \quad \partial_\alpha^\varepsilon = \partial_\alpha \quad \text{and} \quad \partial_3^\varepsilon = \frac{1}{\varepsilon}\partial_3. \tag{2}$$

We introduce the scaled displacement $u(\varepsilon)$, the scaled test function $v(\varepsilon)$, the scaled stress tensor $\sigma(\varepsilon)$ and the scaled contact condition as follow:

$$\begin{cases} u_\alpha^\varepsilon \circ \chi^\varepsilon = \varepsilon^2 u_\alpha(\varepsilon), \quad u_3^\varepsilon \circ \chi^\varepsilon = \varepsilon u_3(\varepsilon) \quad \text{and} \quad v_\alpha^\varepsilon \circ \chi^\varepsilon = \varepsilon^2 v_\alpha(\varepsilon), \quad v_3^\varepsilon \circ \chi^\varepsilon = \varepsilon v_3(\varepsilon) \\ \sigma_{\alpha\beta}^\varepsilon \circ \chi^\varepsilon = \varepsilon^2 \sigma_{\alpha\beta}(\varepsilon), \quad \sigma_{\alpha 3}^\varepsilon \circ \chi^\varepsilon = \varepsilon^3 \sigma_{\alpha 3}(\varepsilon), \quad \sigma_{33}^\varepsilon \circ \chi^\varepsilon = \varepsilon^4 \sigma_{33}(\varepsilon) \quad \text{and} \quad \bar{u}_3(\varepsilon) \leqslant d \end{cases} \tag{3}$$

We also introduce the scaling of the forces:

$$f_\alpha^\varepsilon \circ \chi^\varepsilon = \varepsilon^2 f_\alpha(\varepsilon), \quad f_3^\varepsilon \circ \chi^\varepsilon = \varepsilon^3 f_3(\varepsilon), \quad g_\alpha^\varepsilon \circ \chi^\varepsilon = \varepsilon^3 g_\alpha(\varepsilon), \quad g_3^\varepsilon \circ \chi^\varepsilon = \varepsilon^4 g_3(\varepsilon), \quad \tilde{F}_\alpha^\varepsilon = \varepsilon^2 \tilde{F}_\alpha(\varepsilon) \tag{4}$$

Then we obtain:

$$L^\varepsilon(v^\varepsilon) = \varepsilon^5 L(v) \quad \text{with } L(v) = \int_\Omega f_i v_i \,\mathrm{d}x + \int_{\Gamma_-} g_i \underline{v}_i \,\mathrm{d}\Gamma \tag{5}$$

Therefore we denote by:

$$\begin{cases} V(\Omega) = \{v \in W^{1,4}(\Omega),\ v \text{ independent of } x_3 \text{ on } \Gamma_0\}, \quad V_0(\Omega) = \{v \in W^{1,4}(\Omega),\ v = 0 \text{ on } \Gamma_0\} \\ \vec{V}(\Omega) = V(\Omega) \times V(\Omega) \times V_0(\Omega), \quad K(\Omega) = \{v \in V_0(\Omega),\ \bar{v} \leqslant d \text{ on } \Gamma_+\} \\ \vec{K}(\Omega) = V(\Omega) \times V(\Omega) \times K(\Omega) \end{cases} \tag{6}$$

Using the upper assumptions and notations (1)–(6) lead to the following:

Proposition 1. *The variational problem* (VP$^\varepsilon$) *is equivalent to the following scaled variational problem* (SVP(ε)):

$$\begin{cases} \textit{Find } u(\varepsilon) \in \vec{K}(\Omega) \textit{ such that}: \\ \int_\Omega \sigma_{ij}(\varepsilon)\partial_j v_i \,\mathrm{d}x + \int_\Omega \sigma_{ij}(\varepsilon)\partial_i u_3(\varepsilon)\partial_j v_3 \,\mathrm{d}x + \varepsilon^2 \int_\Omega \sigma_{ij}(\varepsilon)\partial_i u_\alpha(\varepsilon)\partial_j v_\alpha \,\mathrm{d}x \\ \quad = L(v) + \langle \hat{\sigma}_{33}(\varepsilon), \bar{v}_3 \rangle + \int_\gamma \tilde{F}_\alpha (\int_{-1}^{+1} v_\alpha \,\mathrm{d}x_3) \,\mathrm{d}\gamma + \langle \sigma_{\alpha 3}(\varepsilon), \bar{v}_\alpha \rangle + \varepsilon^2 \langle \sigma_{k3}(\varepsilon)\partial_k u_\alpha, \bar{v}_\alpha \rangle, \quad \forall v \in \vec{V}(\Omega) \\ \langle \hat{\sigma}_{33}(\varepsilon), \bar{v}_3 - \bar{u}_3(\varepsilon) \rangle \geqslant 0, \quad \forall v_3 \in K(\Omega) \\ \langle \sigma_{\alpha 3}(\varepsilon), \bar{v}_\alpha - \bar{u}_\alpha(\varepsilon) \rangle + \varepsilon \langle \Lambda |\hat{\sigma}_{33}(\varepsilon)|, |\bar{v}_T| - |\bar{u}_T(\varepsilon)| \rangle + \varepsilon^2 \langle \sigma_{k3}(\varepsilon)\partial_k u_\alpha(\varepsilon), \bar{v}_\alpha - \bar{u}_\alpha(\varepsilon) \rangle \geqslant 0, \quad \forall v_\alpha \in V(\Omega) \end{cases}$$

3.2. The two-dimensional problem

We assume that the scaled displacement-stress $(u(\varepsilon), \sigma(\varepsilon))$ admit a formal asymptotic expansion of the form:

$$\big(u(\varepsilon), \sigma(\varepsilon)\big) = \big(u^0, \sigma^0\big) + \varepsilon\big(u^1, \sigma^1\big) + \varepsilon^2\big(u^2, \sigma^2\big) + \cdots \tag{7}$$

We introduce the Kirchhoff–Love space of admissible displacements $V_{\mathrm{KL}}(\Omega) = \{v = (v_i),\ v_\alpha = \eta_\alpha - x_3\partial_\alpha\eta_3,\ v_3 = \eta_3 \text{ such that: } \eta_\alpha \in H^1(\omega),\ \eta_3 \in H_0^2(\omega)\}$, and the space $L_s^2(\Omega) = \{\tau = (\tau_{ij}) \in L^2(\Omega);\ \tau_{ij} = \tau_{ji}\}$. Substituting expansion (7) into the scaled variational problem (SVP(ε)), we obtain:

Proposition 2. *Assume that* $\partial_3 u_3^0 \in C^0(\bar{\Omega})$ *then the leading term* (u^0, σ^0) *of the expansion* (7) *is solution of the problem*:

$$(\mathrm{SVP}(0))\quad \begin{cases} \text{Find } (u^0,\sigma^0) \in V_{\mathrm{KL}}(\Omega) \cap \vec{K}(\Omega) \times L_s^2(\Omega) \text{ such that:} \\ \int_\Omega \sigma^0_{\alpha\beta}\partial_\beta v_\alpha \,\mathrm{d}x + \int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha u_3^0 \partial_\beta v_3 \,\mathrm{d}x = L(v) + \langle \sigma^0_{33}, \bar{v}_3\rangle + \int_\gamma \tilde{F}_\alpha (\int_{-1}^{+1} v_\alpha \,\mathrm{d}x_3)\,\mathrm{d}\gamma \\ \quad \forall v \in V_{\mathrm{KL}}(\Omega) \\ \langle \sigma^0_{33}, \bar{v}_3 - \bar{u}_3^0\rangle \geqslant 0, \quad \forall v_3 \in K(\Omega) \end{cases}$$

where $\sigma^0_{\alpha\beta} = \frac{2\lambda\mu}{\lambda+2\mu} E^0_{\gamma\gamma}(u^0)\delta_{\alpha\beta} + 2\mu E^0_{\alpha\beta}(u^0)$ *and* $E^0_{\alpha\beta}(u^0) = \frac{1}{2}(\partial_i u_j^0 + \partial_j u_i^0 + \partial_i u_3^0 \partial_j u_3^0)$.

We deduce that the leading term (u^0, σ^0) is characterized by an unilateral contact problem without friction.

Proposition 3. *Let* $u^0 \in V_{\mathrm{KL}}(\Omega) \cap \vec{K}(\Omega)$ *be such that* $u^0_\alpha = \xi_\alpha - x_3\partial_\alpha \xi_3$ *and* $u_3^0 = \xi_3$, *where* ξ_α, ξ_3 *sufficiently regulars. Then the problem* (SVP(0)) *can be formulated in the classical form as two-dimensional problem*:

$$(\mathrm{P}^b(0))\quad \begin{cases} \text{Find } (\xi, \sigma^0_{33}) \in (H_0^1(\omega))^2 \times K_0(\omega) \times H^{-2}(\omega), \text{ where } K_0(\omega) = \{v \in H_0^2(\omega), v \leqslant d\}, \text{ such that} \\ k\Delta^2 \xi_3 - \partial_\beta(n_{\alpha\beta}\partial_\alpha \xi_3) = h_1^1 + h_2^1 + h_3^0 + \sigma^0_{33} \quad \text{on } \omega \\ -\partial_\beta n_{\alpha\beta} = h^0_\alpha \quad \text{on } \omega \\ n_{\alpha\beta}\nu_\beta = 2\tilde{F}_\alpha \quad \text{on } \gamma \\ \sigma^0_{33}(d - \xi_3) = 0 \quad \text{and} \quad \sigma^0_{33} \leqslant 0 \quad \text{in } \omega \end{cases}$$

where: $n_{\alpha\beta} = \frac{4\lambda\mu}{\lambda+2\mu} E^0_{\gamma\gamma}(\xi)\delta_{\alpha\beta} + 4\mu E^0_{\alpha\beta}(\xi)$, $k = \frac{8}{3}\mu\frac{\lambda+\mu}{\lambda+2\mu}$, $h_i^0 = \int_{-1}^1 f_i \,\mathrm{d}x_3 + g_i^-$, $h_i^1 = \int_{-1}^1 x_3 \partial_i f_i \,\mathrm{d}x_3 - \partial_i g_i^-$, $g_i^- = g_i(x_1, x_2, -1)$.

We deduce that the displacement u^0 is characterized by a two-dimensional problem without friction. Then, our three-dimensional Signorini problem with Coulomb friction offers toward a two-dimensional problem without friction.

Remark 1. The loss of the friction term in (SVP(0)) and $(\mathrm{P}^b(0))$ results owing to the fact that the friction force behaves like $\mathrm{O}(\varepsilon^3)$ whereas the contact pressure force scales as $\mathrm{O}(\varepsilon^4)$. Since the two measures are connected by Coulomb law via $|\hat{\sigma}_{\alpha 3}(\varepsilon)| \leqslant \varepsilon\Lambda|\hat{\sigma}_{33}(\varepsilon)|$. Therefore, at least formally when ε tends towards zero, the friction force must be canceled. In the absence of unilateral contact the problem $(\mathrm{P}^b(0))$ is reduced to non-linear von Kármán plate model.

References

[1] J.C. Paumier, Modélisation asymptotique d'un problème de plaque mince en contact unilatéral avec frottement contre un obstacle rigide, Prépublication L.M.C., http://www-lmc.imag.fr/~paumier/signoplaque.ps, 2002.

[2] J.-C. Paumier, Le problème de Signorini dans la thé orie des plaques minces de Kirchhoff–Love, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 567–570.

[3] A. Léger, B. Miara, Mathematical justification of the obstacle problem in the case of a shallow shell, J. Elasticity 90 (2008) 241–257.

[4] P.G. Ciarlet, P. Rabier, Les équations de von Kármán, Lecture Notes in Mathematics, vol. 826, Springer, Berlin, 1980.

[5] P.G. Ciarlet, Plates and Junctions in Elastic Multistructures, Masson, 1990.