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Analytical solution for the 3D steady state conduction in a solid subjected to a moving rectangular heat source and surface cooling

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Abstract

Heating of solids, e.g. by friction or welding, plays an important role in the thermomechanical behaviour of materials. An analytical model to compute the three-dimensional temperature distribution in a solid, subjected to a moving rectangular heat source with surface cooling is proposed in this article. A frequential integral transform and a finite cosine Fourier integral transform are used to solve the advection–diffusion equation related to this problem. The obtained solution is explicit and does not impose any restriction on the speed, the dimensions and the heat convection coefficient. It is presented in series form which converges rapidly. ***To cite this article: T. Osman, A. Boucheffa, C. R. Mecanique 337 (2009).***

Résumé

Solution analytique pour la conduction en régime permanent 3D dans un solide soumis à une source de chaleur rectangulaire mobile et à un refroidissement surfacique. L'échauffement des solides sous l'action du frottement ou du soudage par exemple joue un rôle important dans le comportement thermomécanique des matériaux. Nous proposons dans cet article un modèle analytique pour le calcul de la répartition tridimensionnelle de la température dans un solide soumis à une source de chaleur rectangulaire mobile et à un refroidissement surfacique. On utilise une transformée intégrale fréquentielle et une transformée intégrale de Fourier cosinus finie pour résoudre l'équation d'advection–diffusion associée à ce problème. La solution obtenue est explicite et n'impose aucune restriction en termes de vitesse, de dimensions et de coefficient d'échange. Elle se présente sous la forme de séries qui convergent rapidement. ***Pour citer cet article : T. Osman, A. Boucheffa, C. R. Mecanique 337 (2009).***

Keywords: Heat transfer; Moving heat sources; Advection–diffusion problem; Analytical solution

Mots-clés : Transferts thermiques ; Sources de chaleur mobiles ; Problème d'advection–diffusion ; Solution analytique

1. Introduction

The temperature distribution in a solid, subjected to localised heating, plays an important role in its thermomechanical behaviour. Indeed, the temperature gradients due to such heating lead to mechanical stresses which can damage

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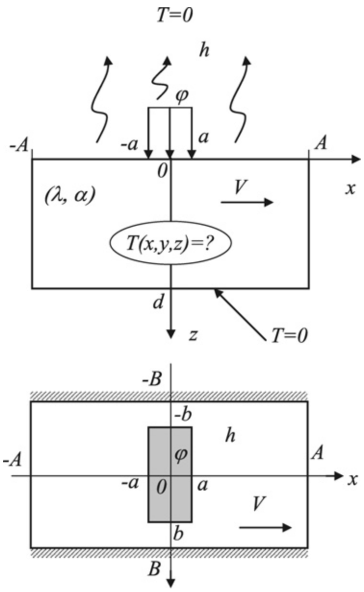

Fig. 1. Physical model.

Fig. 1. Modèle physique.

the concerned systems (fissures, ...). This is the case, e.g., in braking systems or material forming (rolling processes, forging, ...). Other industrial applications, such as welding or laser treatment necessitate a good control of the temperature distribution. Therefore, the temperature determination for given operating conditions is needed for the design of systems as well as the interpretation of their behaviour.

Some models to determine the temperature of solids subjected to a moving heat sources have been developed in the literature [1–6]. The influence of geometry has been also examined [7]. For fast phenomena, some precautions are needed to carry out the numerical computation [8,9]. Some authors are concerned with the effect of thermal contact between solids through a rough interface [10–13]. In certain industrial applications, the solids are covered with a thin film. The effect of coating nature and the source speed were studied in Ref. [14]. Inverse techniques were used to estimate the heat flux generated by friction and its location at the interface [15].

The models developed in the literature are often applied to semi-infinite solids with one or more heat sources, but without surface cooling. This cooling should be taken into consideration, since it plays an important role on the solid surface temperature levels.

An analytical solution is proposed in this article to compute the three-dimensional temperature field in a solid, subjected to a moving rectangular heat source and surface cooling. A circular or elliptical shaped source can be easily approached by using, respectively, a square or a “long” rectangle geometry. No restrictions are imposed on the source speed, the dimensions and the heat convection coefficient.

2. Description of the problem

Consider a solid of width d and dimensions $2A \times 2B$ in the xy-plane as shown on Fig. 1. This solid is subjected to a rectangular heat source of dimensions $2a \times 2b$, located at $z = 0$. The source dissipates a uniform heat flux of constant density φ. The solid is moving in the x-direction with a constant speed V relative to the source. The surface $2A \times 2B$, at $z = 0$, is entirely cooled by convection of coefficient h. This surface is subjected to an ambient temperature $T = 0$ whereas the surface $z = d$ is maintained at the same temperature $T = 0$. The side surfaces $y = \pm B$ are adiabatic while the surfaces $x = \pm A$ are subjected to periodic conditions. This is the case, e.g., of rolling processes and roller bearings.

The solid thermal conductivity λ and thermal diffusivity α are assumed to be independent of the local temperature.

3. Mathematical model and solution

A steady state three-dimensional heat advection–diffusion model is considered with the following governing equations:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}-\frac{V}{\alpha}\frac{\partial T}{\partial x}=0 \tag{1}$$

$$(T)_{x=-A}=(T)_{x=A},\quad \left(\frac{\partial T}{\partial x}\right)_{x=-A}=\left(\frac{\partial T}{\partial x}\right)_{x=A} \tag{2}$$

$$\left(\frac{\partial T}{\partial y}\right)_{y=0}=0,\quad \left(\frac{\partial T}{\partial y}\right)_{y=B}=0 \tag{3}$$

$$-\lambda\left(\frac{\partial T}{\partial z}\right)_{z=0}=\begin{cases}\varphi & \text{for } |x|\leqslant a,\ |y|\leqslant b\\ -h(T)_{z=0} & \text{for } |x|\leqslant A, |y|\leqslant B\end{cases},\quad (T)_{z=d}=0 \tag{4}$$

In Eq. (1), the term $\frac{V}{\alpha}\frac{\partial T}{\partial x}$ represents the energy transport phenomena due to the solid movement in the x-direction relative to the source. This term plays an important role on the heat diffusion in the solid. Indeed, when the speed $V=0$, the problem becomes symmetric and the diffusivity α does not interfere. When $V\neq 0$, the diffusivity becomes an important parameter in the solution. When $V\to\infty$, it can be seen from Eq. (1) that the temperature becomes independent of the movement direction x: $\frac{\partial T}{\partial x}\to 0$.

Taking into account the periodic conditions expressed by Eq. (2), a frequential integral transform can be applied to Eqs. (1)–(4) such as:

$$\tilde{T}_m=\frac{1}{2A}\int_{-A}^{A} Te^{-jm\pi x/A}\,\mathrm{d}x \tag{5}$$

where j is an imaginary unit.

The symmetry conditions in the y-direction, expressed by Eq. (3), allow us to use the finite cosine Fourier integral transform:

$$\tilde{\tilde{T}}_{mn}=\frac{1}{B}\int_{0}^{B}\tilde{T}_m\cos\left(\frac{n\pi y}{B}\right)\mathrm{d}y \tag{6}$$

The double transformation of T leads to the following simple second order differential equation:

$$\frac{\mathrm{d}^2\tilde{\tilde{T}}_{mn}}{\mathrm{d}z^2}-\gamma_{mn}^2\tilde{\tilde{T}}_{mn}=0 \tag{1a}$$

$$-\lambda\left(\frac{\mathrm{d}\tilde{\tilde{T}}_{mn}}{\mathrm{d}z}\right)_{z=0}=\tilde{\tilde{\varphi}}_{mn}-h\big(\tilde{\tilde{T}}_{mn}\big)_{z=0},\quad \big(\tilde{\tilde{T}}_{mn}\big)_{z=d}=0 \tag{4a}$$

where:

$$\gamma_{mn}=\left[\left(\frac{m\pi}{A}\right)^2+\left(\frac{n\pi}{B}\right)^2+j\left(\frac{m\pi}{A}\right)\frac{V}{\alpha}\right]^{1/2}\quad\text{and}\quad \tilde{\tilde{\varphi}}_{mn}=\varphi\frac{\sin(m\pi a/A)\sin(n\pi b/B)}{(m\pi)(n\pi)} \tag{7}$$

Two cases should be noted to build up the solution of Eq. (1a). The first one is that when $m=0$ and $n=0$ (i.e. $\gamma=0$). In this case, the solution is linear and represents the bulk temperature. The second case is concerned with $m\neq 0$ and/or $n\neq 0$ (i.e. $\gamma\neq 0$), which represents the fluctuating term.

The solution of Eq. (1a), with the boundary conditions (4a), is given by:

$$\tilde{\tilde{T}}_{00}=\varphi\frac{ab}{AB}\frac{(d-z)}{hd+\lambda}\quad \text{(for: } m=0 \text{ and } n=0\text{)} \tag{1b}$$

$$\tilde{\tilde{T}}_{mn}=\frac{\tilde{\tilde{\varphi}}_{mn}sh[\gamma_{mn}(d-z)]}{hsh(\gamma_{mn}d)+\lambda\gamma_{mn}ch(\gamma_{mn}d)}\quad \text{(for: } m\neq 0 \text{ and/or } n\neq 0\text{)} \tag{1c}$$

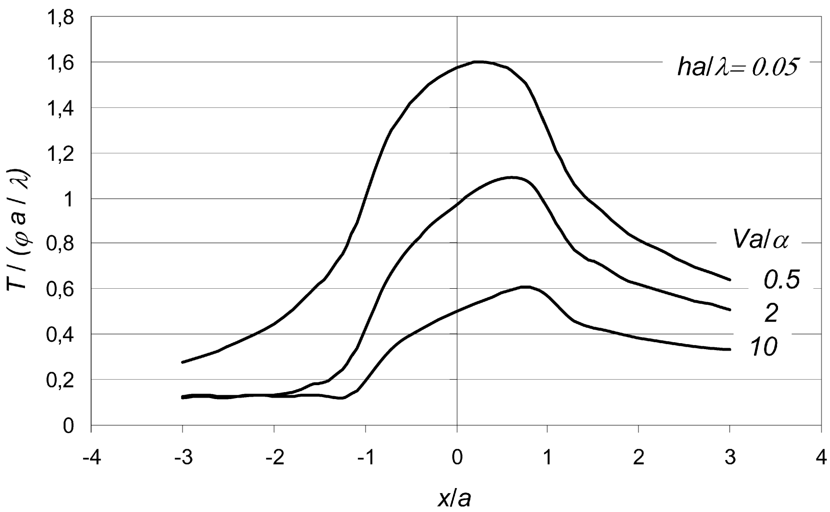

Fig. 2. Dimensionless temperature at $y = 0$ and $z = 0$ for: $A/B = 1$, $a/b = 1$, $a/A = 0.01$, $d/A = 1$.

Fig. 2. Température adimensionnelle en $y = 0$ et $z = 0$ pour : $A/B = 1$, $a/b = 1$, $a/A = 0.01$, $d/A = 1$.

The expression for $T(x, y, z)$ can then be obtained by using the following inverses integral transforms:

$$T = \sum_{m=0}^{\infty} \varepsilon_m \Re_e\left\{\tilde{T}_m e^{jm\pi x/A}\right\} \tag{5a}$$

$$\tilde{T}_m = \sum_{n=0}^{\infty} \varepsilon_n \tilde{\tilde{T}}_{mn} \cos\left(\frac{n\pi y}{B}\right) \tag{6a}$$

with: $\varepsilon_k = 1$ (for: $k = 0$) and $\varepsilon_k = 2$ (for: $k = 1, 2, \ldots, \infty$).

We get:

$$T(x, y, z) = \varphi \frac{ab}{AB} \frac{(d - z)}{hd + \lambda} + 2\sum_{m=1}^{\infty} \Re_e\left\{\tilde{\tilde{T}}_{m0} e^{jm\pi x/A}\right\} + 2\sum_{n=1}^{\infty} \tilde{\tilde{T}}_{0n} \cos\left(\frac{n\pi y}{B}\right) + 4\sum_{n=1}^{\infty}\left[\cos\left(\frac{n\pi y}{B}\right) \sum_{m=1}^{\infty} \Re_e\left\{\tilde{\tilde{T}}_{mn} e^{jm\pi x/A}\right\}\right] \tag{1d}$$

The analytical solution given by Eq. (1d) is explicit and does not necessitate any iterative calculation. It contains two simple series and one double series, all of them are infinite. The number of terms necessary to ensure the convergence of these series depends on the geometric ratios a/A and b/B. This number increases if the geometric ratios are small. For example, if the surface ratio is less than 10^{-4} so, about 1000 terms would be needed to attain the convergence. On the other hand, only 50 terms are needed if the surface ratio is greater than 10^{-2}.

When $V = 0$, γ_{mn} and $\tilde{\tilde{T}}_{mn}$ become real. In this case, the solution is the same as that given by Eq. (1d) with the two terms $\Re_e\{\tilde{\tilde{T}}_{m0} e^{jm\pi x/A}\}$ and $\Re_e\{\tilde{\tilde{T}}_{mn} e^{jm\pi x/A}\}$ respectively replaced by $\tilde{\tilde{T}}_{m0} \cos(m\pi x/A)$ and $\tilde{\tilde{T}}_{mn} \cos(m\pi x/A)$.

When $V \to \infty$, the two terms $\Re_e\{\tilde{\tilde{T}}_{m0} e^{jm\pi x/A}\}$ and $\Re_e\{\tilde{\tilde{T}}_{mn} e^{jm\pi x/A}\}$ approach zero. The temperature becomes independent of the x-direction. This situation is similar to a solid subjected to a strip source of width $2b$ covering the range $[-A, +A]$.

Fig. 2 shows an example of the dimensionless surface temperature evolution at $y = 0$ for different dimensionless speed Va/α. The other parameters are fixed as follows: $A/B = 1$, $a/b = 1$, $a/A = 0.01$, $d/A = 1$, $ha/\lambda = 0.05$. It can be noted that the dimensionless surface temperature $T/(\varphi a/\lambda)$ is higher for small values of Va/α. Also, by increasing Va/α, the maximum dimensionless surface temperature moves towards the contact exit, i.e. $x/a = 1$. This tendency is coherent with the results presented in [5].

4. Conclusions

Many industrial systems such as those of power transmission, braking, material forming, machining, etc. are subjected to local friction and surface cooling. The analytical solution presented in this paper allows – in a simple way – the determination of temperatures and their gradients at any point in the solid, without restriction on the speed and the surface exchange. In this solution, the difficulties related with a 3D numerical calculations, are avoided. Indeed, a numerical solution necessitates particular precautions due to the relative motion and the relatively small dimensions of the heat source. The infinite series converges rapidly, usually with small number of terms. Due to the simplicity of this solution, it can be implemented in a thermomechanical computational code to calculate the stress and deformation fields in a solid.

References

[1] N. Laraqi, An exact explicit analytical solution for steady-state temperature of a half space subjected to a circular moving heat source, ASME Journal of Tribology 125 (2003) 859–862.
[2] J.C. Jaeger, Moving sources of heat and the temperature at sliding contacts, Proc. Royal Soc. NSW 76 (1942) 203–224.
[3] N. Alilat, A. Baïri, N. Laraqi, Three-dimensional calculation of temperature in a rotating disk subjected to an eccentric circular heat source and surface cooling, Numerical Heat Transfer 46 (2) (2004) 167–180.
[4] H.A. Francis, Interfacial temperature distribution within a sliding Hertzian contact, ASLE Trans. 14 (1970) 41–54.
[5] N. Laraqi, N. Alilat, J.M. Garcia-de-Maria, A. Baïri, Temperature and division of heat in a pin-on-disc frictional device – Exact analytical solution, Wear 266 (2009) 765–770.
[6] N.R. DesRuisseaux, R.D. Zerkle, Temperature in semi-infinite and cylindrical bodies subject to moving heat sources and surface cooling, ASME J. Heat Transfer (1970) 456–464.
[7] N. Laraqi, A. Baïri, L. Ségui, Temperature and thermal resistance in frictional devices, Applied Thermal Engineering 24 (17) (2004) 2567–2581.
[8] A. Baïri, N. Laraqi, Diagrams for fast transient conduction in sphere and long cylinder subject to sudden and violent thermal effects on its surface, Applied Thermal Engineering 23 (2003) 1373–1390.
[9] A. Baïri, N. Laraqi, N. Alilat, Z. Zouaoui, Fast transient conduction in infinite plate subject to violent thermal effects, Applied Thermal Engineering 24 (1) (2004) 1–15.
[10] F.F. Ling, T.E. Simkins, Measurement of point-wise juncture condition of temperature at the interface of two bodies in sliding contact, ASME Journal of Basic Engineering 85 (1963) 481–486.
[11] N. Laraqi, A. Baïri, Theory of thermal resistance between solids with randomly sized and located contacts, Int. J. Heat Mass Transfer 45 (20) (2002) 4175–4180.
[12] J.P. Bardon, Bases physiques des conditions de contact thermique imparfait entre milieux en glissement relatif, Revue Générale de Thermique Fr. 386 (1994) 86–91.
[13] B. Salti, N. Laraqi, 3-D numerical modelling of heat transfer between two sliding bodies. Temperature and thermal contact resistance, Int. J. Heat Mass Transfer 42 (13) (1999) 2363–2374.
[14] A. Baïri, J.M. Garcia-de-Maria, N. Laraqi, Effect of thickness and thermal properties of film on the thermal behaviour of moving rough interfaces, European Physical Journal – Applied Physics 26 (1) (2004) 29–34.
[15] J.G. Bauzin, N. Laraqi, Simultaneous estimation of frictional heat flux and two thermal contact parameters for sliding solids, Numerical Heat Transfer 45 (4) (2004) 313–328.