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Algorithm to refine a finite volume mesh admissible for parabolic problems

Florence Hubert [a], Marie-Claude Viallon [b,*]

[a] *Université de Provence, CMI, Technopôle de Château Gombert, 39, rue F. Joliot Curie, 13453 Marseille cedex 13, France*
[b] *UFR Sciences de Saint-Etienne, departement de mathématiques, 23, rue du docteur Paul-Michelon, 42023 Saint-Etienne, France*

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Abstract

We present a simple algorithm to refine a finite volume bidimensional mesh admissible to solve elliptic or parabolic partial differential equations. The approximation of the Laplace operator reduces to the one of the normal fluxes along the edges of control volumes. These normal fluxes can be computed in a consistent way by a classical two points flux approximation simple if the mesh is admissible in the finite volume sense. The originality of the mesh refinement technique that we propose, is to preserve the admissibility property of the meshes. Therefore it can be used in a wide classic context. ***To cite this article: F. Hubert, M.-C. Viallon, C. R. Mecanique 337 (2009).***

Résumé

Algorithme de raffinement de maillage volumes finis admissible. On présente un algorithme simple de raffinement de maillage bidimensionnel de type volumes finis adapté à la résolution d'équations aux dérivées partielles elliptiques ou paraboliques. L'approximation du Laplacien se ramène à celle de flux normaux sur les arêtes des volumes de contrôle. Le calcul du flux est simple si on choisit les centres des mailles de telle sorte que la droite qui joint deux centres voisins soit toujours orthogonale à leur arête commune (maillage admissible). Le processus de raffinement proposé est original car il permet la construction d'un maillage admissible pouvant être utilisé dans un cadre classique très répandu. ***Pour citer cet article : F. Hubert, M.-C. Viallon, C. R. Mecanique 337 (2009).***

Keywords: Computational fluid mechanics; Computer science; Finite volume; Mesh refinement; Laplace operator

Mots-clés : Mécanique des fluides numériques ; Algorithmique ; Volumes finis ; Raffinement de maillage ; Laplacien

* Corresponding author.
E-mail addresses: fhubert@cmi.univ-mrs.fr (F. Hubert), canon@univ-st-etienne.fr (M.-C. Viallon).

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1. Introduction

We are interested in two-dimensional boundary value problems in which the Laplace operator is used, that is to say elliptic or parabolic problems. For simplicity, let us consider the Laplace problem:

$$-\Delta u(x,y) = f(x,y), \quad (x,y) \in \Omega \tag{1}$$

(we don't give any boundary condition to be short), where Ω is an open bounded polygonal subset of $\mathbb{R}^2$, f is a regular function defined on Ω. We discretize (1) by the finite volume method. Let $\mathcal{T}$ be a mesh of Ω such that $\bar{\Omega} = \bigcup_{K \in \mathcal{T}} \bar{K}$, K being open polygonal convex subsets of Ω. We denote by $\mathcal{P}$ a family of points of Ω, $\mathcal{P} = (x_K)_{K \in \mathcal{T}}$. We will refer to x_K as the center of K.

The principle of the finite volume method is to integrate (1) over each cell K of the mesh. This yields:

$$-\int_{\partial K} \operatorname{grad} u . n_K = \int_K f \tag{2}$$

where n_K is the normal to the boundary ∂K, outward to K. Let u_K denote an approximation of $u(x_K)$. If K and L are two adjacent control volumes, if $\sigma_{K/L}$ is the common edge and $m(\sigma_{K/L})$ its length, and $d_{K/L}$ is the distance between the center of the cells K and L, then, the approximation of the normal flux

$$-\int_{\sigma_{K/L}} \operatorname{grad} u . n_K \simeq -m(\sigma_{K/L}) \frac{u_L - u_K}{d_{K/L}} \tag{3}$$

gives a conservative and consistent approximation of the flux if the mesh is admissible, that means $x_K \neq x_L$ and the straight line going through x_K and x_L is orthogonal to $\sigma_{K/L}$ (see [1,2]). This two point flux approximation (TPFA) is not consistent along any edge that does not satisfy the orthogonality condition see [3,9]. If theses "atypical edges" are located along a curve, it is proved in [4] that this TPFA still converges with an order $\frac{1}{2}$ instead of the order 1 classically obtained on admissible mesh. In general the scheme no more converges. In particular, the meshes refined by AMR (Automatic Mesh Refinement) are in general not admissible (see [5,6]) and can not be used with a TPFA scheme.

To cope with this problem, several finite volume approach have been developed within the last 10 years. These methods are all computationally more expensive because of the use of additional unknowns. In return these methods can be used to approximate anisotropic diffusion. A comparison of all these new techniques can be found in [7].

We present in Section 2 a new refinement technique (NRT) of unstructured meshes that connects a coarse Cartesian grid and a fine Cartesian grid and respects the admissibility of the mesh. The refined mesh that we obtain is an admissible mesh and can be used to approximate a diffusion problem with the TPFA scheme with a low cost. Note that the NRT can be used to connect any kind of meshes, even unstructured meshes. We compare in Section 3 the TPFA method on the NRT mesh, on Voronoi meshes and on non-admissible meshes. The NRT meshes reveal to be very efficient for the diffusion approximation.

2. The New Refinement Technique – NRT meshes

Let $(h,l,s) \in \mathbb{R}_+^{*3}$, $h \ll l$. Let $\mathcal{F} = (-l, -\frac{l}{2}] \times (-\frac{l}{2}, \frac{l}{2})$, $\mathcal{R} = (-\frac{l}{2}, 0] \times (-\frac{l}{2}, \frac{l}{2})$, and $\mathcal{C} = (0,s) \times (-\frac{l}{2}, \frac{l}{2})$. We assume $\mathcal{F} \cup \mathcal{R} \cup \mathcal{C} \subset \Omega$. The fine zone $\mathcal{F}$ is discretized with a fine grid (mesh size equal to h) whereas $\mathcal{C}$ is meshed with a coarse grid (mesh size l). We want to connect these two grids with an admissible mesh in the buffer zone $\mathcal{R}$. For sake of simplicity, we assume l (resp. s) is divisible by h (resp. l) and $\frac{l}{h}$ is an even integer. We set $\frac{l}{h} = 2n$, $n \in \mathbb{N}^*$. We will construct the cells in a symmetrical way on both sides of the x axis. That is why in the sequel, we will just consider one part of $\mathcal{R}$ corresponding to $y > 0$.

Let $\mathcal{S}$ be the set of squared cells of size h, and $\mathcal{H}$ be the set of rectangular isosceles triangles of size h. We denote by $\mathcal{S}$-cell an element of $\mathcal{S}$, and $\mathcal{HS}$-cell an element of $\mathcal{H}$ ("half" S-cell). Let us define $S_- = (0, -\frac{l}{2})$, $S_+ = (0, \frac{l}{2})$. Let K_0 be the isosceles triangle $((-h,0), S_-, S_+)$. Let $\mathcal{A}$ be the set of triangles whose S_+ is one vertex, with an horizontal or vertical side or a side parallel to the bisecting line.

The NRT algorithm strategy consists in using in the buffer zone $\mathcal{R}$ as many $\mathcal{S}$- and $\mathcal{HS}$-cells as possible near the fine zone $\mathcal{F}$.

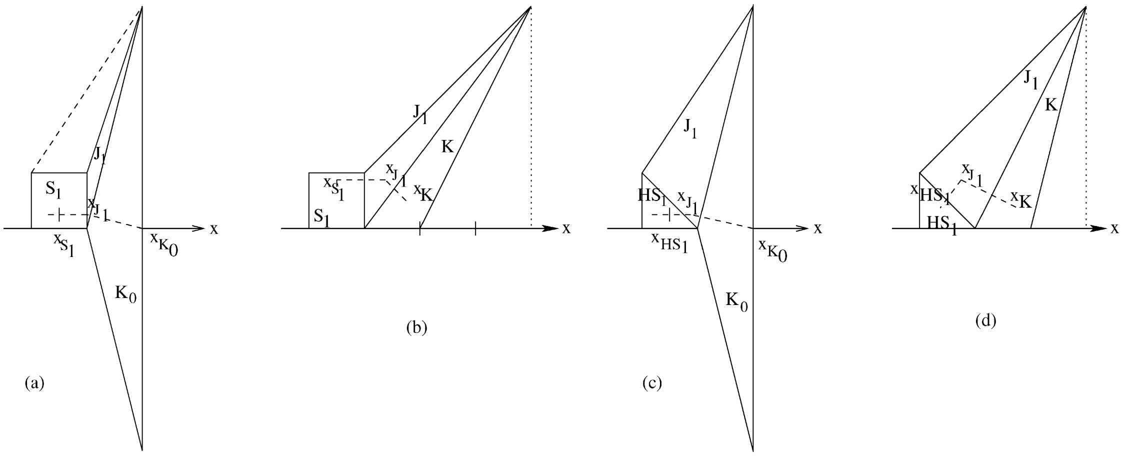


Fig. 1. From left to right: Non-admissible (a) and admissible (b) construction of an $\mathcal{S}$-cell. Non-admissible (c) and admissible (d) construction of an $\mathcal{HS}$-cell.

2.1. Admissible connexion between an $\mathcal{A}$-cell and $\mathcal{S}$- or $\mathcal{HS}$-cells

We first try to connect K_0 to a $\mathcal{S}$-cell or to a $\mathcal{HS}$-cell through a $\mathcal{A}$-cell. We choose $x_{K_0} = (0,0)$.

Definition 2.1. Let $K \in \mathcal{A}$, $x_K \in K$. We say that the cell K can be connected to $J \in \mathcal{S} \cup \mathcal{HS}$ in an admissible way through a $\mathcal{A}$-cell J_1, if K and J have a common vertex S, K (resp. J) and J_1 have a common edge σ_{KJ_1} (resp. σ_{JJ_1}), and there exist $x_J \in J$ and $x_{J_1} \in J_1$ such that the construction is admissible.

Lemma 2.2. *Let $K \in \mathcal{A}$ and $x_K \in K$ its center and $J \in \mathcal{S} \cup \mathcal{HS}$. Assume that S is a common vertex of K and J and note σ an edge of K such that $S \in \partial\sigma$. We introduce $D_{K\sigma}$ the line going through the point x_K perpendicular to σ and D_S^0 the horizontal line going through the point S. The point I denotes the intersection $D_{K\sigma} \cup D_S^0$. We set $r = \frac{h}{SI}$ and $\alpha_{K\sigma}$ the angle between D_S^0 and $D_{K\sigma}$. The cell K can be connected through a $\mathcal{A}$-cell J_1, to $J \in \mathcal{S}$ (resp. $J \in \mathcal{HS}$) if $\alpha_{K\sigma} \geqslant \operatorname{Arctan}(r)$ (resp. if $\operatorname{Arctan}(\frac{r}{2}) \leqslant \alpha_{K\sigma} < \operatorname{Arctan}(r)$).*

If the cell $K \in \mathcal{A}$ can not be connected to an $\mathcal{S}$-cell or $\mathcal{HS}$-cell (i.e. $\alpha_{K\sigma} < \operatorname{Arctan}(\frac{r}{2})$), we construct a new cell $J_1 \in \mathcal{A}$ with an horizontal side of size h and try again to connect this cell to an $\mathcal{S}$- or $\mathcal{HS}$-cell.

In Figs. 1(a) and 1(c), the cell K_0 can not connected neither to a $\mathcal{S}$ nor a $\mathcal{HS}$, the angle $\alpha_{K_0\sigma_0}$ is too small. The examples Figs. 1(b) and 1(d), exhibit successful connections.

In particular in the case $n = 5$, we can not be connected, at the very beginning of the process, the cell K_0 to an $\mathcal{S}$-cell nor to an $\mathcal{HS}$-cell (indeed $r_0 = 1$ and $\alpha_{K_0\sigma_0} < \operatorname{Arctan}(1/2)$). We introduce a new cell K_1 in $\mathcal{A}$ whose vertice are $(-2h, 0)$, S_+, $(-h, 0)$ (see Fig. 2(a) and notations of Lemma 2.2). For any choice of the point $x_{K_1} \in D_{K_0\sigma_0}$, we observe that the new angle $\alpha_{K_1\sigma_1}$ is still small (but $r_1 = \frac{5}{6}$ with for instance $x_{K_1} = (-\frac{8}{5}h, \frac{8}{25}h)$). So we construct another cell K_2 in $\mathcal{A}$ on the left of K_1 and so on. We remark at each step the angle $\alpha_{K_i\sigma_i}$ increases while the ratio r_i becomes smaller. Remark, on Fig. 2(a), that $\operatorname{Arctan}(\frac{r_2}{2}) \leqslant \alpha_{K_2\sigma_2} < \operatorname{Arctan}(r_2)$ so that K_2 can be connected to an $\mathcal{HS}$-cell.

2.2. Refinement algorithm

Let us define $A_i^j = (-ih, jh)$ for $i = 1, \ldots, 2n$ and $j = -n, \ldots, n$. K_0 is the triangle (A_1^0, S_-, S_+).

- **Initialization step.** We define $K_0^0 = K_0$. Set $i_{\max}^0 = 0$, $j = 1$.
- **Main step.** While $i_{\max}^{j-1} \leqslant n - 2$, we proceed as follows:
  - **Step 1.** Set $i = i_{\max}^{j-1}$ for short. Assume that K_i^{j-1} and its center $x_{K_i^{j-1}}$ are already constructed. We naturally introduce an angle α_i^{j-1} and a ratio r_i^{j-1} following Lemma 2.2. If K_i^{j-1} can be connected to an $\mathcal{S}$-

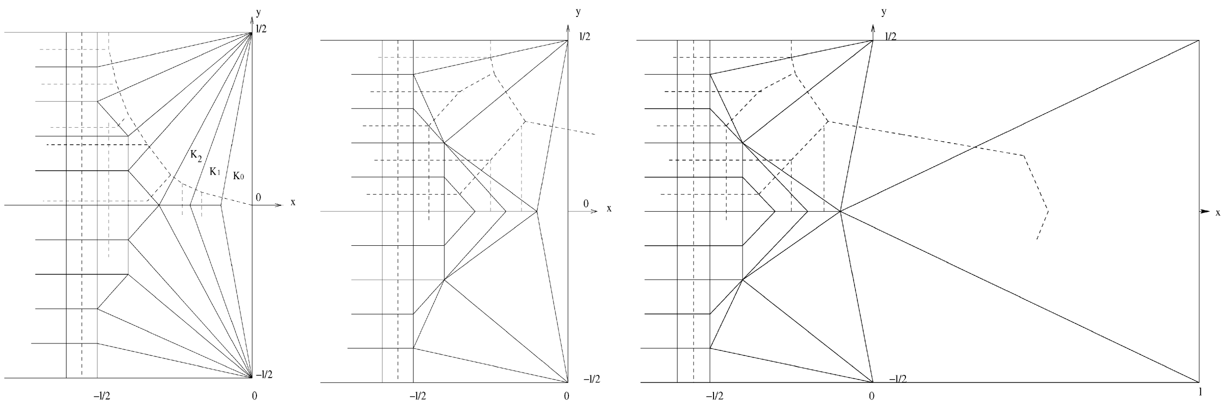


Fig. 2. Comparison of the NRT refined grid and the Voronoi mesh associated to the same nonconformal grid.

or $\mathcal{HS}$-cell then we pass to step 2. Otherwise, if $\alpha_i^{j-1} < \mathrm{Arctan}(\frac{r_i^{j-1}}{2})$, we construct $K_{i+1}^{j-1} \in \mathcal{A}$, the triangle $(S_+, A_{i+1}^{j-1}, A_{i+2}^{j-1})$, and define its center $x_{K_{i+1}^{j-1}}$ to be any point of $D_i^{j-1} \cap K_{i+1}^{j-1}$ and set $i_{\max}^{j-1} = i+1$.

- **Step 2.** Thanks to step 1, the cell $K_{i_{\max}^{j-1}}^{j-1}$ can be connected to an $\mathcal{S}$- or $\mathcal{HS}$-cells through the new cell $K_{i_{\max}^{j}}^{j}$. Set again $i = i_{\max}^{j-1}$ for short.
  - ► If $\mathrm{Arctan}(\frac{r_i^{j-1}}{2}) \leqslant \alpha_i^{j-1} < \mathrm{Arctan}(r_i^{j-1})$, then $i_{\max}^j = i+1$, $K_{i_{\max}^j}^j = (S^+, A_{i+1}^{j-1}, A_{i+2}^j)$ and K_i^{j-1} is connected to the $\mathcal{HS}$-cell $HS_{i+1}^{j-1} = (A_{i+1}^{j-1}, A_{i+2}^{j-1}, A_{i+2}^j)$.
  - ► If $\alpha_i^{j-1} \geqslant \mathrm{Arctan}(r_i^{j-1})$, then $i_{\max}^j = i$, $K_{i_{\max}^j}^j = (S^+, A_{i+1}^{j-1}, A_{i+1}^j)$, and K_i^{j-1} is connected to the $\mathcal{S}$-cell $S_{i+1}^{j-1} = (A_{i+1}^{j-1}, A_{i+2}^{j-1}, A_{i+2}^j, A_{i+1}^j)$.

  We complete the connexion at the "level" $y = (j-1)h$ by the $\mathcal{S}$-cells $S_{k+1}^{j-1} = (A_{k+1}^{j-1}, A_{k+2}^{j-1}, A_{k+2}^j, A_{k+1}^j)$ for $k = i+1, \dots, n-2$.

  We choose the centers $x_{K_{i_{\max}^j}^j}$ in $K_{i_{\max}^j}^j \cap D_i^{j-1}$ and the centers of the $\mathcal{S}$-cells of level $j-1$ to be on the horizontal line going through $x_{K_{i_{\max}^j}^j}$ (respectively $x_{HS_{i+1}^{j-1}}$) if $\alpha_i^{j-1} \geqslant \mathrm{Arctan}(r_i^{j-1})$ (respectively if $\alpha_i^{j-1} < \mathrm{Arctan}(r_i^{j-1})$, and $x_{HS_{i+1}^{j-1}}$ such that $x_{K_{i_{\max}^j}^j} x_{HS_{i+1}^{j-1}}$ and A_{i+1}^{j-1}, A_{i+2}^j are orthogonal). $j = j+1$.

– **Final step.** While $j \leqslant n$, we complete the connexion with $\mathcal{A}$-cells $K_{n-1}^j = (S^+, A_n^{j-1}, A_n^j)$ with centers on the lines D_{n-1}^{j-1}.

Remark 1. Of course, any unstructured mesh may be connected on Fig. 2(a). The method could be generalized in the tridimensional case by a simple geometric process.

3. Numerical tests

We consider Eq. (1) with homogeneous Dirichlet boundary conditions on the domain $\Omega =]-1,1[\times]-1,1[$. The source term f is chosen in such a way that $u_e(x,y) = (1-x^2)(1-y^2)$ is solution of this problem. We compare in the sequel the TPFA approximation on NRT, Voronoi and non-admissible meshes.

3.1. The meshes

Let $N \in \mathbb{N}^*$. We want to connect a coarse square grid (mesh size $\frac{1}{N}$) on $\Omega \cap \{x \geqslant 0\}$, to a fine grid on $\Omega \cap \{x < -\frac{1}{2N}\}$ (mesh size $h = \frac{1}{2nN}$).

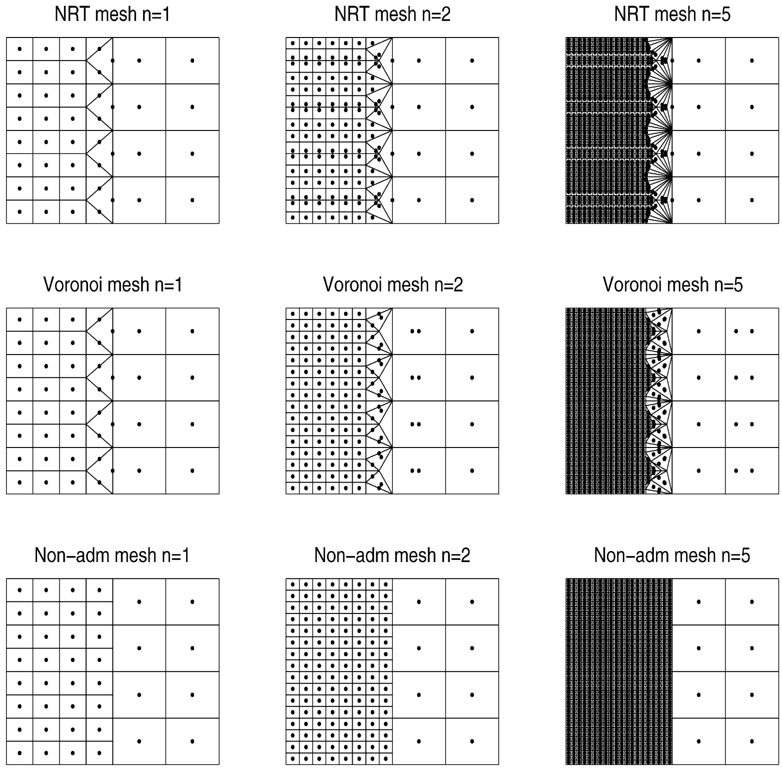


Fig. 3. The meshes corresponding with $N = 2$.

In the "buffer" domain $\Omega \cap \{\frac{-1}{2N} \leqslant x \leqslant 0\}$, we repeat $2N$ times the NRT construction according to the value of the parameter n as far the NRT meshes are concerned. For the non-admissible grids, the fine grid is naturally extended to $\Omega \cap \{x < 0\}$. For the Voronoi meshes, recall that the construction of the mesh consists to determine polygonal regions around some finite set of given point-sites. We construct here the dual mesh whose point-sites coincide with the vertices of the NRT meshes (see [8]).

The three different meshes corresponding with $N = 2$, $n = 1$, $n = 2$ and $n = 5$, are shown on Fig. 3. A zoom of the buffer zone is proposed for the case $n = 5$ in Figs. 2(a) and 2(b). For $n = 1$, the Voronoi and NRT meshes coincide. For $n = 2$, the cells of the two meshes are the same, but the centers of the cells are different.

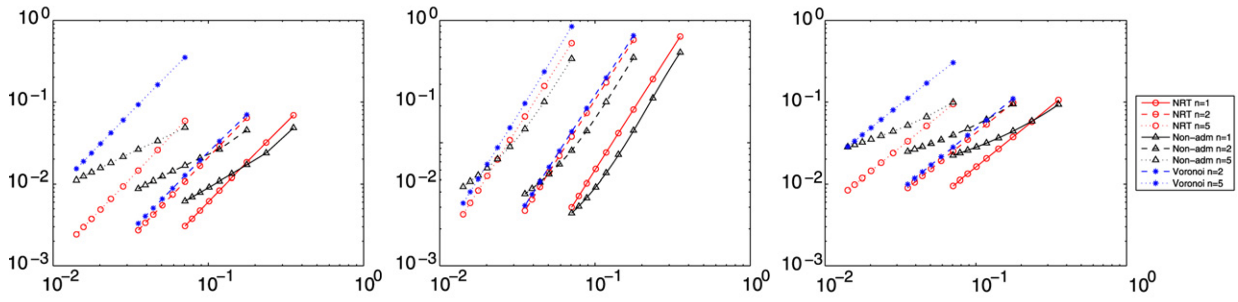
3.2. The comparison

We compare in Fig. 4 the relative error between the two functions

$$u^{\mathcal{T}} = \sum_{K \in \mathcal{T}} u_K \mathbf{1}_K \quad \text{and} \quad u_e^{\mathcal{T}} = \sum_{K \in \mathcal{T}} u_e(x_K) \mathbf{1}_K$$

for the above family of meshes (Fig. 3) as a function of the fine mesh size $\sqrt{2}h$.

Table 1 shows that we obtain a similar order of convergence on both NRT and Voronoi meshes, whereas, as predicted by the theory, the TPFA scheme is less precise on non-admissible meshes. If we compare more carefully the errors on NRT and Voronoi meshes, Fig. 4 shows that the NRT meshes provided a more precise solution. Note also that on coarse grids the non admissible meshes gives in L^2 norm the more precise results. When $n = 5$, let remark that the order of convergence of the scheme is not improved when we use, as shown on Fig. 2(c), the Voronoi mesh with a larger "buffer" zone $\Omega \cap \{\frac{-1}{2N} \leqslant x \leqslant \frac{1}{N}\}$, by adding the points of the coarse grid whose abscissa is $\frac{1}{N}$ in the set of point-sites ($1,438$ in H^1 norm).



(a) The L^∞ norm (b) The L^2 norm (c) The discrete H^1 norm

Fig. 4. Relative error obtained for the TPFA scheme on the different meshes.

Table 1
Order of convergence.

	NRT/Voronoi $n = 1$	NRT $n = 2$	NRT $n = 5$	Non-adm $n = 1$	Non-adm $n = 2$	Non-adm $n = 5$	Voronoi $n = 2$	Voronoi $n = 5$
L^∞-norm	1.936	1.961	1.975	1.229	0.980	0.913	1.902	1.955
L^2-norm	1.990	1.990	1.993	1.890	1.571	1.463	1.982	2.056
H^1-norm	1.496	1.483	1.499	0.873	0.814	0.766	1.489	1.477

4. Conclusion

The construction we proposed is very simple and gives better results than a non-admissible mesh. It is a valuable alternative to Voronoi meshes that are less easy to construct and fit not well narrow "buffer" domains between two grids of different size.

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References

[1] R. Eymard, T. Gallouët, R. Herbin, Finite volume methods, in: P.G. Ciarlet, J.L. Lions (Eds.), Handbook of Numerical Analysis, 2000.
[2] I.D. Mishev, Finite volume methods on Voronoï meshes, Numer. Methods Partial Differential Equations 14 (1998) 193–212.
[3] I. Faille, A control volume method to solve an elliptic equation on a 2D irregular meshing, Comp. Meth. Appl. Mech. Engrg. 100 (1992) 275–290.
[4] R. Cautrés, R. Herbin, F. Hubert, The Lions domain decomposition algorithm on non matching cell-centered finite volume meshes, IMA J. Numer. Anal. 24 (2004) 465–490.
[5] W.J. Coirier, An adaptively-refined, Cartesian, cell-based scheme for the Euler and Navier–Stokes equations. Ph.D. thesis, Michigan Univ., NASA Lewis Research Center, 1994.
[6] M.J. Berger, P. Colella, Local adaptive mesh refinement for shock hydrodynamics, J. Comput. Phys. 82 (1989) 64–84.
[7] R. Herbin, F. Hubert, Benchmark on discretization schemes for anisotropic diffusion problems on general grids, in: Proceedings of the 5th International Symposium on Finite Volumes for Complex Applications, 2008.
[8] F. Hjelle, M. Daehlen, Triangulations and Applications, Springer, 2006, pp. 47–71.
[9] I. Faille, Modélisation bidimensionnelle de la génèse et la migration des hydrocarbures dans un bassin sédimentaire. Thesis, Univ. Grenoble, 1992.