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C. R. Mecanique 337 (2009) 158–165

Comptes Rendus Mecanique

A simple eddy viscosity formulation for turbulent boundary layers near smooth walls

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Received 23 January 2009; accepted after revision 18 March 2009

Available online 11 April 2009

Presented by Jean-Baptiste Leblond

Abstract

The aim of this study is to improve the prediction of near-wall mean streamwise velocity profile U^+ by using a simple method. The U^+ profile is obtained by solving the momentum equation which is written as an ordinary differential equation. An eddy viscosity formulation based on a near-wall turbulent kinetic energy k^+ function [R. Absi, Analytical solutions for the modeled k-equation, ASME J. Appl. Mech. 75 (2008) 044501] and the van Driest mixing length equation [E.R. van Driest, On turbulent flow near a wall, J. Aero. Sci. 23 (1956) 1007] is used. The parameters obtained from the k^+ profiles are used for the computation of U^+ (variables with the superscript of $+$ are those nondimensionalized by the wall friction velocity u_τ and the kinematic viscosity ν). Comparisons with DNS data of fully-developed turbulent channel flows for $109 < Re_\tau < 2003$ show good agreement (where Re_τ denotes the friction Reynolds number defined by u_τ, ν and the channel half-width δ). ***To cite this article: R. Absi, C. R. Mecanique 337 (2009).***

Keywords: Fluid mechanics; Eddy viscosity formulation

Nomenclature

A_k^+, A_l^+, B, C, C_ν	coefficients
k	turbulent kinetic energy
l_m	mixing length
P	pressure
Re_τ	friction Reynolds number ($= \delta u_\tau/\nu$)
x, y	coordinates in respectively the streamwise and wall normal directions
U, V	mean velocity components respectively in the x and y directions
u_τ	wall friction velocity
δ	channel half-width
κ	Kármán constant (≈ 0.41)

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doi:10.1016/j.crme.2009.03.010

ν	kinematic viscosity
ν_t	eddy viscosity
ρ	density
τ	shear stress
τ_w	wall shear stress ($=\rho u_\tau^2$)

All variables with the superscript of $+$ are those nondimensionalized by u_τ and ν.

1. Introduction

Turbulent flows are significantly affected by the presence of walls [1]. Successful predictions of turbulence models used for wall-bounded turbulent flows depend on accurate description of the flow in the near-wall region. Numerous experiments of fully-developed turbulent channel flows, show that the near-wall region can be subdivided into three layers. A viscous sublayer (for a distance from the wall $y^+ < 5$), where the mean velocity U^+ can be approximated by $U^+ = y^+$ and the turbulent kinetic energy k^+ by a quadratic variation $k^+ \approx y^{+2}$ [2]. A fully-turbulent layer or log-law layer (for $y^+ > 30$ until an upper limit), where U^+ can be correctly approximated by the logarithmic profile [3] and k^+ by an exponential decaying function [4]. Between these two layers, a buffer layer, where k^+ can be accurately predicted by an analytical solution [4].

The aim of this Note is to improve the prediction of U^+ by using a simple and accurate method. The U^+ profile will be obtained from the resolution of the momentum equation. An eddy viscosity formulation based on a near-wall turbulent kinetic energy k^+ analytical solution [4], which was validated by DNS data for $109 < Re_\tau < 642$ for $y^+ < 20$, and the van Driest [5] mixing length equation will be used. The values of U^+ and k^+ at an upper limit of the buffer layer could be used as boundary conditions for a turbulence closure model applied in the outer layer.

The test case is the fully developed plane channel flow which is considered to be the simplest and most idealized boundary layer flow. Reynolds number effects on wall turbulence have been investigated by many experimental and computational studies. A review of turbulence closure models for wall-bounded shear flows was presented in Patel et al. (1985) [6], and experiments in the range of $190 < Re_\tau < 1900$ were performed by Wei and Willmarth (1989) [7] to investigate the effects of the Reynolds number very near the wall. There are several DNS studies of plane channel flows which have allowed to improve the knowledge of the boundary layer dynamics. DNS were performed at $Re_\tau = 180$ by Kim et al. (1987) [8], up to $Re_\tau = 590$ by Moser et al. (1999) [9], up to $Re_\tau = 642$ by Iwamoto et al. (2002) [10], up to $Re_\tau = 950$ by del Álamo et al. (2004) [11], and recently at $Re_\tau = 2003$ by Hoyas and Jiménez (2006) [12].

2. Model equations

We consider a steady uniform fully developed incompressible plane channel flow (i.e. the flow between two infinitely large plates, Fig. 1), where x and y are respectively the coordinates in the streamwise and wall normal directions and the corresponding mean velocity components are respectively U and V. The channel half width is δ (can represent the boundary layer thickness), and the flow is driven by a pressure gradient in the streamwise direction.

2.1. Momentum equation

DNS data (Fig. 2) [10] for $109 < Re_\tau < 642$ show that $V \approx 0$ for $y^+ < 20$. By taking $V = 0$, the streamwise momentum equation becomes

$$(1/\rho)\partial_x P = \partial_y\big((\nu + \nu_t)\partial_y U\big) \tag{1}$$

where ν_t is the eddy viscosity, P the pressure and ρ the density. With the shear stress τ, we write Eq. (1) as

$$\partial_x P = \partial_y \tau \tag{2}$$

where $\tau = \rho\,(\nu + \nu_t)\partial_y U$. For a constant $\partial_x P$, by integrating Eq. (2) between $\tau(y = 0) = \tau_w$ and $\tau(y = \delta) = 0$, we obtain $\tau_w = -\delta\,\partial_x P$ and therefore $u_\tau = \sqrt{(\delta/\rho)(-\partial_x P)}$.

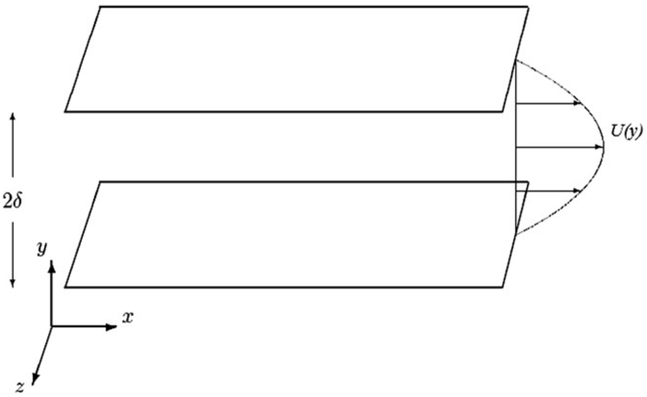


Fig. 1. Sketch of the flow geometry for plane channel flow.

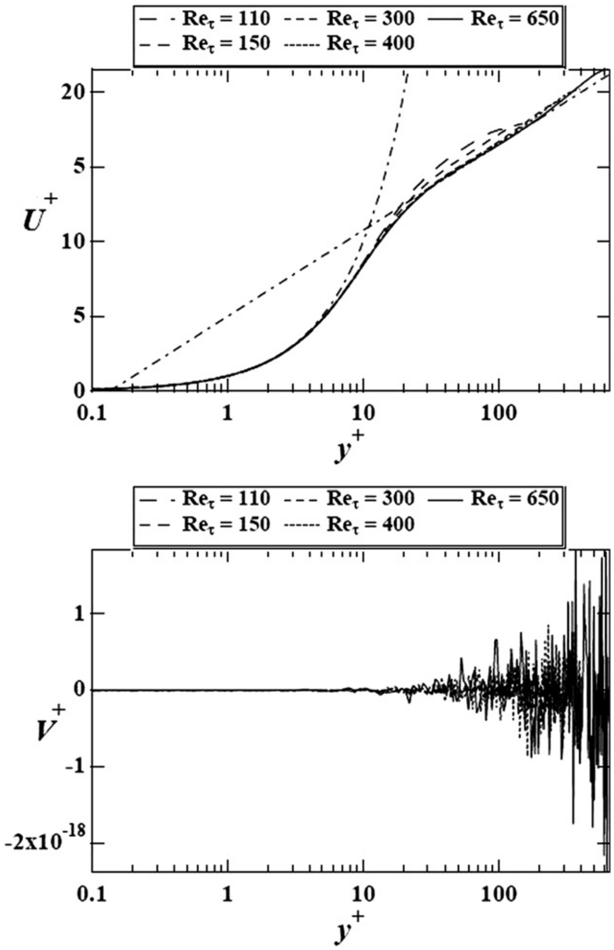


Fig. 2. DNS data [10] of mean velocity profiles for $109 < Re_\tau < 642$. Top figure, $U^+(y^+)$; Bottom figure, $V^+(y^+)$; dash-dotted lines, $U^+ = y^+$ and $U^+ = 2.5\ln(y^+) + 5.0$ (figure adapted from [13]).

By integrating Eq. (1) between $y = 0$ and $y = \delta$, we obtain

$$\frac{\mathrm{d}U}{\mathrm{d}y} = \frac{u_\tau^2}{\nu + \nu_t}\left(1 - \frac{y}{\delta}\right) \tag{3}$$

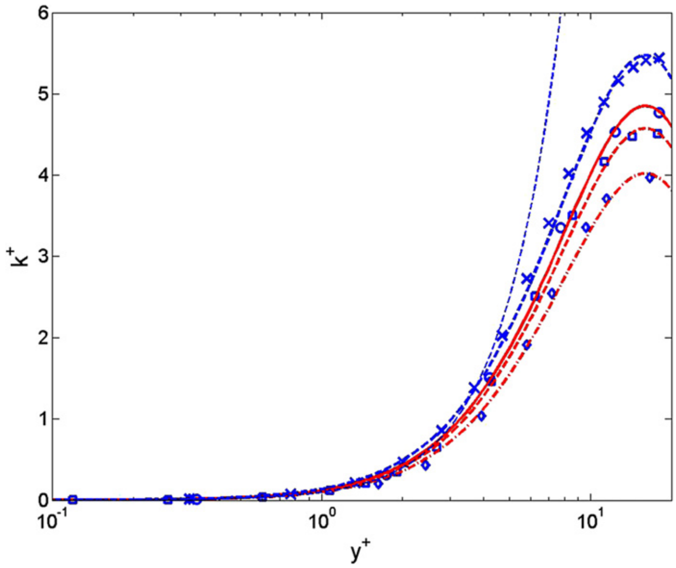


Fig. 3. Turbulent kinetic energy $k^+(y^+)$ for different Reynolds numbers (for $y^+ < 20$). Symbols, DNS data. $Re_\tau = 150$, diamonds [10], dash-dotted line Eq. (9) with $A_k^+ = 8$ and $B = 0.116$; $Re_\tau = 395$, squares [10], dashed line Eq. (9) with $A_k^+ = 8$ and $B = 0.132$; $Re_\tau = 642$, circles [10], solid line Eq. (9) with $A_k^+ = 8$ and $B = 0.14$; $Re_\tau = 2003$, $\times$ [12], dashed line Eq. (9) with $A_k^+ = 8$ and $B = 0.158$; Thin dashed line, $k^+ = 0.1 y^{+2}$.

Or in wall units

$$\frac{\mathrm{d}U^+}{\mathrm{d}y^+} = \frac{1}{1+\nu_t^+}\left(1 - \frac{y^+}{Re_\tau}\right) \tag{4}$$

where $U^+ = U/u_\tau$, $y^+ = y u_\tau/\nu$ and $\nu_t^+ = \nu_t/\nu$. The resolution of the ordinary differential equation (4) needs the dimensionless eddy viscosity ν_t^+.

2.2. A near-wall eddy viscosity formulation

The eddy viscosity is given by

$$\nu_t = C_\nu \sqrt{k}\, l_m \tag{5}$$

where l_m is the mixing length and C_ν a coefficient.

On the one hand, the mixing length is given by the van Driest [5] equation

$$l_m = \kappa y\left(1 - e^{-y^+/A_l^+}\right) \tag{6}$$

where κ is the Kármán constant (≈ 0.41) and $A_l^+ = 26$. We write ν_t^+ from Eqs. (5) and (6) as

$$\nu_t^+ = C_\nu \sqrt{k^+}\, l_m^+ \tag{7}$$

where $k^+ = k/u_\tau^2$ and $l_m^+ = \kappa y^+(1 - e^{-y^+/A_l^+})$.

On the other hand, we obtained a general analytical solution for the modeled k-equation [4]. For steady uniform channel flows, we write the k-equation as $\partial_y(\nu_t \partial_y k) = -(G + \partial_y(\nu \partial_y k) - \epsilon)$, where G and ϵ are respectively the energy production and dissipation. With an approximation for the right-hand side as $(G + d_y(\nu d_y k) - \epsilon) \approx 1/y^2$ and by integrating, we obtained [4]

$$k^+ = B y^{+2C} e^{(-y^+/A_k^+)} \tag{8}$$

where A_k^+, B and C are coefficients. Examination of Eq. (8) by DNS data of channel flows shows that for $y^+ \leqslant 20$, $C = 1$, $A_k^+ = 8$ and B is Re_τ-dependent. We write therefore k^+ for $y^+ \leqslant 20$ as

Table 1
Values of coefficient $B(Re_\tau)$ obtained from Eq. (9) and DNS data.

Re_τ	109	150	298	395	642	2003
B	0.11	0.116	0.127	0.132	0.14	0.158

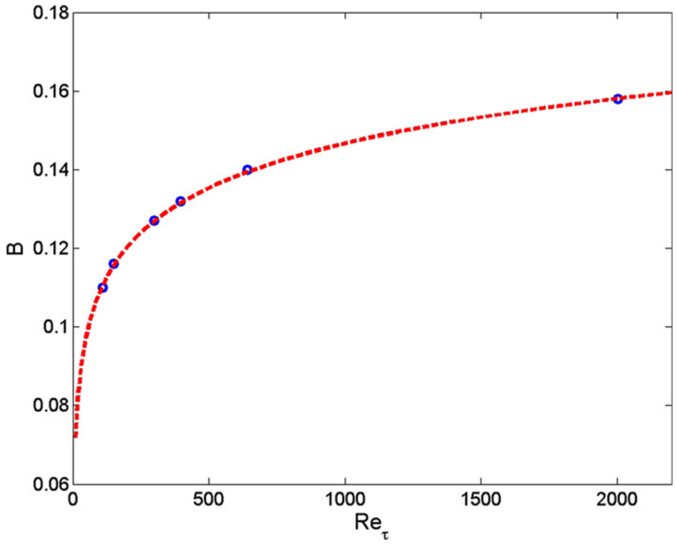


Fig. 4. Dependency of the coefficient B on the Reynolds number Re_τ. ○, values obtained from DNS data; Curve, proposed function (10).

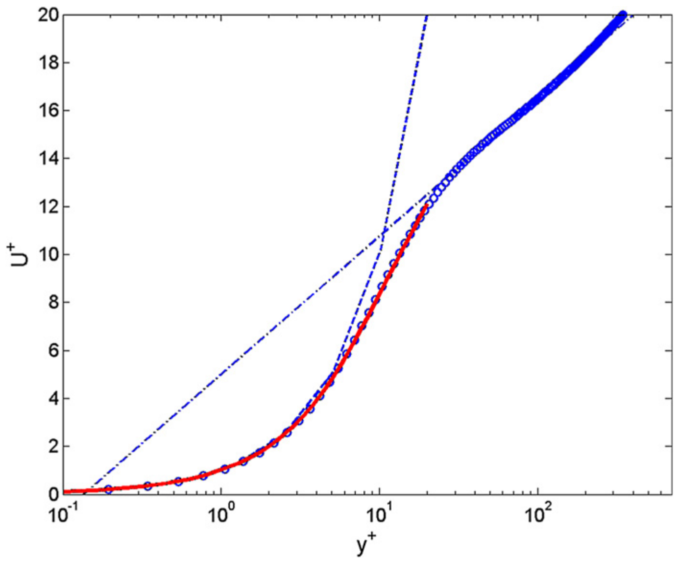


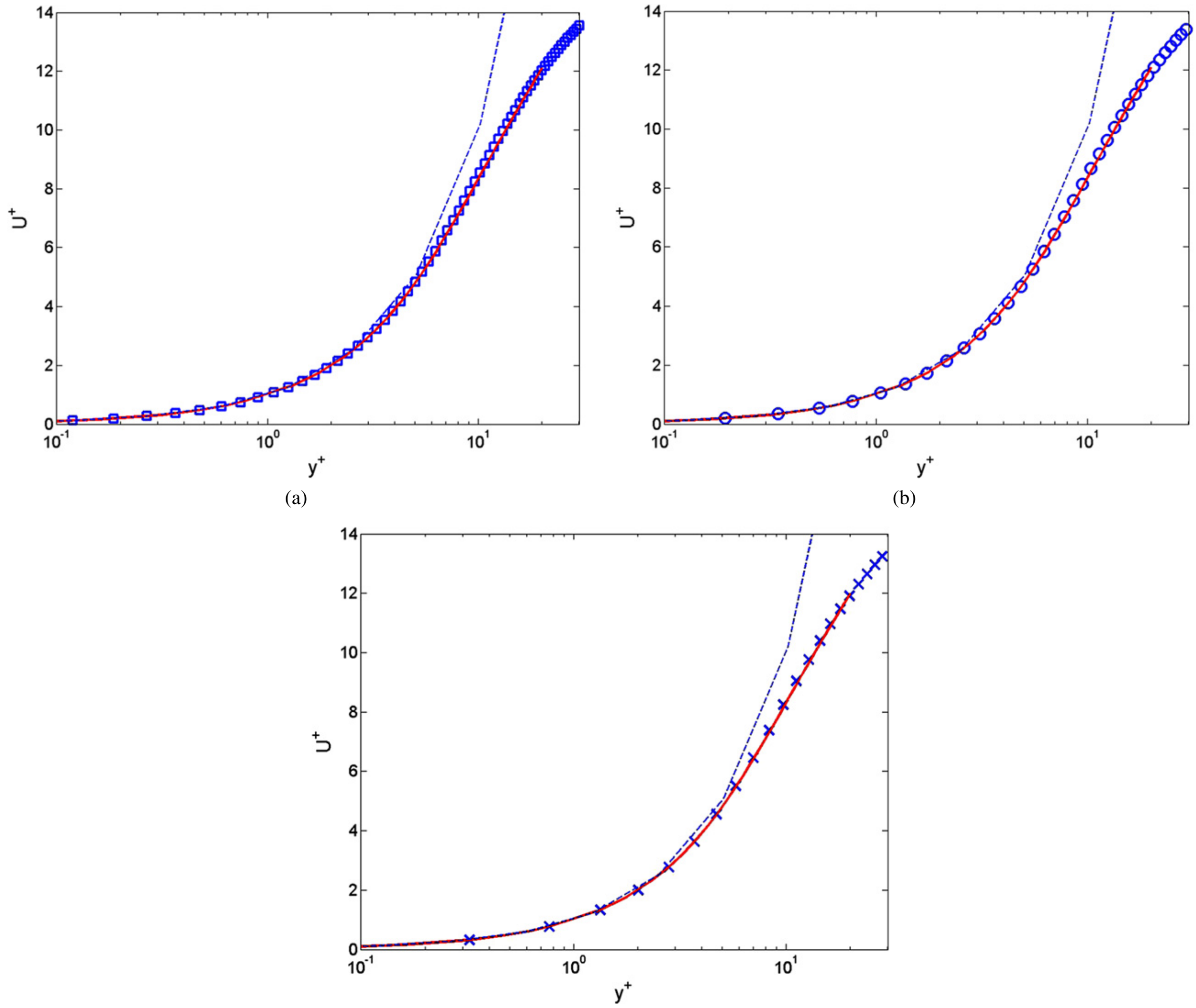
Fig. 5. Mean streamwise velocity profile $U^+(y^+)$ for $Re_\tau = 642$. ○, DNS data [10]. Curves: Bold solid line, solution of Eq. (4) with Eq. (11) ($C_\nu = 0.3$, $A_l^+ = 26$, $A_k^+ = 8$ and $B = 0.14$); Dashed line, $U^+ = y^+$; Dash-dotted line, $U^+ = 2.5\ln(y^+) + 5.0$.

$$k^+ = B y^{+2} e^{(-y^+/A_k^+)} \tag{9}$$

Table 1 gives values of $B(Re_\tau)$ obtained from Eq. (9) and DNS data [10,12].

We propose the following function Eq. (10) for the coefficient B

$$B(Re_\tau) = C_{B1}\ln(Re_\tau) + C_{B2} \tag{10}$$



(c)

Fig. 6. Mean streamwise velocity profiles $U^+(y^+)$ for $Re_\tau \geqslant 395$. Symbols, DNS data. Curves: Thin dashed line, $U^+ = y^+$; Solid lines, solution of Eq. (4) with Eq. (11) ($A_l^+ = 26$, $A_k^+ = 8$); (a) $Re_\tau = 395$, squares [10], solid line ($C_\nu = 0.3$, $B = 0.132$); (b) $Re_\tau = 642$, circles [10], solid line ($C_\nu = 0.3$, $B = 0.14$); (c) $Re_\tau = 2003$, × [12], solid line ($C_\nu = 0.3$, $B = 0.158$).

where C_{B1} and C_{B2} are constants. The calibration (Fig. 4) gives $C_{B1} = 0.0164$ and $C_{B2} = 0.0334$.

We noticed that the series expansion of the exponential in Eq. (9) at the first order gives $k^+ = By^{+2} - (B/A_k^+)y^{+3}$. This equation is similar to the approximation deduced from the continuity equation and the no-slip condition [2] (page 608). However, the quadratic variation of k (first term in the right-hand side) is valid only in the immediate vicinity of the wall ($y^+ < 5$). Eq. (9) is therefore a more general and more accurate solution (Fig. 3).

With Eq. (9), we write the dimensionless eddy viscosity (Eq. (7)) as

$$\nu_t^+ = C_\nu \kappa B^{0.5} y^{+2} e^{-y^+/(2A_k^+)} \left(1 - e^{-y^+/A_l^+}\right) \tag{11}$$

3. Results and discussions

Predicted mean streamwise velocity $U^+(y^+)$ profiles are obtained from Eq. (4) and Eq. (11). Fig. 5 presents $U^+(y^+)$ profile for $Re_\tau = 642$. Solution of Eq. (4) with Eq. (11) (solid line), where $A_l^+ = 26$, $A_k^+ = 8$, $B = 0.14$ and

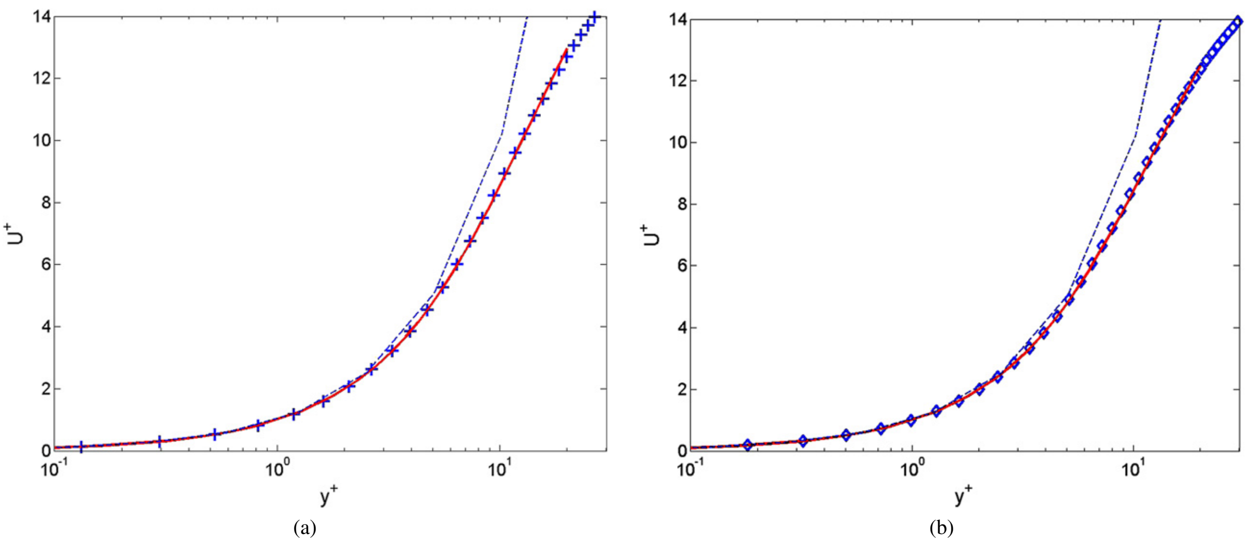


Fig. 7. Mean streamwise velocity profiles $U^+(y^+)$ for $Re_\tau < 395$. Symbols, DNS data; Curves: Thin dashed line, $U^+ = y^+$; Solid lines, solution of Eq. (4) with Eq. (11) ($A_l^+ = 26$, $A_k^+ = 8$); (a) $Re_\tau = 109$, + [10], solid line ($C_\nu = 0.2$, $B = 0.11$); (b) $Re_\tau = 150$, diamonds [10], solid line ($C_\nu = 0.25$, $B = 0.116$).

$C_\nu = 0.3$, is compared to DNS data [10]. The predicted $U^+(y^+)$ profile (solid line) shows good agreement with DNS data. Values of $A_k^+ = 8$ and $B = 0.14$ are those of the k^+ profile (Fig. 3).

In order to verify the dependency of the coefficient C_ν on the Reynolds number Re_τ, we present predicted $U^+(y^+)$ profiles for different Re_τ (Fig. 6). Profiles of Fig. 6 for $Re_\tau = 395$, $Re_\tau = 642$ and $Re_\tau = 2003$ were obtained with $C_\nu = 0.3$ and values of $A_k^+ = 8$ and B (Eq. (10)) obtained from the k^+ profiles (Fig. 3). It seems that C_ν is independent of the Reynolds number for $Re_\tau \geqslant 395$ and is equal to 0.3. The values of $B(Re_\tau)$ obtained from the k^+ profiles (Eq. (10)) are suitable for computation of $U^+(y^+)$ profiles. However, for $Re_\tau = 150$ and $Re_\tau = 109$ (Fig. 7) the required values of C_ν are respectively 0.25 and 0.2. Therefore, C_ν seems to be Re_τ-dependent for Re_τ less than 395. This dependency seems to be associated to low-Reynolds-number effects. Indeed, Moser et al. [9] showed that low-Reynolds-number effects are absent for $Re_\tau > 390$. We notice that for $y^+ < 20$, the required C_ν is different from $C_\mu^{1/4}$ (with C_μ is the empirical constant in the k–ϵ model, equal to 0.09). For $Re_\tau \geqslant 395$, $C_\nu = C_\mu^{1/2}$.

4. Conclusion

In summary, mean streamwise velocity profiles U^+ were obtained by solving a momentum equation which is written as an ordinary differential equation. The analytical eddy viscosity formulation is based on a near-wall analytical solution for the turbulent kinetic energy k^+ and the van Driest mixing length equation. The parameters obtained from the calibration of k^+ were used for the computation of U^+. Comparisons with DNS data of fully-developed turbulent channel flows show good agreement. Our simulations show that for $Re_\tau \geqslant 395$ the coefficient of proportionality C_ν in the eddy viscosity equation is independent of Re_τ and equal to 0.3. However, for $Re_\tau < 395$, the coefficient C_ν is Re_τ-dependent. The values of $k^+(y^+ = 20)$ and $U^+(y^+ = 20)$ could be used as boundary conditions for a turbulence closure model applied for $y^+ \geqslant 20$.

Acknowledgements

The author would like to thank N. Kasagi, K. Iwamoto, Y. Suzuki from the University of Tokyo and J. Jimenez, S. Hoyas from Universidad Politécnica de Madrid, for providing the DNS data.

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