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C. R. Mecanique 337 (2009) 173–178
COMPTES RENDUS
MECANIQUE

Homogenization of $p_\varepsilon(x)$-Laplacian in perforated domains with a nonlocal transmission condition

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Received 24 May 2008; accepted after revision 20 March 2009

Available online 16 April 2009

Presented by Évariste Sanchez-Palencia

Abstract

We study the asymptotic behavior, as $\varepsilon \to 0$, of u^ε solutions to a nonlinear elliptic equation with nonstandard growth condition in domains containing a grid-type microstructure $\mathcal{F}^\varepsilon$ that is concentrated in an arbitrary small neighborhood of a given hypersurface Γ. We assume that $u^\varepsilon = A^\varepsilon$ on $\partial\mathcal{F}^\varepsilon$, where A^ε is an unknown constant. The macroscopic equation and a nonlocal transmission condition on Γ are obtained by the variational homogenization technique in the framework of Sobolev spaces with variables exponents. This result is then illustrated by a periodic example. ***To cite this article: B. Amaziane et al., C. R. Mecanique 337 (2009).***

Résumé

Une condition de transmission non locale dans l'homogénéisation du $p_\varepsilon(x)$-Laplacian dans des domaines perforés. On étudie le comportement asymptotique, lorsque $\varepsilon \to 0$, des solutions u^ε d'une équation elliptique non linéaire de croissance non standard dans des domains qui contiennent une microstructure ayant la forme d'une grille. Cette microstructure est concentrée dans un petit voisinage arbitraire d'une hypersurface Γ. On suppose que $u^\varepsilon = A^\varepsilon$ sur $\partial\mathcal{F}^\varepsilon$, où A^ε est une constante inconnue. L'équation macroscopique et une condition de transmission non locale sur Γ sont obtenues par la technique de l'homogénéisation variationnelle dans le cadre des espaces de Sobolev avec des exposants variables. On présente un exemple périodique pour illustrer le résultat obtenu. ***Pour citer cet article : B. Amaziane et al., C. R. Mecanique 337 (2009).***
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Keywords: Fluid mechanics; Homogenization; Nonlinear variational problem; Nonstandard growth

Mots-clés : Mécanique des fluides ; Homogénéisation ; Problème variationnel non linéaire ; Croissance non standard

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1631-0721/$ – see front matter
doi:10.1016/j.crme.2009.03.011

1. Introduction

In this Note we study the homogenization of the following nonlinear boundary value problem:

$$\begin{cases} -\mathrm{div}\big(\big|\nabla u^\varepsilon\big|^{p_\varepsilon(x)-2}\nabla u^\varepsilon\big)+\big|u^\varepsilon\big|^{\sigma(x)-2}u^\varepsilon=g(x) \quad \text{in } \Omega^\varepsilon \\ u^\varepsilon=A^\varepsilon \quad \text{on } \partial\mathcal{F}^\varepsilon; \quad u^\varepsilon=0 \quad \text{on } \partial\Omega; \quad \displaystyle\int_{\partial\mathcal{F}^\varepsilon}\big|\nabla u^\varepsilon\big|^{p_\varepsilon(x)-2}\frac{\partial u^\varepsilon}{\partial\nu}\,\mathrm{d}s=0 \end{cases} \tag{1}$$

where $\varepsilon>0$; $\Omega^\varepsilon=\Omega\setminus\overline{\mathcal{F}^\varepsilon}$ is a perforated domain in $\mathbb{R}^n$ ($n\geqslant 3$) with Ω being a bounded Lipschitz domain and $\mathcal{F}^\varepsilon$ being an open connected subset of Ω like a net that is concentrated near a hypersurface $\Gamma\Subset\Omega$; A^ε is an unknown constant; the growth functions p_ε and σ satisfy some conditions that will be specified in Section 2; g is a given function. These equations are known as $p_\varepsilon(x)$-Laplacian equations.

In recent years, there has been an increasing interest in the study of such equations (in the case where there is non dependance on the small parameter) motivated by their applications to the mathematical modeling in continuum mechanics. These equations arise, for example, from the modeling of non-Newtonian fluids with thermo-convective effects (see for instance [1]), the modeling of electro-rheological fluids (see, e.g., [2]), and the motion of a compressible fluid in a heterogeneous anisotropic porous medium obeying to the nonlinear Darcy law (see, e.g., [3,4]).

In this Note we deal with the variational problem corresponding to the nonlinear equation (1). For a review of homogenization problems of the Lagrangians with variable exponents and rapidly oscillating coefficients, we refer for instance to [5] and the bibliography therein. The Dirichlet homogenization problem for Lagrangians of $p_\varepsilon(x)$ growth in perforated domains has been studied recently in [6].

Following the approach developed in [7–9], instead of a classical periodicity assumption on the structure of the perforated domain, we impose certain conditions on the so-called *local energy characteristics* of Ω^ε. It will be shown that the asymptotic behavior of u^ε solution of (1) (as $\varepsilon\to 0$) is described by an elliptic boundary value problem with a nonlocal transmission condition on the hypersurface Γ. Note that various transmission conditions were constructed in [9] by the method of local energy characteristics. Let us also mention that nonlocal homogenized models were already obtained for a class of nonlinear elliptic equations in divergence form with non-uniformly bounded coefficients, the elasticity equations, and for some linear problems in the electrostatics, see for instance [10–12,9] and the references therein.

The proof of the main result is based on the variational homogenization technique which is nowadays widely used in the homogenization theory (see, e.g., [13,9,14] and the references therein).

2. Statement of the problem and the main result

Let Ω be a bounded domain in $\mathbb{R}^n$ ($n\geqslant 3$) with sufficiently smooth boundary and let $\mathcal{F}^\varepsilon$ be an open subset in Ω. Here $\varepsilon>0$ is a small parameter characterizing the scale of the microstructure. We set $\Omega^\varepsilon=\Omega\setminus\overline{\mathcal{F}^\varepsilon}$. We assume that $\mathcal{F}^\varepsilon$ is distributed in an asymptotically regular way in an arbitrary small neighborhood of a hypersurface $\Gamma\Subset\Omega$, i.e., for any ball $B(y,r)$ of radius r centered at $y\in\Gamma$ and $\varepsilon>0$ small enough ($\varepsilon\leqslant\varepsilon_0(r)$), $B(y,r)\cap\mathcal{F}^\varepsilon\neq\emptyset$ and $B(y,r)\cap\Omega^\varepsilon\neq\emptyset$.

Let p_ε be a continuous function defined in $\overline{\Omega}$. We assume that, for any $\varepsilon>0$, it satisfies the conditions:

(i) p_ε is bounded in $\overline{\Omega}$, i.e., $1<\mathsf{p}^-\leqslant p_\varepsilon^-\equiv\min_{x\in\overline{\Omega}}p_\varepsilon(x)\leqslant p_\varepsilon(x)\leqslant\max_{x\in\overline{\Omega}}p_\varepsilon(x)\equiv p_\varepsilon^+\leqslant\mathsf{p}^+\leqslant n$.
(ii) p_ε is log-continuous, i.e., for any $x,y\in\Omega$, $|p_\varepsilon(x)-p_\varepsilon(y)|\leqslant\omega_\varepsilon(|x-y|)$, where $\overline{\lim}_{\tau\to 0}\omega_\varepsilon(\tau)\ln(\frac{1}{\tau})\leqslant C$.
(iii) p_ε converges uniformly in Ω to a function p_0, where p_0 is log-continuous.
(iv) p_ε satisfies the inequality: $p_\varepsilon(x)\geqslant p_0(x)$ in Ω.

Let σ be a log-continuous function in Ω such that, for any $\varepsilon>0$,

(v) $1<\sigma^-\equiv\min_{x\in\overline{\Omega}}\sigma(x)\leqslant\sigma(x)\leqslant\max_{x\in\overline{\Omega}}\sigma(x)\equiv\sigma^+\leqslant np_0(x)/(n-p_0(x))$ in $\overline{\Omega}$.

In what follows we refer to [3] (see also the bibliography therein) for the properties of Sobolev spaces with variable exponents. Following [3], for any $\varepsilon>0$, we introduce the Sobolev space $W^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon)$ with a variable exponent

p_ε defined by $W^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon)=\{\phi\in L^{p_\varepsilon(\cdot)}(\Omega^\varepsilon)\colon\ |\nabla\phi|\in L^{p_\varepsilon(\cdot)}(\Omega^\varepsilon)\}$. Here by $L^{p_\varepsilon(\cdot)}(\Omega^\varepsilon)$ we denote the space of measurable functions ϕ in Ω^ε such that $\Upsilon_{p_\varepsilon(\cdot),\Omega^\varepsilon}(\phi)=\int_{\Omega^\varepsilon}|\phi(x)|^{p_\varepsilon(x)}\,\mathrm{d}x<+\infty$. This space equipped with the norm $\|\phi\|_{L^{p_\varepsilon(\cdot)}(\Omega^\varepsilon)}=\inf\{\lambda>0\colon\ \Upsilon_{p_\varepsilon(\cdot),\Omega^\varepsilon}(\phi/\lambda)\leqslant 1\}$ is a Banach space.

Consider the variational problem:

$$J^\varepsilon[u]\equiv\int_{\Omega^\varepsilon}\mathsf{F}_\varepsilon(x,u,\nabla u)\,\mathrm{d}x\longrightarrow\inf,\quad u^\varepsilon\in W^{1,p_\varepsilon(\cdot)}\big(\Omega^\varepsilon\big);\quad u^\varepsilon=A^\varepsilon\quad\text{on }\partial\mathcal{F}^\varepsilon\quad\text{and}\quad u^\varepsilon=0\quad\text{on }\partial\Omega \tag{2}$$

where $\mathsf{F}_\varepsilon(x,u,\nabla u)=\frac{1}{p_\varepsilon(x)}|\nabla u|^{p_\varepsilon(x)}+\frac{1}{\sigma(x)}|u|^{\sigma(x)}-g(x)u$; A^ε is an unknown constant; $g\in C(\Omega)$. It is known from [3] that, for each $\varepsilon>0$, there exists a unique solution $u^\varepsilon\in W^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon)$ of problem (2).

We extend u^ε by the equality $u^\varepsilon=A^\varepsilon$ in $\mathcal{F}^\varepsilon$ and we keep for it the same notation. Thus, we obtain the family $\{u^\varepsilon\}\subset W^{1,p_\varepsilon(\cdot)}(\Omega)$. We study the asymptotic behavior of the family $\{u^\varepsilon\}$ as $\varepsilon\to 0$.

Instead of the classical periodicity assumption on the microstructure of the perforated domain Ω^ε, we impose a condition on the local (or mesoscopic) characteristic of massiveness of the sets $\mathcal{F}^\varepsilon$ (for more details see [9], p. 336). To this end, for any piece S of Γ, we introduce a layer $T_h(S)$ generated by the surfaces $\Gamma_h^-(S)$, $\Gamma_h^+(S)$. These surfaces are formed by the ends of the normal vectors of the length $h>0$ taken on the both sides of S. Consider the functional:

$$c^{\varepsilon,h}(S)=\inf_{v^\varepsilon}\int_{T_h(S)}\left\{\frac{1}{p_\varepsilon(x)}\big|\nabla v^\varepsilon\big|^{p_\varepsilon(x)}+h^{-\mathsf{p}^+-\gamma}\big|v^\varepsilon-1\big|^{p_\varepsilon(x)}\right\}\mathrm{d}x \tag{3}$$

where $\gamma>0$ and the infimum is taken over $v^\varepsilon\in W^{1,p_\varepsilon(\cdot)}(T_h(S))$ that equal zero in $\mathcal{F}^\varepsilon$.

We make the following further assumption:

(C.1) *for any arbitrary piece $S\subset\Gamma$, there exist the limits*:

$$\lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}}\,c^{\varepsilon,h}(S)=\lim_{h\to 0}\underline{\lim_{\varepsilon\to 0}}\,c^{\varepsilon,h}(S)=\int_S\mathsf{c}(x)\,\mathrm{d}S \tag{4}$$

where c *is a nonnegative continuous function on* Γ.

To formulate the homogenization result for the variational problem (2) we introduce the functional:

$$J_{hom}[v,B]\equiv\int_\Omega\mathsf{F}_0(x,v,\nabla v)\,\mathrm{d}x+\int_\Gamma\mathsf{c}(x)|v-B|^{p_0(x)}\,\mathrm{d}S$$

where $\mathsf{F}_0(x,v,\nabla v)=\frac{1}{p_0(x)}|\nabla v|^{p_0(x)}+\frac{1}{\sigma(x)}|v|^{\sigma(x)}-g(x)\,v$. It is clear that the functional $J_{hom}[v,B]$ is strictly convex in v, B. Moreover, it is continuous in the space $W_0^{1,p_0(\cdot)}(\Omega)$ with respect to the variable v.

The main result of the Note is the following:

Theorem 2.1. *Let u^ε be a solution of* (2) *extended by the equality $u^\varepsilon=A^\varepsilon$ in $\mathcal{F}^\varepsilon$. Let assumptions* (i)–(v) *and* (C.1) *hold. Then u^ε converges weakly in $W^{1,p_0(\cdot)}(\Omega)$ to a function u such that the pair $\{u(x),A\}$, where $A=\lim_{\varepsilon\to 0}A^\varepsilon$, is a solution of*

$$J_{hom}[v,B]\longrightarrow\inf,\quad\{v,B\}\in W_0^{1,p_0(\cdot)}(\Omega)\times\mathbb{R} \tag{5}$$

Remark 1. It is important to notice that the constant A in (5) remains unknown. When $p_0,\sigma\geqslant 2$ in Ω, Euler's equation for the functional in (5) reads:

$$\begin{cases}-\operatorname{div}\big(|\nabla u|^{p_0(x)-2}\nabla u\big)+|u|^{\sigma(x)-2}u=g(x)\quad\text{in }\Omega\setminus\Gamma\\ u=0\quad\text{on }\partial\Omega;\quad[u]_\Gamma^\pm=0,\quad\left[|\nabla u|^{p_0(x)-2}\dfrac{\partial u}{\partial\nu}\right]_\Gamma^\pm=c'_u(x,u-A)\quad\text{and}\quad\displaystyle\int_\Gamma c'_u(x,u-A)\,\mathrm{d}S=0\end{cases} \tag{6}$$

where ν is a normal vector to Γ, $[\cdot]_\Gamma^\pm$ is the jump of the corresponding function on Γ, $c(x,u-A)=\mathsf{c}(x)|u-A|^{p_0(x)}$, c'_u is the partial derivative of c with respect to u. This means that problem (5) contains a nonlocal transmission condition.

3. Sketch of the proof of Theorem 2.1

It follows from the definition of J^ε that the solution of (2) satisfies the bound: $\|u^\varepsilon\|_{W^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon)} \leqslant C$, where C is a constant that does not depend on ε. We extend u^ε by A^ε to $\mathcal{F}^\varepsilon$ and consider $\{u^\varepsilon\}$ as a sequence in $W^{1,p_\varepsilon(\cdot)}(\Omega)$. It is clear that $\|u^\varepsilon\|_{W^{1,p_\varepsilon(\cdot)}(\Omega)} \leqslant C$. Now the condition (iv) implies that $\|u^\varepsilon\|_{W^{1,p_0(\cdot)}(\Omega)} \leqslant C$. Following the ideas of [12], one can also prove that $|A^\varepsilon| \leqslant C$, where C does not depend on ε. Therefore, there is a subsequence $\{u^\varepsilon, \varepsilon = \varepsilon_k \to 0\}$ that converges weakly to a function u in the space $W^{1,p_0(\cdot)}(\Omega)$ and $A = \lim_{\varepsilon=\varepsilon_k\to 0} A^\varepsilon$. We will show that the pair $\{u, A\}$ is a solution of the variational problem (5). The proof will be done in two mains steps.

Step 1. Upper bound. Let us cover Γ by a finite number of sets S_i' ($i = 1, \dots, N$) with nonintersecting interiors. We suppose that the diameters $d_i(N)$ of S_i' are small such that $\delta = \max_i d_i(N) \to 0$ as $N \to +\infty$. For any $i = 1, \dots, N$, we introduce a convex set $\widetilde{S}_i$ with a piecewise smooth boundary that satisfies the following properties: (a) $\overline{S}_i \subset S_i' \subset \widetilde{S}_i$, where $S_i = S_i' \setminus (\bigcup_{i\neq j} \widetilde{S}_j)$; (b) diam $\widetilde{S}_i \leqslant C\,\delta$; (c) the number of intersections $\widetilde{S}_i \cap \widetilde{S}_j$ is bounded by an integer M that does not depend on N; (d) $\sum_{i=1}^N \text{meas}(\widetilde{S}_i \setminus S_i) = o(1)$ as $\delta \to 0$.

We associate with the covering $\{\widetilde{S}_i\}$ a partition of unity $\{\varphi_i(\bar{x}), \bar{x} \in \Gamma\}$ satisfying the following conditions: $0 \leqslant \varphi_i(\bar{x}) \leqslant 1$ in Γ; $\varphi_i(\bar{x}) = 0$ for $\bar{x} \notin \widetilde{S}_i$; $\varphi_i(\bar{x}) = 1$ for $\bar{x} \in S_i$; $\sum_i \varphi_i(\bar{x}) \equiv 1$ in Γ; $|D^\beta \varphi_i(\bar{x})| \leqslant C\rho^{-1-\gamma/\mathsf{p}^+}$, where ρ is the distance between ∂S_i and $\partial \widetilde{S}_i$.

Let now w be a smooth function in Ω such that $w = 0$ on $\partial\Omega$ and let B be an arbitrary constant. Denote by $\mathfrak{L}_\theta$ a subset of layers $T_h(\widetilde{S}_i)$ covering Γ such that $|w(x) - B| > \theta > 0$ for any $x \in T_h(\widetilde{S}_i)$. We set $b_i = w(x^i) - B$, where $x^i \in S_i$, for $T_h(\widetilde{S}_i) \in \mathfrak{L}_\theta$ and $b_i = 1$ for $T_h(\widetilde{S}_i) \notin \mathcal{L}_\theta$.

Let now w be a smooth function in Ω such that $w = 0$ on $\partial\Omega$ and let B be an arbitrary constant. We denote by $v_i^\varepsilon(x)$ the minimizer of the functional in (3) with $S = \widetilde{S}_i$. In the domain Ω^ε we introduce the function:

$$w_h^{(\varepsilon)}(x) = \left(w(x) + \sum_{i=1}^{N_\delta}\big[w(x) - B\big]\big(v_i^\varepsilon(x) - 1\big)\varphi_i(\bar{x})\right)\psi(x) + w(x)\big[1 - \psi(x)\big]$$

Here $\psi \in C^\infty(\overline{\Omega})$ is a cut-off function such that $0 \leqslant \psi(x) \leqslant 1$ in $\overline{\Omega}$, $\psi(x) = 1$ in $T_{h'}(\Gamma)$, $\psi(x) = 0$ in $\Omega \setminus T_h(\Gamma)$, and $|\nabla\psi(x)| \leqslant Cr^{-1}$, where $T_{h'}(\Gamma)$ is the layer with the middle surface Γ and of thickness $2h' = 2(h - r)$ with $r = h^{1+\gamma}$.

It follows from the properties of the functions $v_i^\varepsilon(x)$, φ_i, and ψ that $w_h^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon)$. Moreover, $w_h^s = 0$ on $\partial\Omega$ and $w_h^s = B$ on $\partial\mathcal{F}^\varepsilon$. Since u^ε is the minimizer of the functional J^ε, then we have that $J^\varepsilon[u^\varepsilon] \leqslant J^\varepsilon[w_h^\varepsilon]$. Estimating the right-hand side of this inequality we get:

$$\lim_{\delta\to 0}\lim_{h\to 0}\lim_{\varepsilon\to 0} J^\varepsilon\big[u^\varepsilon\big] \leqslant \int_\Omega \mathsf{F}_0(x, w, \nabla w)\,\mathrm{d}x + \int_\Gamma \mathsf{c}(x)|w - B|^{p_0(x)}\,\mathrm{d}S = J_{hom}[w] \tag{7}$$

Inequality (7) is obtained for $w \in C_0^\infty(\Omega)$. Then it follows from density arguments and the continuity of the functional in $W_0^{1,p_0(\cdot)}(\Omega)$ that it remains true for any $w \in W_0^{1,p_0(\cdot)}(\Omega)$.

Step 2. Lower bound. Let $u \in W_0^{1,p_0(\cdot)}(\Omega)$ be a weak limit in $W^{1,p_0(\cdot)}(\Omega)$ of the sequence $\{u^\varepsilon\} \subset W_0^{1,p_\varepsilon(\cdot)}(\Omega)$ (extended by $u^\varepsilon = A^\varepsilon$ in $\mathcal{F}^\varepsilon$) by a subsequence $\varepsilon = \varepsilon_k$. For any $\varrho > 0$, we introduce a function $u_\varrho \in C_0^\infty(\Omega)$ such that $\|u - u_\varrho\|_{W^{1,p_0(\cdot)}(\Omega)} < \varrho$. One can show that there is a sequence $\{w_\varrho^\varepsilon\} \subset W_0^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon)$ with $w_\varrho^\varepsilon = 0$ in $\mathcal{F}^\varepsilon$ that converges weakly in $W^{1,p_0(\cdot)}(\Omega)$ to $(u - u_\varrho)$. We set $u_\varrho^\varepsilon = u^\varepsilon + w_\varrho^\varepsilon$. Then $\lim_{\varrho\to 0}\overline{\lim}_{\varepsilon=\varepsilon_k\to 0}\|u_\varrho^\varepsilon - u^\varepsilon\|_{W^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon)} = 0$ and it follows from the continuity of the functional in $W_0^{1,p_0(\cdot)}(\Omega)$ that $\lim_{\varrho\to 0}\overline{\lim}_{\varepsilon=\varepsilon_k\to 0}|J^\varepsilon[u_\varrho^\varepsilon] - J^\varepsilon[u^\varepsilon]| = 0$ and $\lim_{\varrho\to 0} J_{hom}[u_\varrho] = J_{hom}[u]$. Now it is clear that the bound,

$$\underline{\lim}_{\varepsilon=\varepsilon_k\to 0} J^\varepsilon\big[u^\varepsilon\big] \geqslant J_{hom}[u] \tag{8}$$

immediately follows from the inequality: $\underline{\lim}_{\varepsilon=\varepsilon_k\to 0} J^\varepsilon[u_\varrho^\varepsilon] \geqslant J_{hom}[u_\varrho]$ (*ϱ-lower bound*).

Let us prove the *ϱ-lower bound*. To this end, we cover the layer $T_h(\Gamma)$ by the layers $T_h(S_i')$ with nonintersecting interiors (see the definition of $T_h(S_i')$ in the paragraph "Upper bound") and introduce the notation: $T_\theta^\pm = \{x \in T_h(\Gamma)\colon\ \pm(u_\varrho - A) > \theta > 0\}$; $T_\theta = T_\theta^+ \cup T_\theta^-$; $T_\theta^\varepsilon = T_\theta \cap \Omega^\varepsilon$; $\widetilde{T}_{\theta,h}^\pm = \{\bigcup_i T_h(S_i')\colon\ T_h(S_i') \Subset T_\theta^\pm\}$;

$\widetilde{T}_{\theta,h}=\widetilde{T}^{+}_{\theta,h}\cup\widetilde{T}^{-}_{\theta,h}$; $\widetilde{T}^{\varepsilon}_{\theta,h}=\widetilde{T}_{\theta,h}\cap\Omega^{\varepsilon}$; $\mathcal{O}_\theta=T_h(\Gamma)\setminus T_\theta$; $\mathcal{O}^{\varepsilon}_\theta=\mathcal{O}_\theta\cap\Omega^{\varepsilon}$. Notice that since u_ϱ is a smooth function in Ω, then $\lim_{h\to 0}\operatorname{meas}(T_\theta\setminus\widetilde{T}_{\theta,h})=0$.

We rewrite $J^{\varepsilon}[u^{\varepsilon}_{\varrho}]$ as follows:

$$J^{\varepsilon}\big[u^{\varepsilon}_{\varrho}\big]=\int\limits_{\Omega\setminus T_h(\Gamma)}\mathsf{F}_{\varepsilon}\big(x,u^{\varepsilon}_{\varrho},\nabla u^{\varepsilon}_{\varrho}\big)\,\mathrm{d}x+\int\limits_{\widetilde{T}^{\varepsilon}_{\theta,h}}\mathsf{F}_{\varepsilon}\big(x,u^{\varepsilon}_{\varrho},\nabla u^{\varepsilon}_{\varrho}\big)\,\mathrm{d}x+\int\limits_{T^{\varepsilon}_{\theta}\setminus\widetilde{T}^{\varepsilon}_{\theta,h}}\mathsf{F}_{\varepsilon}\big(x,u^{\varepsilon}_{\varrho},\nabla u^{\varepsilon}_{\varrho}\big)\,\mathrm{d}x$$
$$+\int\limits_{\mathcal{O}^{\varepsilon}_{\theta}}\mathsf{F}_{\varepsilon}\big(x,u^{\varepsilon}_{\varrho},\nabla u^{\varepsilon}_{\varrho}\big)\,\mathrm{d}x \tag{9}$$

Conditions (i)–(iv) imply the inequality:

$$\lim_{h\to 0}\varliminf_{\varepsilon=\varepsilon_k\to 0}\int\limits_{(T^{\varepsilon}_{\theta}\setminus\widetilde{T}^{\varepsilon}_{\theta,h})\cup\mathcal{O}^{\varepsilon}_{\theta}}\mathsf{F}_{\varepsilon}\big(x,u^{\varepsilon}_{\varrho},\nabla u^{\varepsilon}_{\varrho}\big)\,\mathrm{d}x\geqslant\int\limits_{\mathcal{O}_\theta}\mathsf{F}_0(x,u_\varrho,\nabla u_\varrho)\,\mathrm{d}x \tag{10}$$

Finally, following the lines of [7] and using the definition (3), for any $T_h(S'_i)\subset T^{\pm}_{\theta}$, as $h\to 0$, we have:

$$\varliminf_{\varepsilon=\varepsilon_k\to 0}\int\limits_{T_h(S'_i)\cap\Omega^{\varepsilon}}\mathsf{F}_{\varepsilon}\big(x,u^{\varepsilon}_{\varrho},\nabla u^{\varepsilon}_{\varrho}\big)\,\mathrm{d}x\geqslant\int\limits_{T_h(S'_i)}\mathsf{F}_0(x,u_\varrho,\nabla u_\varrho)\,\mathrm{d}x+|b_i|^{p_0(x^i)}\varliminf_{\varepsilon=\varepsilon_k\to 0}c^{\varepsilon,h}\big(S'_i\big)+o\big(h^n\big) \tag{11}$$

where $b_i=u_\varrho(x^i)-A$ with $x^i\in S'_i$.

Taking into account (9)–(11), and the condition (C.1), we pass to the limit first as $h\to 0$, then as $\varrho\to 0$ and $\delta\to 0$, and finally as $\theta\to 0$. This leads to the ϱ-lower bound and, therefore, to the lower bound (8).

Now it follows from (7), (8) that $J_{hom}[u]\leqslant J_{hom}[w]$ for any $w\in W^{1,p_0(\cdot)}_0(\Omega)$, where u is the weak limit of the solution of (2), extended by $u^{\varepsilon}=A^{\varepsilon}$ in $\mathcal{F}^{\varepsilon}$. This completes the proof of Theorem 2.1.

4. A periodic example

As an application of the previous general result, we now give an example of a perforated medium, where the set $\mathcal{F}^{\varepsilon}$ and the growth function p_ε are given explicitly.

Theorem 2.1 provides sufficient conditions for the existence of the homogenized problem (5). The goal of this section is to prove that, for an appropriate example, all the conditions of Theorem 2.1 are satisfied and to compute the function $\mathsf{c}(x)$ in the homogenized problem (5) explicitly.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. We suppose that the set $\mathcal{F}^{\varepsilon}$ consists of thin intersecting cylinders of radius $r^{(\varepsilon)}=e^{-1/\varepsilon}$. Moreover, the axes of the cylinders belong to a plane $\Gamma\Subset\Omega$ and form an ε-periodic lattice in $\mathbb{R}^2$. We set $\Omega^{\varepsilon}=\Omega\setminus\overline{\mathcal{F}^{\varepsilon}}$.

Let $\{p_\varepsilon\}_{(\varepsilon>0)}$ be a class of smooth functions in $\overline{\Omega}$ given by:

$$p_\varepsilon(x)=\begin{cases}2+\varepsilon\ell(x) & \text{in }\mathcal{N}(\mathcal{F}^{\varepsilon},\varepsilon^2)\\ 2+\ell_\varepsilon(x) & \text{elsewhere}\end{cases} \tag{12}$$

where $\mathcal{N}(\mathcal{F}^{\varepsilon},\varepsilon^2)$ denotes the cylindrical ε^2-neighborhood of the set $\mathcal{F}^{\varepsilon}$ and where ℓ, ℓ_ε are smooth strictly positive functions in $\overline{\Omega}$, moreover, $\max_{x\in\overline{\Omega}}\ell_\varepsilon(x)=o(1)$ as $\varepsilon\to 0$. It is clear that p_ε satisfies conditions (i)–(iv), and converges uniformly in $\overline{\Omega}$ to the function $p_0\equiv 2$.

We study the asymptotic behavior of the solution of problem (1), where the growth p_ε is given by (12), the growth σ satisfies condition (v) of Theorem 2.1 with $\sigma^-\geqslant 2$ and $g\in C(\Omega)$. We show that the homogenized model in this case is given by (6), where

$$\mathsf{c}(x)=4\pi\mu(x),\quad A=\bigg(\int\limits_{\Gamma}\mu(x)\,\mathrm{d}S\bigg)^{-1}\int\limits_{\Gamma}\mu(x)u(x)\,\mathrm{d}S\quad\text{with }\mu(x)=\frac{e^{l(x)}-1}{l(x)} \tag{13}$$

We end this section with the following remark:

Remark 2. It is known from [9] that in the case of a surface distribution of $\mathcal{F}^\varepsilon$, with a constant growth $p_\varepsilon(x) = 2 + \alpha$, where $\alpha > 0$ is a parameter independent of ε, there is no 3D lattice for the corresponding problem which leads to homogenization because the capacity of the lattice goes to infinity as $\varepsilon \to 0$. However this gives an example of the growth $p_\varepsilon \sim 2 + \varepsilon$ (in a small neighborhood of the lattice) which leads to a nontrivial homogenization result.

Acknowledgements

This paper was completed when L. Pankratov was visiting the Applied Mathematics Laboratory, CNRS–UMR 5142, of the University of Pau. He is grateful for the invitation and hospitality. V. Prytula acknowledges the support from MEC through the grants AP2005-4528, FIS2007-29090-E and the grant from Junta de Comunidades de Castilla-La Mancha (PCI08-0093).

References

[1] S.N. Antontsev, J.F. Rodrigues, On stationary thermo-rheological viscous flows, Ann. Univ. Ferrara, Sez., Sci. Mat. 52 (2006) 19–36.
[2] M. Růžička, Electrorheological Fluids: Modeling and Mathematical Theory, Springer, Berlin, 2000.
[3] S.N. Antontsev, S.I. Shmarev, Elliptic equations with anisotropic nonlinearity and nonstandard growth conditions, in: M. Chipot, P. Quittner (Eds.), Handbook of Differential Equations: Stationary Partial Differential Equations, Elsevier, Amsterdam, 2006, pp. 1–100.
[4] S.N. Antontsev, S.I. Shmarev, Existence and uniqueness of solutions of degenerate parabolic equations with variable exponents of nonlinearity, J. Math. Sci. 150 (2008) 2289–2301.
[5] B. Amaziane, S. Antontsev, L. Pankratov, A. Piatnitski, Γ-convergence and homogenization of functionals on Sobolev spaces with variable exponents, J. Math. Anal. Appl. 342 (2008) 1192–1202.
[6] B. Amaziane, S. Antontsev, L. Pankratov, Homogenization of a class of nonlinear elliptic equations with nonstandard growth, C. R. Mécanique 335 (2007) 138–143.
[7] M. Goncharenko, V. Prytula, Homogenization of the electrostatic problems in nonlinear medium with thin perfectly conducting grids, J. Math. Phys. Anal. Geom. 2 (2006) 424–448.
[8] E. Khruslov, L. Pankratov, Homogenization of the Dirichlet variational problems in Orlicz–Sobolev spaces, Fields Inst. Commun. 25 (2000) 345–366.
[9] V.A. Marchenko, E.Ya. Khruslov, Homogenization of Partial Differential Equations, Birkhäuser, Berlin, 2006.
[10] M. Bellieud, G. Bouchitté, Homogenization of elliptic problems in a fiber reinforced structure. Nonlocal effects, Ann. Scuola Norm. Sup. Cl. Sci. IV 26 (1998) 407–436.
[11] M. Briane, J. Casado-Diaz, Two-dimensional div-curl results. Application to the lack of nonlocal effects in homogenization, Comm. Partial Differential Equations 32 (2007) 935–969.
[12] E. Khruslov, Homogenization of Maxwell's equations in domains with dense perfectly conducting grids, Ukrainskii Matematicheskii Vestnik 2 (2005) 119–151.
[13] A. Braides, A. Defranceschi, Homogenization of Multiple Integrals, Oxford Lecture Series in Mathematics and its Applications, vol. 12, Clarendon Press, Oxford, 1998.
[14] V.V. Zhikov, S.M. Kozlov, O.A. Oleinik, Homogenization of Differential Operators and Integral Functionals, Springer-Verlag, Berlin, 1994.