Available online at www.sciencedirect.com

ScienceDirect

C. R. Mecanique 337 (2009) 30–33

Comptes Rendus Mecanique

An incremental constitutive law for ageing viscoelastic materials: a three-dimensional approach

Claude François Chazal, Rostand Moutou Pitti *

Groupe d'étude des matériaux hétérogènes – axe génie civil et durabilité (GEMH-GCD), Université de Limoges, centre universitaire génie civil d'Egletons, 19300 Egletons, France

Received 13 June 2008; accepted after revision 10 December 2008

Available online 23 January 2009

Presented by Jean-Baptiste Leblond

Abstract

A new approach to a viscoelastic medium is presented and the constitutive equations of ageing viscoelastic materials are developed. The formulation is equivalent to ageing Kelvin or Maxwell chain models with age dependent elastic moduli and viscosities. It is shown that by using such models, the retaining of the complete past history of stress and strain in the memory of a digital computer is avoided. Finally, the formulation is generalized to deal with three-dimensional problems. ***To cite this article: C.F. Chazal, R. Moutou Pitti, C. R. Mecanique 337 (2009).***

Résumé

Une formulation incrémentale tridimensionnelle pour les matériaux viscoélastiques vieillissants. Une nouvelle approche incrémentale propre aux milieux viscoélastiques vieillissants est présentée. La formulation analytique proposée est équivalente aux modèles de Kelvin Voigt ou de Maxwell dont les modules élastiques et visqueux dépendent du temps. Grâce à ces modèles, il est démontré que le stockage complet, dans la mémoire de l'ordinateur, de toute l'histoire des contraintes et des déformations est évité. Finalement, la formulation développée est généralisée aux problèmes tridimensionnels. ***Pour citer cet article : C.F. Chazal, R. Moutou Pitti, C. R. Mecanique 337 (2009).***

Keywords: Computational solid mechanics; Incremental formulation; Ageing viscoelastic materials; Finite element

Mots-clés : Mécanique des solides numérique ; Formulation incrémentale ; Matériaux viscoélastiques vieillissants ; Eléments finis

Version française abrégée

Le phénomène de vieillissement affecte globalement les propriétés physiques et le comportement mécanique de l'ensemble des matériaux induisant, dans la plus part des cas, leur ruine brutale. Ce constat est encore plus marqué dans les matériaux viscoélastiques, dont les champs mécaniques dépendent du temps. Il apparaît donc indispensable

* Corresponding author.
E-mail address: rostand.moutou_pitti@yahoo.fr (R. Moutou Pitti).

1631-0721/$ – see front matter
doi:10.1016/j.crme.2008.12.002

d'incorporer dans les différentes formulations du comportement viscoélastique ce phénomène complexe. Pour ces raisons, il est présenté une approche analytique, mémorisant toute l'histoire passée des déformations, et adaptée au vieillissement des matériaux soumis au fluage. Tout d'abord, en s'aidant d'un ensemble de ressorts $E(t)$, d'amortisseurs $\eta(t)$ [8] et de la loi de comportement (1) [4,6,7,10], l'équation différentielle (2), propre au modèle de Kelvin Voigt vieillissant, est établie [4]. La solution incrémentale (5) de l'équation différentielle (2) est obtenue en prenant en compte dans l'expression (4) l'approximation linéaire incrémentale (3) du module élastique $E(t)$ défini durant le laps de temps $\Delta t = t_{n+1} - t_n$. Dans ce cas, l'équation (5) peut s'écrire conjointement à l'aide des coefficients de la complaisance viscoélastique $\alpha_k(t_n, \Delta t)$, expression (6), et du terme $\beta_k(t_n)$ représentant l'influence antérieure de toute l'histoire des contraintes. En supposant le coefficient de Poisson υ constant, la généralisation de la formulation incrémentale au cas tridimensionnel (10) est finalement obtenue en combinant la forme incrémentale (9) de la loi de comportement viscoélastique isotrope (8) avec la relation (5). La résolution de l'équation (10) se fait par éléments finis en s'aidant de l'équation d'équilibre (12) issue du principe des déplacements virtuels [4–6]. La fonction de fluage $J(t, t_0)$, traduite par les relations (1) et (14) est, quant à elle, introduite par des séries de Dirichlet et peut être déterminée expérimentalement [1,2].

1. Introduction

Ageing in viscoelastic materials characterizes generally the phenomenon whereby changes occur in properties of the material under zero load. This ageing may modify the physical and mechanical behaviour of a material and must be taken into account when dealing with real design. In the present Note, our intention is to investigate the ageing of the mechanical properties of a material and to develop a new technique which extracts ageing characteristics from creep data. As they are known, the classical viscoelastic constitutive models which account for the effect of ageing [1], are based on a Volterra history – integral equations. These models require the retaining of the complete past history of stress and strain in the memory of a digital computer. To overcome these difficulties, we use a Dirichlet series expansion of the creep function [2]. This is equivalent to ageing a generalized Kelvin Voigt or Maxwell [3] model with ageing spring moduli and dashpot viscosities such that the retardation times are constants [4,5]. Then the resulting differential equations are integrated using finite difference techniques and the incremental constitutive equation is then obtained. The identification of the ageing moduli appearing in the mathematical structure of the model is determined by using all the experimental data assumed to be given in terms of creep tests performed at various age of the material. Finally, the formulation is generalized in order to deal with three-dimensional problems and the solutions of particular time are found from those at previous time [4], this leads to a great savings in the amount of computer storage requirements needed to solve real problems involving three-dimensional loading.

2. The ageing viscoelastic problem

In the context of a linear theory of deformation, the uniaxial constitutive equation of an ageing, linear, viscoelastic material has the following form [4–7]:

$$\varepsilon(t) = \int_{-\infty}^{t} J(t, \tau) \frac{\partial \sigma(\tau)}{\partial \tau} \, d\tau \tag{1}$$

Here $\varepsilon(t)$ is the actual strain at time t, $\sigma(\tau)$ is the history of stress and $J(t, \tau)$ is a creep function and represents the strain at time t caused by a unit uniaxial stress acting since the time τ. The constitutive equation given by (1) is not well suited for numerical computation. To overcome the problem of computer storage and the computer time, we use a Kelvin chain model with ageing spring moduli $E_i(t)$ and dashpot viscosities $\eta_i(t)$ such that the retardation times $\lambda_i = \eta_i(t)/E_i(t)$ are constants [8]. The differential equations which govern the Kelvin ageing model is given by:

$$\dot{\varepsilon}(t) + \lambda \varepsilon(t) = \lambda \frac{\sigma(t)}{E(t)} \tag{2}$$

where the dot over a variable denotes the time derivative [4]. The solution process of a step-by-step nature can now be described. Consider the time step $\Delta t = t_{n+1} - t_n$ and consider also a linear approximation of spring moduli $E(t)$ [9]:

$$\frac{1}{E(\tau)} = \frac{1}{E(t_n)} + \frac{\tau - t_n}{\Delta t} \Delta\left(\frac{1}{E}\right) \quad \forall \tau \in [t_n, t_n + \Delta t] \tag{3}$$

The solution of the differential equation (2), using the linear approximation given by (3), can be written as:

$$\Delta\varepsilon = \alpha(t_n, \Delta t_n)\Delta\sigma + \beta(t_n) \quad \forall t_n, \Delta t \tag{4}$$

where α and β are viscoelastic compliance coefficients. In the case of a generalized Kelvin Voigt model, Eq. (4) can be expressed as:

$$\Delta\varepsilon = \sum_{k=0}^{K} \alpha_k(t_n, \Delta t_n)\Delta\sigma + \sum_{k=0}^{K} \beta_k(t_n) \quad \forall t_n, \Delta t \tag{5}$$

$\alpha_k(t_n, \Delta t)$ are viscoelastic compliance coefficients and $\beta_k(t_n)$ represents the influence of the complete past history of stress and strain, they are given by:

$$\alpha_k(t_n, \Delta t) = \frac{1+\Psi_k}{E_k(t_n)}\left[1 - \frac{1}{\lambda_k \Delta t}\cdot\left(1 - \mathrm{e}^{-\lambda_k\cdot\Delta t}\right)\right] \tag{6}$$

$$\beta_k(t_n) = \left[\Psi_k + \left(1 - \frac{\Psi_k}{\lambda_k \Delta t}\right)\left(1 - \mathrm{e}^{-\lambda_k\cdot\Delta t}\right)\right]\frac{\sigma_k(t_n)}{E_k(t_n)} - \left(1 - \mathrm{e}^{-\lambda_k\cdot\Delta t}\right)\varepsilon_k(t_n) \tag{7}$$

We have assumed that $\Psi_k = E_k(t_n)/E_k(t_n + \Delta t) - 1 < 0\ \forall t_n, \Delta t_n$.

3. Three-dimensional incremental formulation

The formulation proposed in the previous section can be extended to deal with three-dimensional ageing, viscoelastic problems. We assume that the Poisson's coefficient υ is constant. Thus, in the isotropic viscoelastic behaviour, we can write:

$$\varepsilon_{ij}(t) = (1+\upsilon)J(t, t_0)\sigma_{ij} - \upsilon J(t, t_0)\sigma_{kk}\delta_{ij} \tag{8}$$

The incremental form of Eq. (8) can be obtained using Eq. (5) for uniaxial behaviour, one finds:

$$\Delta\varepsilon_{ij} = (1+\upsilon)\left[\alpha\Delta\sigma_{ij} + \beta_{ij}(t_n)\right] - \upsilon\left[\alpha\Delta\sigma_{kk} + \beta_{kk}(t_n)\right]\delta_{ij} \tag{9}$$

In matrix notation, constitutive equations given by (9) reduce to:

$$\{\Delta\varepsilon\} = [\Theta]\{\Delta\sigma\} + \left\{\tilde{\beta}(t_n)\right\} \tag{10}$$

where

$$[\Theta] = [\upsilon]\sum_{k=0}^{K}\alpha_k \quad \text{and} \quad \left\{\tilde{\beta}(t_n)\right\} = [\upsilon]\sum_{k=0}^{K}\beta_k(t_n) \tag{11}$$

$[\upsilon]$ is a matrix of 6×6 of Poisson's coefficient. Note that the constitutive equations given by (10) are highly suitable for numerical computation using finite element method. This will be detailed in the next section.

4. Variational formulation in linear ageing viscoelasticity

In the context of a finite element model, the governing equations of the discretized system are derived from the principle of virtual displacement [4]. Assuming small displacements within finite elements, strains are derived from shape functions in a standard manner. The equilibrium equations can be written as [4,5,7]:

$$[K_T]_n\{\Delta q\}_n = \{\Delta F\}_n + \{\Delta F_{\mathrm{vis}}\}_{n-1} \tag{12}$$

where $[K_T]_n$ is the viscoelastic stiffness matrix, $\{\Delta q\}_n$ is the displacement vector increment. Also, $\{\Delta F\}_n = \{F\}_n - \{F\}_{n-1}$ is the external load vector increment and $\{\Delta F_{\mathrm{vis}}\}_{n-1}$ is the viscous load vector corresponding to the complete past history of strains and stresses [5,7,10].

5. Creep spectral decomposition and fitting of creep data

A given creep function $J(t, t_0)$ can be approximated by the Dirichlet series with any desired accuracy:

$$J(t - t_0) = \frac{1}{E_0} + \sum_{i=1}^{N} \frac{1}{E_i}\left(1 - \mathrm{e}^{-(t-t_0)/\theta_i}\right) \tag{13}$$

The retardation times cannot be determined from experimental data and must be chosen in advance [2,8]. For each age t_0 we should choose (E_i, θ_i) and then minimization conditions must be used to reduce error approximation.

$$J(t - t_0) = \frac{1}{E_0(t_0)} + \sum_{i=1}^{N} \frac{1}{E_i(t_0)}\left(1 - \mathrm{e}^{-(t-t_0)/\theta_i}\right) \tag{14}$$

where

$$E_i(t_0) = \Gamma_{0,i} + \sum_{k=1}^{M} \Gamma_{k,i}\left(\mathrm{e}^{-(t_0-1)/\gamma_k}\right) \tag{15}$$

This approach was used to evaluate strains in concrete structures belonging to E.D.F. (French National Electricity Company).

6. Conclusion

A new formulation is developed in the time domain for linear, ageing, viscoelastic materials undergoing small mechanical deformation. The constitutive ageing viscoelastic equations are established in an incremental form, thus computer storage requirements are avoided. The response of the structure is obtained by using finite element method, thus the model maintains all the merit and accuracy of numerical computation. The formulation can easily be extended to deal with thermo-viscoelastic and dynamic analysis.

References

[1] A. Drozdov, On constitutive laws for ageing viscoelastic materials at finite strains, Eur. J. Mech. Solids 12 (1993) 305–324.

[2] P. Bâzant, J.C. Chen, Rate-type concrete creep law with reduced time, J. Eng. Mech. 110 (1984) 329–340.

[3] F. Corte's, M.J. Elejabarrieta, Modelling viscoelastic materials whose storage modulus is constant with frequency, Int. J. Solids Structures 43 (2006) 7721–7726.

[4] G. Ghazlan, S. Caperaa, C. Petit, An incremental formulation for the linear analysis of thin viscoelastic structures using generalized variables, Int. J. Numer. Meth. Eng. 38 (1995) 3315–3333.

[5] G. Ghazlan, F. Dubois, A new incremental formulation in the time domain of crack initiation in an orthotropic linearly viscoelastic solid, Mech. Time-Dependent Mat. 5 (2001) 229–253.

[6] F. Dubois, C. Chazal, C. Petit, A finite element analysis of creep-crack growth in viscoelastic media, Mech. Time-Dependent Mat. 2 (1999) 269–286.

[7] R. Moutou Pitti, F. Dubois, C. Petit, N. Sauvat, Mixed mode fracture separation in viscoelastic orthotropic media: numerical and analytical approach by the $M\theta v$, Int. J. Fracture 145 (2007) 181–193.

[8] I. Carol, P. Bâzant, Viscoelasticity with ageing caused by solidification of nonageing constituent, J. Eng. Mech. 119 (1993) 2252–2269.

[9] J.F. Destrébecq, G. Ghazlan, B. Jurkiewiez, Evaluation des effets du fluage dans un aéroréfrigérant atmosphérique, Rapport de Recherche n°4, Laboratoire 3MSGC, Université de Limoges, France, 1995.

[10] R. Moutou Pitti, F. Dubois, C. Petit, N. Sauvat, O. Pop, A new M integral parameter for mixed-mode crack growth in orthotropic viscoelastic material, Eng. Fracture Mech. 75 (2008) 4450–4465.