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A mean flow field solution to a moderately under/over-expanded turbulent supersonic jet

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Abstract

The linearized solution for an inviscid imperfectly-expanded supersonic axisymmetric jet has been extended to the case of a turbulent flow, by taking into account the mean Reynolds stresses. The analytical results agree reasonably well with experimental data available in the literature, and so indicate that the solution is a good approximation to the near-field of an imperfectly-expanded jet. This analytical solution could be used to improve semi-empirical models of broadband shock-associated noise in aeronautics. ***To cite this article: B. Emami et al., C. R. Mecanique 337 (2009).***

Résumé

Une solution à champ d'ècoulement moyen pour un jet supersonique turbulent modérément sous/sur-expansé. La solution linéarisée pour un jet à la fois axisymétrique, supersonique, non-visqueux et imparfaitement expansé a été portée au cas d'un ècoulement turbulent en tenant compte des contraintes Reynolds moyennes. Les résultats analytiques correspondent plutôt bien aux données expérimentales disponibles dans la literature. Ils indiquent donc que la solution est une bonne approximation au quasi-champ d'un jet imparfaitement expansé. Cette solution analytique pourrait servir à améliorer les modéles semi-empiriques de bruits de large bande liés aux chocs dans de domaine de l'aéronautique. ***Pour citer cet article : B. Emami et al., C. R. Mecanique 337 (2009).***

Keywords: Analytical solution; Supersonic; Jet

Mots-clés : Solution analytique ; Supersonique ; Jet

1. Introduction

The study of under/over-expanded jets is of importance to various engineering applications, including the design of aeronautic vehicles and the reduction of screech noise in civilian aircraft. When the pressure of a supersonic jet at a

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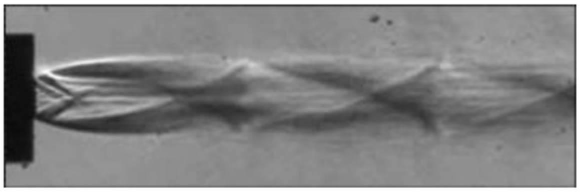

Fig. 1. Schlieren image of the decaying shock-cell structure of an under-expanded jet [1]. Exit Mach number is 2.53; exit pressure ratio is 3.12.

nozzle exit is higher/lower than the ambient pressure, a multi-cell shock structure (that consists of shock and expansion waves) forms, causing the pressure of the flow field to decrease/increase to the ambient value. Under/over-expanded free jets involve a simple flow geometry, yet very complicated phenomena, because of these shock waves and because the shock cells decay with distance from the nozzle exit due to the interaction with turbulence (see Fig. 1).

The solution of the linearized compressible Euler equations for an inviscid imperfectly-expanded supersonic axisymmetric jet is well known. In the early work of Pack [2], the velocity potential equation for a compressible inviscid flow was linearized and solved. Later, Howe and Ffowcs-Williams [3] and Tam [4] linearized and solved the pressure perturbation equation. More recently, the role of screech noise in self-exciting jets was studied using the linearized solution [5]. In all of this work, solutions were obtained for the inviscid equations, neglecting turbulent and molecular diffusion. Tam et al. [6] addressed this problem by applying the method of multiple-scales to the linearized Navier–Stokes equations, reducing them to an inhomogeneous ODE eigenvalue problem that, except for the first mode of the eigenvalue problem, is solved numerically.

Here, we provide an analytical solution by extending the linearized solution to the Navier–Stokes equations, taking into account the mean Reynolds stresses, where the empirical turbulence model of Witze [8] is used to determine the value of the eddy viscosity. The results are compared to the inviscid solution as well as some available experimental data.

2. Methodology

The Favre-averaged continuity and momentum equations for a turbulent compressible flow in vector form, neglecting the molecular viscous terms, are

$$\nabla \cdot (\bar{\rho}\tilde{\mathbf{u}}) = 0 \tag{1}$$

and

$$\bar{\rho}\tilde{\mathbf{u}} \cdot \nabla\tilde{\mathbf{u}} \simeq -\nabla\bar{P} + \nabla \cdot \mathbf{R} \tag{2}$$

where $\bar{\rho}$ is Reynolds-averaged density, $\tilde{\mathbf{u}}$ is Favre-averaged velocity, $\bar{P}$ is Reynolds-averaged pressure, and $\mathbf{R}$ is the Reynolds stress dyad that can be calculated, in tensor form, as

$$R_{ij} = \mu_t\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial \tilde{u}_k}{\partial x_k}\right) - \frac{2}{3}\delta_{ij}\bar{\rho}\tilde{k} \tag{3}$$

where μ_t is the turbulent eddy viscosity, and $\tilde{k}$ is the turbulent kinetic energy. By assuming a constant eddy viscosity μ_t, and by neglecting the isotropic contribution of the turbulent kinetic energy $\tilde{k}$ with respect to the mean pressure $\bar{P}$,

$$\nabla \cdot \mathbf{R} \simeq \boldsymbol{\mu_t}\nabla^{\mathbf{2}}\tilde{\mathbf{u}} + \frac{\mathbf{1}}{\mathbf{3}}\boldsymbol{\mu_t}\nabla(\nabla \cdot \tilde{\mathbf{u}}) \tag{4}$$

Eq. (2) then reduces to

$$\bar{\rho}\tilde{\mathbf{u}} \cdot \nabla\tilde{\mathbf{u}} = -\nabla\bar{P} + \mu_t\nabla^2\tilde{\mathbf{u}} + \frac{1}{3}\mu_t\nabla(\nabla \cdot \tilde{\mathbf{u}}) \tag{5}$$

The assumption of constant eddy viscosity is according to the work of Kleinstein [7] and Witze [8], who concluded that the eddy viscosity remains almost constant in the flow field of a high speed jet, based on a wide range of related experimental data. Kleinstein [7] suggested

$$\mu_t = 0.00915(\rho_\infty \rho_e)^{0.5} U_e D_e \tag{6}$$

where ρ_∞ is the ambient density, ρ_e is the nozzle exit density, U_e is the nozzle exit velocity, and D_e is the nozzle exit diameter. Later, Witze [8] proposed an improved correlation for eddy viscosity by looking at a wider range of measurements,

$$\mu_t = 0.01\rho_\infty^{0.28}\rho_e^{0.72} U_e D_e (1 - 0.16 M_e) \tag{7}$$

where M_e is the Mach number at the nozzle exit. In the present results, the eddy viscosity is calculated using Eq. (7), unless otherwise mentioned, and the molecular diffusion is assumed negligible.

The above equations (for an axisymmetric jet exiting from a nozzle of diameter D_e) are linearized by ρ_e and U_e as

$$\rho_e \nabla \cdot \tilde{\mathbf{u}} + U_e \frac{\partial \bar{\rho}}{\partial x} = 0 \tag{8}$$

and

$$\rho_e U_e \frac{\partial \tilde{\mathbf{u}}}{\partial x} = -\nabla \bar{P} + \mu_t \nabla^2 \tilde{\mathbf{u}} + \frac{1}{3}\mu_t \nabla(\nabla \cdot \tilde{\mathbf{u}}) \tag{9}$$

As can be seen, the radial derivatives of $\bar{\rho}$ and $\tilde{\mathbf{u}}$ on the left-hand sides of Eqs. (8) and (9) respectively, are deemed negligible because $\tilde{\mathbf{u}}$ was set to $(U_e, 0)$. This can also be physically justified, as the radial component of velocity is much smaller than the axial component in the near-field of a jet. Taking the divergence of both sides of Eq. (9), substituting Eq. (8), and introducing a linearized equation of state, $\bar{P} = \bar{\rho} a_e^2/\gamma$ (γ is the specific heat ratio and a_e is the speed of sound at the nozzle exit), we obtain an equation for pressure

$$A\left(\frac{\partial^3 p}{\partial x^3} + \frac{\partial^3 p}{\partial x \partial r^2} + \frac{1}{r}\frac{\partial^2 p}{\partial x \partial r}\right) + \left(1 - \gamma M_e^2\right)\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} = 0 \tag{10}$$

where $A = 4\gamma \mu_t U_e/(3\rho_e a_e^2)$, r and x are the radial and axial coordinates in a 2D axisymmetric system, and $p = \bar{P} - P_\infty$ (P_∞ is the ambient pressure) is introduced to create homogeneous boundary conditions. The above equation is subject to

$$p(x = 0, r) = P_e - P_\infty \tag{11}$$

and

$$\frac{\partial p}{\partial x}(x = 0, r) = 0 \tag{12}$$

at the nozzle exit, where Eq. (12) holds as long as the jet is mildly imperfectly-expanded; and

$$p(x, r = D_e/2) = 0 \tag{13}$$

which holds in the near-field of the jet, close to the nozzle exit.

Using separation of variables, $p(x, r) = X(x)R(r)$, Eq. (10) is converted to a Bessel equation for $R(r)$ and a third-order ODE for $X(x)$. Considering the boundary conditions, it can be shown that the solution to the first ODE is a linear combination of $J_0(\lambda_m r)$, where J_0 is the zero-order Bessel function of the first kind, $\lambda_m = 2\beta_m/D_e$, and $J_0(\beta_m) = 0$ $(m = 1, 2, 3, \ldots)$. The second ODE reduces to

$$AX''' + \left(1 - \gamma M_e^2\right)X'' - A\lambda_m X' - \lambda_m^2 X = 0 \tag{14}$$

In the inviscid limit, $A = 0$ and the above equation reduces to a second-order ODE, the solution to which has been shown to be a linear combination of $\cos\{2\beta_m x/[D_e(\gamma M_e^2 - 1)^{1/2}]\}$ (e.g. [3,4]). For $A \neq 0$, we consider a solution of the form $e^{-\theta_m x}[C_1 \sin(\eta_m x) + C_2 \cos(\eta_m x)]$, where θ_m, η_m, C_1 and C_2 are constants to be determined. In other words, the solution is postulated to have the form of a damped oscillation, as suggested by the experimental work of Selezneva et al. [9]. Substituting this into Eq. (14) and considering the boundary conditions, it can be shown that $X(x)$ is a linear combination of $e^{-\theta_m x}[\cos(\eta_m x) + (\theta_m/\eta_m)\sin(\eta_m x)]$, where θ_m and η_m are obtained from a system of two nonlinear equations

$$-A\theta_m^4+\eta_m^4+\left(1-\gamma M_e^2\right)\left(\theta_m^3+\theta_m\eta_m^2\right)+A\lambda_m^2\left(\theta_m^2+\eta_m^2\right)-\lambda_m^2\theta_m=0 \tag{15}$$

$$2A\left(\theta_m^3+\theta_m\eta_m^2\right)-\left(1-\gamma M_e^2\right)\left(\theta_m^2+\eta_m^2\right)-\lambda_m^2=0 \tag{16}$$

that can be solved iteratively using the Jacobian matrix (Newton–Raphson method). Again, in the inviscid limit ($A=0$), it can be shown that $\theta_m=0$ and $\eta_m=2\beta_m/[D_e(\gamma M_e^2-1)^{1/2}]$, and so the solution reduces to that of the inviscid case.

Applying the boundary conditions the solution to Eq. (10) is then

$$p(x,r)=\sum_{m=1}^{\infty}\alpha_m J_0(\lambda_m r)e^{-\theta_m x}\left[\cos(\eta_m x)+\frac{\theta_m}{\eta_m}\sin(\eta_m x)\right] \tag{17}$$

where $\alpha_m=2(P_e-P_\infty)/(\beta_m J_1(\beta_m))$ and J_1 is the first-order Bessel function of the first kind. Substituting this into Eq. (9) and considering Eq. (8), the velocity components can then be calculated. For the axial velocity component u, we obtain

$$\nabla^2 u+B\frac{\partial u}{\partial x}=F(x,r) \tag{18}$$

where $B=-\rho_e u_e/\mu_t$, and

$$F(x,r)=\frac{1}{\mu_t}\frac{\partial p}{\partial x}+\frac{\gamma u_e}{3\rho_e a_e^2}\frac{\partial^2 p}{\partial x^2} \tag{19}$$

subject to

$$u(x=0,r)=u_e \tag{20}$$

and

$$\frac{\partial u}{\partial x}(x=0,r)=0 \tag{21}$$

which holds for mild under/over-expansions.

Eq. (18) can be solved using a Bessel series. Let $u(x,r)=\sum_{m=1}^{\infty}\zeta_m(x)J_0(\lambda_m r)$. Following some tedious calculations, it can be shown that $\zeta_m(x)$ satisfies

$$\zeta_m''+B\zeta_m'-\lambda_m^2\zeta_m=f_m(x) \tag{22}$$

where

$$f_m(x)=\alpha_m e^{-\theta_m x}\left\{-\frac{\gamma u_e}{3\rho_e a_e^2}\left(\theta_m^2+\eta_m^2\right)\cos(\eta_m x)+\left[-\frac{1}{\mu_t}\left(\frac{\theta_m^2}{\eta_m}+\eta_m\right)+\frac{\gamma u_e}{3\rho_e a_e^2}\left(\frac{\theta_m^3}{\eta_m}+\theta_m\eta_m\right)\right]\sin(\eta_m x)\right\} \tag{23}$$

subject to

$$\zeta_m(0)=\frac{2u_e}{\beta_m J_1(\beta_m)} \tag{24}$$

and

$$\zeta_m(\infty)=0 \tag{25}$$

Eq. (22) can be solved analytically,

$$\zeta_m(x)=\left(\zeta_m(0)-C_2\right)e^{\frac{-B-\sqrt{B^2+4\lambda_m^2}}{2}x}+e^{-\theta_m x}\left[C_1\sin(\eta_m x)+C_2\cos(\eta_m x)\right] \tag{26}$$

where C_1 and C_2 are obtained by solving a set of two linear equations,

$$\left(\theta_m^2-\eta_m^2-B\theta_m-\lambda_m^2\right)C_1+(2\theta_m\eta_m-B\eta_m)C_2=f_1 \tag{27}$$

$$(-2\theta_m\eta_m+B\eta_m)C_1+\left(\theta_m^2-\eta_m^2-B\theta_m-\lambda_m^2\right)C_2=f_2 \tag{28}$$

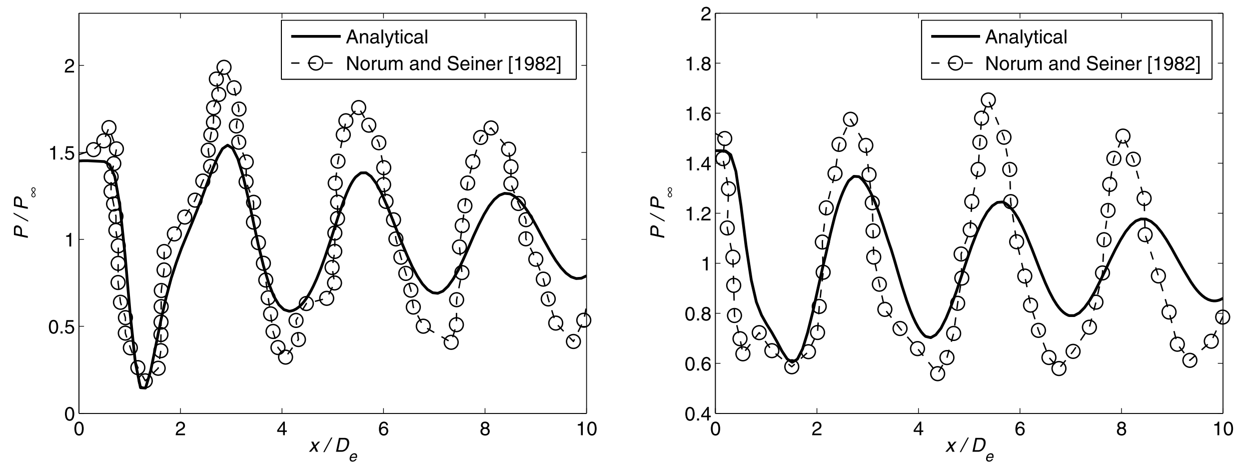

Fig. 2. Comparison of analytical and experimental results; $P_e/P_\infty = 1.45$, $M_e = 2$; $r/D_e = 0$ (left) and $r/D_e = 0.25$ (right).

where

$$f_1 = \alpha_m e^{-\theta_m x}\left[-\frac{1}{\mu_t}\left(\frac{\theta_m^2}{\eta_m} + \eta_m\right) + \frac{\gamma u_e}{3\rho_e a_e^2}\left(\frac{\theta_m^3}{\eta_m} + \theta_m \eta_m\right)\right] \tag{29}$$

and

$$f_2 = -\alpha_m e^{-\theta_m x}\frac{\gamma u_e}{3\rho_e a_e^2}\left(\theta_m^2 + \eta_m^2\right) \tag{30}$$

3. Results

Eq. (10) and Eq. (18) were solved for the mildly under-expanded jets of Norum and Seiner [10,11], corresponding to convergent-divergent nozzles. Three cases were solved: $P_e/P_\infty = 1.45$, $M_e = 2$; $P_e/P_\infty = 1.62$, $M_e = 1.5$; and $P_e/P_\infty = 1.33$, $M_e = 1.5$. The near-field solutions were restricted to the first 10 D_e and 5 D_e downstream from the nozzle exit, for the first case and for the other two, respectively.

Fig. 2 shows the results for the first case; a comparison of the analytical solution and experimental measurements for axial distributions ($r/D_e = 0$ and $r/D_e = 0.25$) of normalized pressure. The first 15 terms of the series were calculated; adding more terms did not significantly change the results. As can be seen, the solution is fairly good, as it estimates the position of the first few shock cells reasonably well, although the amplitudes are not predicted equally well.

Fig. 3 shows the results for the second jet; comparison of the analytical solution and experimental measurement for axial distribution of normalized pressure along the jet centerline. As can be seen, the agreement is reasonably good. For the third jet, as shown in Fig. 4, analytical solution predicts the trend of the experimental data, but the axial positions of the shock cells are over predicted.

The method was also used to predict some available data related to more strongly under-expanded jets [10,12]; the agreement with the data is not as good, as the analytical solution assumes a mild under/over-expansion.

The sensitivity of solutions to the value of eddy viscosity was also studied. All the results that follow are for the first jet ($P_e/P_\infty = 1.45$, $M_e = 2$). Fig. 5 (left) compares viscous ($\mu_t = 0.3$) and inviscid solutions. As expected, the viscous solution is a decaying oscillation, rather than a superposition of periodic functions as in the inviscid limit. Fig. 5 (right) presents viscous solutions for different values of μ_t. Obviously, the greater is μ_t, the more the solution decays.

Finally, Fig. 6 (left) compares the viscous ($\mu_t = 0.3$) and inviscid solutions for the centerline distribution of the axial velocity component. (It should be noted that as Eq. (18) has μ_t in the denominator, the inviscid solution is obtained by setting μ_t to a very small value.) Again, the viscous solution is a decaying oscillation, rather than a superposition of periodic functions as in the inviscid limit, and the average velocity decreases with axial distance from

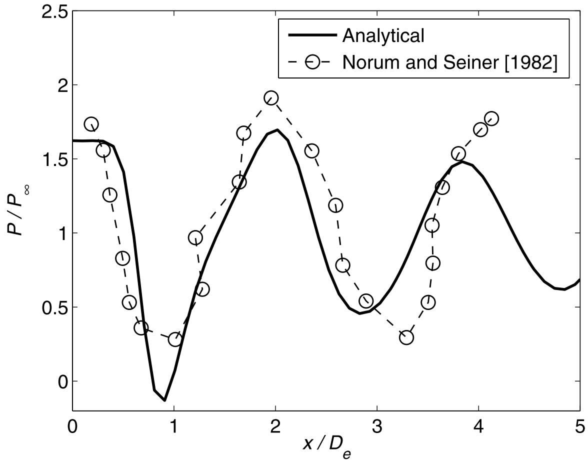

Fig. 3. Comparison of analytical and experimental results; $P_e/P_\infty = 1.62$, $M_e = 1.5$, $r/D_e = 0$.

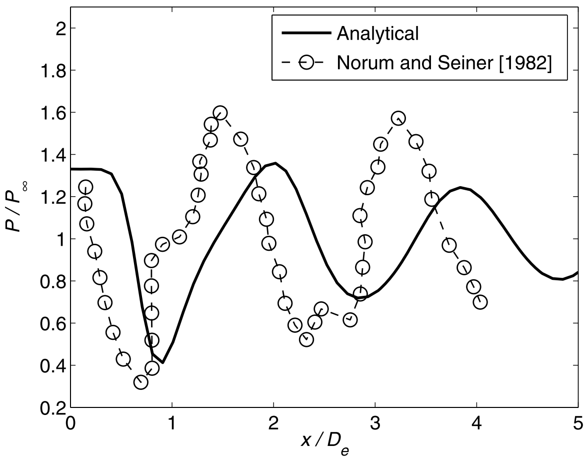

Fig. 4. Comparison of analytical and experimental results; $P_e/P_\infty = 1.33$, $M_e = 1.5$, $r/D_e = 0$.

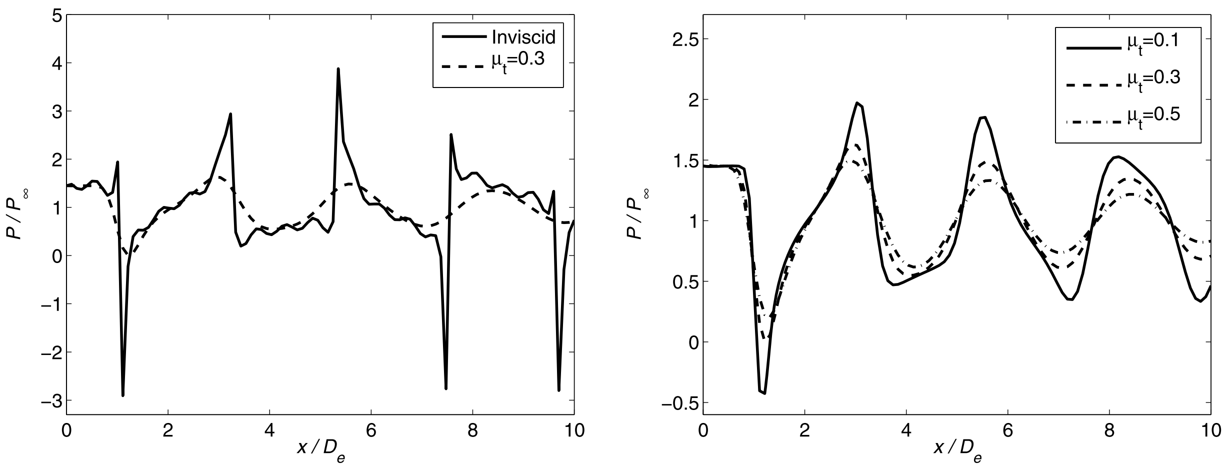

Fig. 5. Comparison of the centerline pressure: inviscid and viscous solutions (left), viscous solutions at different values of μ_t (right).

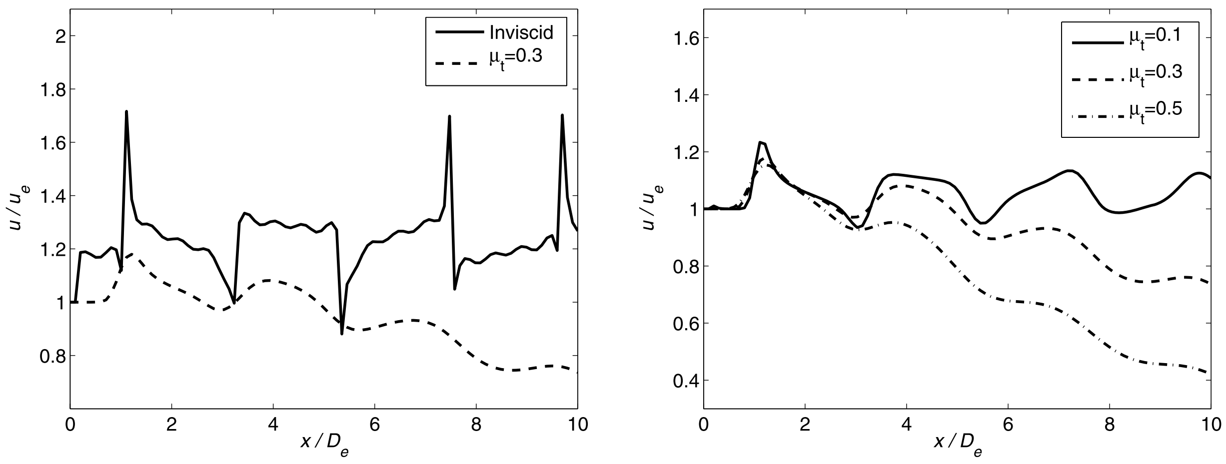

Fig. 6. Comparison of the centerline axial velocity: inviscid and viscous solutions (left), viscous solutions at different values of μ_t (right).

the nozzle exit. Fig. 6 (right) presents viscous solutions at different values of μ_t. Obviously, the greater is μ_t, the more quickly the velocity decreases.

4. Concluding remarks

The linear analytical solution for an inviscid under/over-expanded supersonic axisymmetric jet has been extended to take into account the mean turbulent Reynolds stresses. The solution was compared to some available measurements and yielded satisfactory agreement. Future work could include a "multi-field" extension of this method, by dividing the flow field into several sections and applying the method to each. Using this strategy, the evolution of the mean flow field could be taken into account in the linearization of the equations, and a more complex eddy viscosity model could be adopted, by linearizing and solving the PDEs of a two-equation turbulence model.

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