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ScienceDirect
C. R. Mecanique 337 (2009) 48–52
Comptes Rendus Mecanique

A modified Shkadov's model for thin film flow of a power law fluid over an inclined surface

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Received 10 March 2008; accepted after revision 24 December 2008
Available online 11 February 2009
Presented by Patrick Huerre

Abstract

A new evolution equation coherent up to order one in the long wave parameter is derived to describe the non-linear behavior of a thin film flow down an inclined plane of a power law fluid for small to moderate Reynolds numbers. The method we have used combines the lubrication theory and the weighted residual approach, with a suitable weighting function. That approach was first developed by Ruyer-Quil and Manneville (2000) for Newtonian fluids. The model has the advantages of both the Shkadov type approach far from criticality and that of Benney close to criticality. ***To cite this article: M. Amaouche et al., C. R. Mecanique 337 (2009).***

Résumé

Modèle de Shkadov modifié pour l'écoulement de film mince de fluide en loi de puissance sur une plaque inclinée. Un modèle non linéaire, cohérent à l'ordre un et combinant les avantages de l'approche asymptotique de Benney et de la méthode intégrale de Shkadov est proposée pour décrire le comportement d'un film mince de fluide en loi de puissance pour des nombres de Reynolds petits et modérés. La procédure utilisée est inspirée de la méthode des résidus pondérés développée par Ruyer-Quil et Manneville (2000) dans le cadre des fluides Newtoniens. ***Pour citer cet article : M. Amaouche et al., C. R. Mecanique 337 (2009).***

Keywords: Fluid mechanics; Shkadov's model; Thin film flow

Mots-clés : Mécanique des fluids ; Modèle de Shkadov ; Écoulement de film mince de fluide

1. Introduction

Falling films on inclined planes are driven by a gravitational pressure gradient and become unstable when inertia overcomes hydrostatic pressure effects. The disturbance originates at the free surface where vorticity is produced by the base flow shear stress. Because of its advection by the base flow, the perturbation vorticity becomes out of phase with the disturbed interface so as to cause the amplification of the interface disturbance. The instability manifests itself

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1631-0721/$ – see front matter
doi:10.1016/j.crme.2009.01.002

in surface waves whose wavelength is, except for very small inclination, much larger than the film thickness. Thus, long wave asymptotics of the Benney [1] type allow a description of the flow development near criticality; the flow variables are then all enslaved to the local interface shape. Even though the Benney equation (BE) contains different physical mechanisms and is potentially capable of describing the near critical non-linear behavior, it loses its physical relevance when the convective effects become significant, because of the production of shorter wave components. The solutions of the BE then depart from those of the full Navier–Stokes equations and, at some distance beyond the stability threshold, they exhibit non-physical finite time catastrophic behavior. To overcome some of the drawbacks associated to the BE, several improvements were recently proposed. The regularized procedure developed by Ooshida [2] allows to avoid the occurrence of time blow ups but fails to serve as an accurate model at moderate Reynolds numbers. Another single evolution equation including the second order dissipative effects via a suitable scaling was proposed by Panga and Balakotaiah [3]. Ruyer-Quil and Manneville [4] have shown that Panga and Balakotaiah model can be modified such that its inertial terms correspond to Ooshida's equation. The failure of the long wave models to correctly describe non-linear behaviors far from criticality is partly due to their incapacity to capture all of the inertia effects. The way to improve the modelling would be, according to Ruyer-Quil and Manneville [4] to incorporate the flow rate which becomes a genuine variable just after the wave formation. Such a model was first introduced by Shkadov [5] by using an integral boundary layer (IBL) approach. This theory combines the long wave assumption with the depth averaging method of Karman Pohlhansen type. In spite of its success to describe non-linear regimes for moderate Reynolds numbers the IBL approach does not accurately predict the flow behavior close to the stability threshold as the BE does. This defect is, as we will see later on, due to the fact that the IBL is coherent only up to zeroth order in the long wave parameter near criticality. A better account of the first order convective terms near criticality is therefore required to remove this drawback. The remedy was found by Ruyer-Quil and Manneville [6] by using a weighted residual integral boundary layer (WRIBL) approach.

In the present work, we are concerned with the related problem of thin film flow of a non-Newtonian fluid. There are several practical situations such as plastic manufacturing, coating processes, biological fluid motions, geological flows in which non-Newtonian effects are present and the power law model is quite suitable to describe the rheological behavior of these fluids. Hence, it is important to understand how these specific effects affect the dynamics of such film flows. As for Newtonian fluids, a number of investigations have been made in that area by using the lubrication theory of Benney [7,8] as well as Shkadov's method [9,10]. However, similar limitations to those described above were also encountered when non-Newtonian effects are included. To cure these limitations, we extend the idea developed in [6] for Newtonian fluids to derive a new system of two evolution equations valid up to moderate Reynolds numbers. It will be seen that the derived system is similar to that of Shkadov with slightly different coefficients and is nothing but the BE close to the instability threshold. Marginal stability results will be presented to illustrate the accuracy of the modelling.

2. Governing equations

The physical model of the problem is depicted in Fig. 1. A power law liquid of constant density ρ, consistency K and index n, flows under gravity along an infinitely long flat plate which is inclined at an angle β to the horizontal. A coordinate system (x, y) is adopted with x as the downstream coordinate and y being measured normal to the plate. The surface tension coefficient between the liquid and the surrounding passive medium (with presume $p_a = 0$), is σ

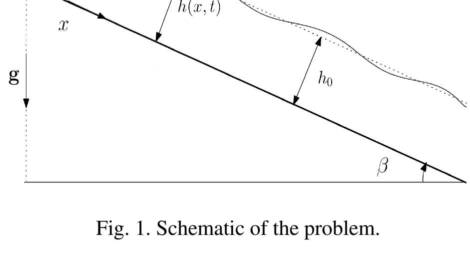

Fig. 1. Schematic of the problem.

and the acceleration due to gravity is $\mathbf{g}$. Denoting by $\mathbf{v} = (u, v)$, p and $\boldsymbol{\tau}$ the velocity field, the pressure and the stress tensor, conservation of mass and momentum then read:

$$\operatorname{div} \mathbf{v} = 0 \tag{1}$$

$$\rho(\partial_t + \mathbf{v} \cdot \mathbf{grad})\mathbf{v} = -\,\mathbf{grad}\, p + \operatorname{div} \boldsymbol{\tau} + \rho \mathbf{g} \tag{2}$$

where $\boldsymbol{\tau} = 2K\dot{\gamma}^{n-1}\mathbf{d}$ with $\dot{\gamma} = \sqrt{2d_{ij}d_{ij}}$ being the second invariant of the strain rate tensor $\mathbf{d}$. The index n indicates the degree of the non-Newtonian behavior and the greater is the departure from unity the more pronounced are the non-Newtonian effects, $n < 1$ corresponds to shear thinning (pseudoplastic) behavior while $n > 1$ represents shear thickening (dilatant) behavior. The above equations are subject to the boundary conditions

$$\text{on } y = 0, \quad u = v = 0 \tag{3}$$

$$\text{on } y = h(x,t), \quad h_t + uh_x - v = 0 \tag{4}$$

$$(-p + \sigma \operatorname{div} \mathbf{n})\mathbf{n} + \boldsymbol{\tau} \cdot \mathbf{n} = 0 \tag{5}$$

Expressing no slip on the rigid plate, the impermeability of the free surface $y = h(x,t)$ and equilibrium of all forces acting on it respectively, $\mathbf{n}$ being the unit normal vector to the free surface. To remove dimensions, we scale lengths transverse to the film by the mean film thickness h_0 and distances downslope by a typical wavelength l. The streamwise velocity u and the transverse velocity v are referred to the depthwise average velocity u_m and ϵu_m respectively, $\epsilon = h_0/l$ being the small gradient parameter. Time and pressure are in units of l/u_m and $Ku_m^n/\epsilon h_0^n$ respectively. We then set

$$\hat{x} = x/l, \quad \hat{y} = y/h_0, \quad \hat{u} = u/u_m, \quad \hat{v} = v/\epsilon u_m, \quad \hat{t} = tu_m/l, \quad \hat{p} = \epsilon p h_0^n/\big(K u_m^n\big)$$

Making use of the shallow flow approximation which amounts to taking $\epsilon \ll 1$ and keeping the other parameters order one or higher, Eqs. (1), (2) become, after a little algebra

$$u_x + v_y = 0 \tag{6}$$

$$Re\,\epsilon(u_t + uu_x + vu_y) = -p_x + \big(u_y^n\big)_y + G \tag{7}$$

$$p_y + \epsilon G \cot\beta = 0 \tag{8}$$

where only terms up to first order in ϵ are retained, the hat decoration is dropped for convenience, $Re = \rho h_0^n u_m^{2-n}/K$ is the Reynolds number and $G = (\frac{2n+1}{n})^n$. Introducing the flow rate $q = \int_0^h u\,\mathrm{d}y$, the kinematic free surface condition may be rewritten in the form

$$q_x + h_t = 0 \tag{9}$$

The normal and tangential balances at the free surface are then reduced to

$$u_y = 0 \tag{10}$$

$$p = -\epsilon W h_{xx} \tag{11}$$

where W is the surface tension parameter defined as $W = \epsilon^2 Re\, W_e$, $W_e = \sigma/\rho h_0 u_m^2$ being the Weber number. Note that W_e is large in practical applications so that $W = \mathcal{O}(1)$. Integrating Eq. (8) and using the condition (11) allows to eliminate the pressure in Eq. (7) which then takes the form

$$\epsilon\big\{Re(u_t + uu_x + vu_y) + G\cot\beta h_x - W h_{xxx}\big\} = \big(u_y^n\big)_y + G \tag{12}$$

The problem at hand is now described by the system (6), (9) and (12) along with the boundary conditions (3) and (10).

3. Derivation of the model

First, we note that the base flow velocity corresponding to the undisturbed flat film has the form

$$u_b = \frac{n}{n+1} G^{1/n}\big\{1 - (1-y)^{1+1/n}\big\} \tag{13}$$

and the dimensional depthwise average velocity is found as

$$u_m = \left(\frac{\rho g \sin\beta}{KG}\right)^{1/n} h_0^{1+1/n} \tag{14}$$

Since q appears in Eq. (9) as a basic variable in the same way as the film thickness, it seems quite natural to transform Eq. (12) to obtain another equation for these two variables. This can be done in the following manner. First, let us substitute to the coordinate y the similarity variable $z = 1 - y/h(x,t)$ which maps the film into a strip and assume the velocity profile in the form

$$u = a(x,t) f_0(z) + \epsilon u_1^* \tag{15}$$

where $f_0(z) = 1 - z^{1+1/n}$ and u_1^* indicates the first order correction of u. As it will be seen later on, the explicit form of u_1^* is not required at all. Then, integrating Eq. (15) through the film depth allows to eliminate the coefficient $a(x,t)$ in favor of q and to rewrite Eq. (15) as

$$u = u_0 + \epsilon u_1 \tag{16}$$

where $u_0 = \frac{2n+1}{n+1}\frac{q}{h} f_0(z)$ and $u_1 = u_1^* - (\frac{2n+1}{n+1}\int_0^1 u_1^*\,\mathrm{d}z) f_0(z)$. Note that u_0 and u_1 are such that $\int_0^h u_0\,\mathrm{d}y = q$ and $\int_0^h u_1\,\mathrm{d}y = 0$. Following the Shkadov procedure, a direct integration of Eq. (12) transforms the dissipative term into $-\tau_w^n$, τ_w being the wall shear which, according to (16), reads

$$\tau_w = \frac{1}{h}\left(\frac{2n+1}{n}\frac{q}{h} - \epsilon u_{1z}|_{z=1}\right) \tag{17}$$

Note that the viscous term is the only one where that correction must be taken into account for consistency. In the Shkadov model, the first order correction of the wall shear is neglected, which makes the model inaccurate near the stability threshold. The way to correct this defect is to introduce a suitable weighting function $F(z)$ in order to avoid an explicit introduction of u_1. Hence, integrating Eq. (12) and performing two successive integrations by parts of the viscous term gives

$$\epsilon\left\{Re\int_0^h (u_{0t} + u_0 u_{0x} + v_0 u_{0y}) F\,\mathrm{d}y + (G\cot\beta h_x - W h_{xxx})\int_0^h F\,\mathrm{d}y\right\}$$
$$= G\int_0^h F\,\mathrm{d}y + \left[u_y^n F\right]_0^h - \left[(u_0 + n\epsilon u_1) u_{0y}^{n-1} F_y\right]_0^h + \int_0^h (u_0 + n\epsilon u_1)\left(u_{0y}^{n-1} F_y\right)_y \mathrm{d}y \tag{18}$$

The integration is carried out by writing, owing to (16), $u_y^n = u_{0y}^{n-1}(u_{0y} + n\epsilon u_{1y}) + \mathcal{O}(\epsilon^2)$. Now, it can be seen that the correction u_1 can be eliminated from the calculation by simply choosing a suitable function F, such that

$$F|_{y=0} = 0, \qquad u_{0y}^{n-1} F_y\big|_{y=h} = 0, \qquad \left(u_{0y}^{n-1} F_y\right)_y = -w(x,t) \tag{19}$$

where $w(x,t)$ is some convenient function that will be specified later on. With the relations $\int_0^h u_0\,\mathrm{d}y = q$ and $\int_0^h u_1 = 0$ in mind, the right-hand side of (18) then reduces to $G\int_0^h F\,\mathrm{d}y - qw$. From the second and third conditions in (19), one obtains, owing to expression of u_0

$$\frac{n+1}{n} G^{1-1/n} \frac{q^{n-1}}{h^{2n}} F_y = w f_{0y} \tag{20}$$

This equation is satisfied by setting $w = \frac{n+1}{n} G^{1-1/n} \frac{q^{n-1}}{h^{2n}}$ and therefore $F = f_0$. Thus Eq. (18) takes the final form

$$\frac{q^n}{h^{2n}} - h\left\{1 + \epsilon\left(\frac{W}{G} h_{xxx} - \cot\beta h_x\right)\right\} + 2\epsilon\frac{Re}{G}\frac{2n+1}{3n+2}\left\{q_t + \frac{11n+6}{4n+3}\frac{q q_x}{h} - 3\frac{2n+1}{4n+3}\frac{q^2}{h^2} h_x\right\} = 0 \tag{21}$$

When added to (9), Eq. (21) completes our first order modified Shkadov model for the two unknowns h and q. It has the same structure as the Shkadov's one that writes (see [9])

$$\frac{q^n}{h^{2n}} - h\left\{1 + \epsilon\left(\frac{W}{G} h_{xxx} - \cot\beta h_x\right)\right\} + 2\epsilon\frac{Re}{G}\frac{2n+1}{3n+2}\left\{q_t + \left(\frac{q^2}{h}\right)_x\right\} = 0 \tag{22}$$

but with slightly different coefficients arising from a better account of inertia terms. Now, assume $q = q_0 + \epsilon q_1$, we obtain at zeroth and first order respectively

$$q_0 = h^{2+1/n} \tag{23}$$

$$q_1 = \frac{h^{2+1/n}}{n}\left\{\left(2\frac{Re\,G^{2/n-1}}{3n+2}h^{1+2/n} - \cot\beta\right)h_x + \frac{W}{G}h_{xxx}\right\} \tag{24}$$

which, when replaced in Eq. (9) gives the following Benney equation (see [8])

$$h_t + \left\{h^{2+1/n} + \frac{\epsilon}{n}\left[\left(\frac{W}{G}h_{xxx} - \cot\beta h_x\right)h^{2+1/n} + \frac{2\,Re}{3n+2}G^{2/n-1}h^{3/n+3}h_x\right]\right\}_x = 0 \tag{25}$$

4. Conclusion

A system of two evolution equations is given for a thin film flow of a power law liquid down an inclined plane. The derivation is based on an expansion at first order in streamwise gradient in order to recover asymptotic results close to the instability threshold. The model however ignores some important physical effects such as dispersion introduced by viscosity and therefore is inappropriate to describe solitary waves. For that reason, a second order model is required. This investigation is in progress.

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