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Gaps in the essential spectrum of infinite periodic necklace-shaped elastic waveguide

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Abstract

We describe a periodic homogeneous elastic waveguide of a particular shape of beads connected by ligaments of diameter $O(h)$ such that the essential spectrum contains gaps, the number of which grows unboundedly when h tends to $+0$. ***To cite this article: S.A. Nazarov et al., C. R. Mecanique 337 (2009).***

Résumé

Gaps dans le spectre essentiel d'un guide d'onde élastique, infini et périodique, ayant la forme d'un collier. Nous décrivons un guide d'ondes élastique homogène et périodique, ayant la forme particulière de collier constitué de grains reliés par des ligaments de diamètre $O(h)$ de telle sorte que le spectre essentiel contienne des gaps dont le nombre augmente infiniment quand h tend vers zéro. ***Pour citer cet article : S.A. Nazarov et al., C. R. Mecanique 337 (2009).***

Keywords: Periodic elastic waveguide; Gap-band structure of the spectrum

Mots-clés : Guide d'onde périodique élastique ; Faille dans le spectre

1. The waveguide

Let ϖ be a convex domain in $\mathbb{R}^3$, with a smooth boundary $\partial\varpi$ and a compact closure $\overline{\varpi} = \varpi \cup \partial\varpi$, such that

$$\varpi \subset \{x = (y, z):\ y = (y_1, y_2) \in \mathbb{R}^2, |z| < H/2\}, \quad \mathcal{O}^{\pm} = (0, 0, \pm H/2) \in \partial\varpi \tag{1}$$

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Fig. 1. The periodic waveguide and the limit case of disjoint beads.

Let also ω be a domain in $\mathbb{R}^2$ such that $\overline{\omega}$ is compact and $\partial\omega$ is smooth, and let $\Omega_h = \omega_h \times \mathbb{R}$ and $\omega_h = \{y \in \mathbb{R}^2\colon\ \eta := h^{-1}y \in \omega\}$. The ratio h/H is small and after rescaling we set $H = 1$ so that $h \in (0, 1]$ and the Cartesian coordinates $x = (x_1, x_2, x_3)$ become dimensionless. The periodic waveguide $\Pi_h = \Omega_h \cup \bigcup_{j\in\mathbb{Z}} \varpi_h^j$ (Fig. 1a) consists of thin infinite straight needle Ω_h and the periodic family of beads

$$\varpi(j) = \{x\colon\ (y, z - j) \in \varpi\}, \quad j \in \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\} \tag{2}$$

The set $\varpi_h = \{x \in \Pi_h\colon\ |z| < 1/2\}$ is called the periodicity cell of the quasi-cylinder Π_h. At $h = 0$ the set Π_h turns into the union of disconnected domains (2) (Fig. 1b).

For $h \in (0, h_0]$ we consider the spectral elasticity problem

$$L(\nabla_x)u^h := \mu\Delta_x u^h - (\lambda + \mu)\nabla_x\nabla_x \cdot u^h = \varrho\Lambda^h u^h \quad \text{in } \Pi_h, \qquad N(x, \nabla_x)u^h := \sigma^{(\nu)}(u) = 0 \quad \text{on } \partial\Pi_h \tag{3}$$

where $\lambda \geqslant 0$, $\mu \geqslant 0$ are the Lamé constants, $\varrho > 0$ is the constant material density, and Λ^h is a spectral parameter (square of the oscillation frequency). Furthermore, $\nabla_x = \text{grad}$, $\nabla\cdot = \text{div}$, $\Delta_x = \nabla_x \cdot \nabla_x$ is the Laplacian in the variables x, $u^h = (u_1^h, u_2^h, u_3^h)$ stands for the displacement vector,

$$\sigma_j^{(\nu)}(u) = \sum_{k=1}^{3} \nu_k(x)\sigma_{jk}(u^h), \quad \sigma^{(\nu)} = (\sigma_1^{(\nu)}, \sigma_2^{(\nu)}, \sigma_3^{(\nu)})$$

$\nu = (\nu_1, \nu_2, \nu_3)$ is the unit vector of the outward normal defined for almost all points of the piecewise smooth surface $\partial\Pi_h$, and the strains and stresses are given by

$$\varepsilon_{jk}(u^h) = \frac{1}{2}(\partial_{x_k}u_j^h + \partial_{x_j}u_k^h), \qquad \sigma_{jk}(u^h) = 2\mu\varepsilon_{jk}(u^h) + \lambda\delta_{j,k}(\varepsilon_{11}(u^h) + \varepsilon_{22}(u^h) + \varepsilon_{33}(u^h))$$

with Kronecker's symbol δ_{jk}. Here $\partial_{x_k} = \partial/\partial x_k$ and further $\partial_z = \partial/\partial z$.

2. The operator formulation of the problem

The problem (3) admits the variational formulation [1,2] with the elastic energy quadratic form $\frac{1}{2}a$,

$$a(u, v; \Pi_h) = \sum_{j,k=1}^{3} (\sigma_{jk}(u), \varepsilon_{jk}(u))_{\Pi_h} \tag{4}$$

where $(\cdot,\cdot)_{\Pi_h}$ is the natural scalar product in the Lebesgue space $L^2(\Pi_h)$. The form (4) is closed and positive Hermitian in the Sobolev space $H^1(\Pi_h)$. The Birman–Krein–Vishik theory (cf. [3] and [4, Ch. 10]) can therefore be applied to transform (3) to the abstract formulation

$$\mathcal{T}^h u^h = \Lambda^h u^h \tag{5}$$

where $\mathcal{T}^h$ is an unbounded self-adjoint positive operator in $L^2(\Pi)$. The spectrum $\Sigma(\mathcal{T})$ lies in $\overline{\mathbb{R}}_+ = [0, \infty)$. Since the embedding $H^1(\Pi_h) \subset L^2(\Pi_h)$ is not compact, the essential spectrum $\Sigma_{\text{ess}}(\mathcal{T})$ is not empty (cf. [4, Th. 10.1.5]. Moreover, it is known (see [5–7] and others) that the spectrum gets the band-gap structure, namely

$$\Sigma(\mathcal{T}^h) = \Sigma_{\text{ess}}(\mathcal{T}^h) = \bigcup_{p=1}^{\infty} \Upsilon_p^h \tag{6}$$

where $\Upsilon_p^h = \{M_p^h(\eta)\colon\ \eta \in [0, 2\pi)\}$ are closed segments and

$$0 \leqslant M_1^h(\eta) \leqslant M_2^h(\eta) \leqslant \cdots \leqslant M_p^h(\eta) \leqslant \cdots \to +\infty \tag{7}$$

constitutes the eigenvalue sequence for the following model spectral problem on the periodicity cell ϖ_h:

$$\begin{aligned} &L(\nabla_y, \partial_z + i\eta)V^h = \varrho M^h v^h && \text{in } \varpi_h \\ &N(x, \nabla_y, \partial_z + i\eta)V^h = 0 && \text{in } \upsilon_h \\ &V^h(y, 1/2) = V^h(y, -1/2), \quad \partial_z V^h(y, 1/2) = \partial_z V^h(y, -1/2) && \text{for } y \in \omega_h \end{aligned} \tag{8}$$

where $\upsilon_h = \partial\varpi_h \setminus \overline{(\omega_h^+ \cup \omega_h^-)}$ is the lateral side of the cell. Notice that the periodicity conditions are imposed only on the small cross-sections $\omega^\pm = \omega_h \times \{\pm 1/2\}$ of the needle Ω_h.

The model problem (8) is derived from (3) using the Gelfand transform

$$v(y, z) \mapsto V(y, z; \eta) = \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{Z}} \exp\bigl(-i\eta(z + m)\bigr) v(y, z + m) \tag{9}$$

(see [8] and, e.g. [6,7] for its properties). Note that $(y, z) \in \Pi_h$ on the left of (9), but $(y, z) \in \varpi$ on the right. For any real η, the problem (8) is associated with the Hermitian positive closed sesquilinear form

$$a_\eta(U, V, \varpi_h) = a\bigl(\exp(i\eta z)U, \exp(i\eta z)V; \varpi_h\bigr), \quad u, V \in H^1_{\text{per}}(\varpi_h)$$

where $H^1_{\text{per}}(\varpi_h)$ is the subspace of $H^1(\varpi_h)$ of functions 1-periodic in z. Hence, (8) is associated with a self-adjoint semi-bounded operator in $L^2(\varpi_h)$ (see [3], [4, Ch. 10] again) and in view of the compact embedding $H^1_{\text{per}}(\varpi_h) \subset L^2(\varpi_h)$, the problem has the discrete spectrum (7) only. It is known that the functions $\mathbb{R} \ni \eta \mapsto M_p^h(\eta)$ are continuous and 2π-periodic so that (6) indeed consists of closed segments.

Remark. The authors do not know, if it is possible in (8) that

$$M_q^h(\eta) = M_{q0}^h = \text{const} \quad \text{for } \eta \in [\eta_0, \eta_1) \subset [0, 2\pi),\ \eta_1 > \eta_0 \tag{10}$$

Under the condition (10), the operator $\mathcal{T}^h$ in (5) gets the eigenvalue M_{q0}^h of infinite multiplicity. If (10) does not occur for any q, the spectrum $\Sigma(\mathcal{T}^h)$ is fully continuous.

3. Opening gaps

The structure (6) does not necessarily provide gaps because the bands Υ_p^h may cover the ray $[0, +\infty)$. However, plenty of examples of opened gaps have been discovered for scalar equations and Maxwell's system in periodic media, infinite in all directions (see [9,10] and others). To open a gap, one usually considers differential operators with piecewise constant contracting coefficients and tunes the parameters.

An approach based on parameter-dependent Korn-type inequalities [11] was proposed in [12]. It permits to detect a gap for periodic homogeneous elastic waveguide of a specific shape with partly clamped surface. This approach was modified in [13] to cover waveguides with traction-free surfaces of various shapes and elastic properties, in particular, the one in Fig. 1a. However, the method [12,13] is able to ensure the detection of only one gap. In this Note we develop a new approach which enables to open as many gaps as we wish when $h \to +0$.

In order to simplify the demonstration here, we have made many assumptions on the waveguide Π_h. In general, the elastic material could be anisotropic and periodically inhomogeneous. The boundaries $\partial\varpi$ and $\partial\omega$ could be Lipschitz, except that $\partial\omega$ should be smooth in the vicinity of the points $\mathcal{O}^\pm$ (see (1)).

Our result on gaps provokes to formulate a hypothesis about the origin of the experimentally known effect of "backward wave", describing the splintering of a brittle rod by a wave reflected from the free end. Indeed, moving from the embedded end the wave may produce a family a salvage cracks (see Fig. 2) which on the way back create gaps in the spectrum. These are inhibitory for waves at certain frequencies, and, thus, cause an energy concentration which usually leads to fracture. Of course, the result below does not yet prove this phenomenon, and a numerical simulation becomes the next task for the authors.

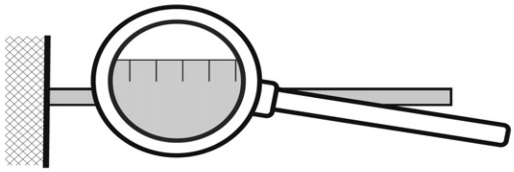

Fig. 2. Family of salvage cracks.

4. The limit model problem

At $h=0$ the ligaments vanish so that the problem (8) loses the periodicity conditions and turns into

$$L(\nabla_y, i\eta + \partial/\partial z)V = MV \quad \text{in } \varpi, \qquad N(x, \nabla_x, i\eta + \partial/\partial z)V = 0 \quad \text{on } \partial\varpi \tag{11}$$

We observe that $M_p(\eta) = \mathbf{M}_p$, $v_{(p)}(x;\eta) = \exp(-i\eta z)\mathbf{V}_{(p)}(x)$ (recall here the Remark), where $\mathbf{V}_{(p)}$ is the eigenvalue of the problem (11) at $\eta = 0$ corresponding to the eigenvalue $\mathbf{M}_p$ in the sequence

$$0 = \mathbf{M}_1 = \cdots \mathbf{M}_6 < \mathbf{M}_7 \leqslant \mathbf{M}_8 \leqslant \cdots \leqslant \mathbf{M}_p \leqslant \cdots \to +\infty \tag{12}$$

Note that (8) is nothing but the standard spectral problem for the isolated elastic body ϖ, and the six null eigenvalues in (12) correspond to rigid motions. The eigenvectors $\mathbf{V}_{(p)}$ can be subject to the conditions

$$(\mathbf{V}_{(p)}, \mathbf{V}_{(q)}) = \delta_{p,q}, \quad p,q = 1,2,\ldots \tag{13}$$

The max–min principle (see, e.g., [4, Th. 10.2.2]) gives the formula

$$M_j(\eta) = \mathbf{M}_j = \max_{\mathcal{H}_j} \inf_{\mathcal{U}\in\mathcal{H}_j\setminus\{0\}} \frac{a_\eta(\mathcal{U},\mathcal{U};\varpi)}{\varrho\|\mathcal{U}; L^2(\varpi)\|^2} \tag{14}$$

where $\mathcal{H}_j$ stands for any subspace in $H^1(\varpi)$ of codimension $j-1$ and hence $\mathcal{H}_1 = H^1(\varpi)$.

5. Comparing eigenvalues in ϖ_h and ϖ

We again have

$$M_j^h(\eta) = \max_{\mathcal{H}_j^h} \inf_{\mathcal{U}^h\in\mathcal{H}_j^h\setminus\{0\}} \frac{a_\eta(\mathcal{U}^h,\mathcal{U}^h;\varpi_h)}{\varrho\|\mathcal{U}^h; L^2(\varpi_h)\|^2} \tag{15}$$

where $\mathcal{H}_1^h = H^1_{\mathrm{per}}(\varpi_h)$ and other notation is similar to (14). Eigenvectors $V_{(p)}^h$ of the model problem (8) in ϖ_h fall into $H^1(\varpi)$ and satisfy the orthogonality and normalization conditions

$$\left(V_{(p)}^h, V_{(q)}^h\right)_{\varpi_h} = \delta_{p,q}, \quad p,q = 1,2,\ldots \tag{16}$$

A result on trimming rough surfaces in [11, Ch. 2] provides the following inequality of Korn's type:

$$\left\|V; L^2(\varpi_h \setminus \varpi)\right\|^2 \leqslant C_K h^2\left(a(V,V;\varpi_h) + \left\|V; L^2(\varpi_h)\right\|^2\right), \quad V \in H^1(\varpi_h) \tag{17}$$

where C_K is independent of $h \in (0,1]$ and V. Let us fix j and choose $h > 0$ such that

$$C_K h^2\left(1 + M_j^h(\eta)\right) \leqslant (2j)^{-1} \quad \text{for } \eta \in [0, 2\pi) \tag{18}$$

Then the eigenvectors $V_{(1)}^h, \ldots, V_{(j)}^h$ still remain linearly independent in $L^2(\varpi)$, and thus any subspace $\mathcal{H}_j$ contains a non-trivial linear combination $\mathcal{U}_{(j)}^h$ of them. Inserting $\exp(-i\eta z)\mathcal{U}_{(j)}^h$ into (14) with $\eta = 0$, we use (16)–(18) to conclude that

$$M_j(\eta) = \mathbf{M}_j \leqslant M_j^h(\eta)\left(1 + 2C_K h j^2\left(1 + M_j^h(\eta)\right)\right) \tag{19}$$

Consider the products $\mathcal{V}_{(p)}^h(x;\eta) = \chi_h(x)V_{(p)}(x;\eta)$, where $V_{(p)}$ is taken from (4) and χ_h is a smooth plateau function which is equal to 1 for $\min|x - \mathcal{O}^p m| > 2c_\chi h$ and 0 for $\min|x - \mathcal{O}^p m| < c_\chi h$ and $c_\chi > 0$ is such that the extension of $\mathcal{V}^h_{(p)}$ to ϖ_h by null remains smooth. The inequalities $|\mathbf{V}_{(p)}(x)| \leqslant c_\varpi(1+\mathbf{M}_p)^{3/2}$ and $|\nabla_x \mathbf{V}_{(p)}(x)| \leqslant c_\varpi(1+\mathbf{M}_p)^2$ follow from local estimates in [14] and the normalization condition (13) with $q = p$, and they can be used to derive the estimates

$$\left\|\mathbf{V}_{(p)} - \mathcal{V}^h_{(p)}; L^2(\varpi)\right\|^2 \leqslant C_\varpi h^3(1+\mathbf{M}_p)^3$$
$$\left\|\nabla_x \mathbf{V}_{(p)} - \nabla_x \mathcal{V}^h_{(p)}; L^2(\varpi)\right\|^2 \leqslant C_\varpi h(1+\mathbf{M}_p)^3\big(1+h^2(1+\mathbf{M}_p)\big)$$

Hence, since $h \leqslant 1$ we find that $\mathcal{V}^h_{(1)}, \ldots, \mathcal{V}^h_{(j)}$ are linearly independent in $L^2(\varpi)$ under the restriction $C_\varpi h^3 j(1+\mathbf{M}_j)^3 \leqslant 1/2$, and applying (15) in the same way as above we obtain the relation

$$M^h_j(\eta) \leqslant \frac{\mathbf{M}_j + C\varpi C_a hj(1+\mathbf{M}_j)^3(1+h^2(1+\mathbf{M}_j))}{1 - C_\varpi h^3 j(1+\mathbf{M}_j)^3} \leqslant \mathbf{M}_j + C_M hj(1+\mathbf{M}_j)^3$$

Notice that the latter particularly provides (18) for $h \in (0, h_j]$, if $\delta > 0$ is small in the formula

$$h_j = \delta j^{-2/3}(1+\mathbf{M}_j)^{-1} \tag{20}$$

We are now in the position to conclude our result:

Theorem 5.1. *Let $\mathbf{M}_j > \mathbf{M}_{j-1}$ in* (12)*. Then, for $h \in (0, h_j]$ with h_j as in* (20)*, the bands $\Upsilon^h_1, \ldots, \Upsilon^h_{j-1}$ and Υ^h_j, $\Upsilon^h_{j+1}, \ldots,$ respectively, belong to the sets $[0, \mathbf{M}_{j-1} + Ahj(1+\mathbf{M}_{j-1})^3]$ and $[\mathbf{M}_j + Ahj(1+\mathbf{M}_j), +\infty)$, where $A > 0$ depends on neither h, nor j. In the case*

$$h < A^{-1} j^{-1}(\mathbf{M}_j - \mathbf{M}_{j-1})\big((1+\mathbf{M}_{j-1})^3 + (1+\mathbf{M}_j)\big)^{-1} \tag{21}$$

the spectrum (6) *gets a gap between the bands Υ^h_{j-1} and Υ^h_j.*

We are not yet able to indicate a gap between the bands corresponding to the same (multiple) eigenvalue in (12) using the method presented above. In [13] it is proven under certain symmetry restrictions that just $\Upsilon^h_1, \ldots, \Upsilon^h_6$ cover an intact segment. Notice that the quantities (20) form a monotone decreasing sequence but the bounds in (21) do not. However, diminishing the parameter h yields any given number of gaps.

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