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Two-dimensional nonlinear models for heterogeneous plates

Erick Pruchnicki

École polytechnique universitaire de Lille, cité scientifique, avenue Paul-Langevin, 59655 Villeneuve d'Ascq cedex, France

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Abstract

We consider a formal asymptotic study of plates with periodically rapidly varying heterogeneities. The asymptotic analysis is performed when both the period of change of the material properties and the thickness of the plate are of the same orders of magnitude. We consider a plate made of Ciarlet–Geymonat type materials (P.G. Ciarlet and G. Geymonat (1982)). Depending on the order of magnitude of the applied loads, we obtain a nonlinear membrane model and a nonlinear membrane inextensional-bending model as announced in E. Pruchnicki (2006). Our approach is based on a sequence of recursive minimization problems. ***To cite this article: E. Pruchnicki, C. R. Mecanique 337 (2009).***

Résumé

Modèles bidimensionnels non linéaires pour les plaques hétérogènes. On considère une étude asymptotique formelle de plaques avec des hétérogénéités variant périodiquement. L'analyse asymptotique est faite lorsque la période de variation des propriétés du matériau de la plaque sont du même ordres de grandeur. On considère une plaque faite de matériaux de Ciarlet–Geymonat (P.G. Ciarlet et G. Geymonat (1982)). En fonction de l'ordre de grandeur des charges appliquées, on obtient un modèle non linéaire membranaire et un modèle non linéaire membranaire flexion-inextensionnelle comme annoncé par E. Pruchnicki (2006). Notre approche est basée sur une suite récursive de problèmes de minimisation. ***Pour citer cet article : E. Pruchnicki, C. R. Mecanique 337 (2009).***

Keywords: Asymptotic expansions; Plates; Nonlinear elasticity; Variational methods; Homogenization

Mots-clés : Développements asymptotiques ; Plaques ; Élasticité non linéaire ; Méthodes variationnelles ; Homogénéisation

1. Introduction

For the homogeneous thin plates structures, the only small parameter to be introduced into the asymptotic analysis is the thickness. The work of P.G. Ciarlet [1] shows that the analysis of the asymptotic expansion with the thickness as a small parameter allows a set of two-dimensional models for such structures to be proven in a systematic way in various orders. Composite materials are characterized by the fact that they contain two or more finely mixed constituents. For heterogeneous plates, the size of heterogeneities highlighting the periodic character of the hetero-

E-mail address: Erick.Pruchnicki@polytech-lille.fr.

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geneous microstructure on the middle surface introduces a second small parameter. T. Lewiński and J.J. Telega [2] showed within the framework of the linear elasticity that we could consider same orders of magnitude for these two parameters because the other cases can thus be deduced and this is confirmed by the rigorous results of Γ-convergence. By assuming a nonlinear constitutive law of Saint Venant–Kirchoff, E. Pruchnicki [3,4] showed respectively for plates ans shells that the model of the highest possible order is of membrane type. In the present note, we are interested in the important case of a nonlinear material of P.G. Ciarlet and G. Geymonat [5] because this material is more realistic than a Saint Venant–Kirchoff material. Indeed he does not allow the possibility of squashing a volume into a point thanks to the logarithmic term in the stored energy function. Moreover the stored energy function is polyconvex with respect to the invariants of the deformation gradient which ensures powerful results for the minimization problem of the potential energy (J.M. Ball [6], P.G. Ciarlet [7]). Moreover, we consider composite plates for which the stiffness of each material is of the same order of magnitude. When this assumption is false, the asymptotic analysis depends on another small parameter which is the ratio of the mechanical characteristics of the matrix to that of the reinforcement. This leads to non-classical models of homogenization (G. Panasenko [8]).

The outline of this paper is as follows. We adapt the method developed previously by O. Pantz [9,10] to solve a sequence of minimization problems for heterogeneous plates. This method is also used by N. Meunier [11], K. Trabelsi [12] for the cases of homogeneous rods and plates respectively. We obtain a hierarchy of two-dimensional models depending on the order of magnitude of the applied loads. For loading of order one with respect to the small parameter, we obtain a two-dimensional energy minimization problem modelling nonlinear membrane behaviour of plate and whose the solution is the leading term in the expansion of the displacement field. We are thus led to define a generalization of the space of nonlinear inextensional displacements for heterogeneous plates. When this space does not reduce to {0} and for loading of order three with respect to the small parameter, the displacement field can be identified as a solution of a two-dimensional nonlinear membrane inextensional-bending model. This model generalizes the ones proposed by P. Giroud [13], D. Caillerie and E. Sanchez-Palencia [14] and K. Trabelsi [15].

2. The nonlinear boundary-value problem of plates

Let ω be a bounded, open, connected subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary $\partial\omega$. Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ denotes the basis of the space $\mathbb{R}^3$. We shall write $\mathbf{x} = (x_1, x_2, x_3)$ for vectors in $\mathbb{R}^3$, and $\mathbf{x}' = (x_1, x_2)$ for vectors in $\mathbb{R}^2$. The following notations are used: Greek index α, $\beta = 1, 2$; Latin index $i, j, k = 1, 2, 3$. Let us consider a plate of thickness ε and mid-surface ω. The set $\Omega^\varepsilon = \omega \times]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[$ is called the reference configuration of this plate. The boundary of the plate is divided into two parts: the first is composed of the lower and upper boundaries $\omega \times (\{-\frac{\varepsilon}{2}\} \cup \{\frac{\varepsilon}{2}\})$; the second is the lateral boundary $\Gamma^\varepsilon = \partial\omega \times]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[$. This plate is heterogeneous and the size of the heterogeneities, which is assumed to be of the same order of magnitude of the thickness, is very small with regard to the global length-scale $\mathbf{x}' = (x_1, x_2)$. Thus we can define the ratio between both the local length-scale of the plate $\mathbf{y}' = (y_1, y_2)$ and the global one $\varepsilon = \frac{x'_\alpha}{y'_\alpha}$. We note that the heterogeneous microstructure of the plate is periodic with respect to global coordinates and it is sufficient to define the distribution of the constituents on the smallest period $Y^\varepsilon = Y' \times]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[$ ($Y' = (0, 1) \times (0, 1)$), which is also called the unit cell. A current point in Y^ε is defined by $\mathbf{y}^\varepsilon = (y_1, y_2, x_3^\varepsilon)$. Since the thickness of the plate is of order of magnitude ε, we introduce y_3 such that $x_3^\varepsilon = \varepsilon y_3$. The boundary ∂Y^ε of the unit cell is divided into the lateral boundary $\partial Y_L^\varepsilon = \partial Y' \times [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$, and into both the lower and the upper boundaries $\partial Y_\pm^\varepsilon = Y' \times (\{-\frac{\varepsilon}{2}\} \cup \{\frac{\varepsilon}{2}\})$. The plate is submitted to dead loading which is periodic with respect to the local length-scale $\mathbf{y}'$, and it is sufficient to define this dead loading only over one unit cell. The density of body force is the vector denoted $f_i^\varepsilon \mathbf{e}_i$. Let us consider the surface force per unit area $h_i^\varepsilon \mathbf{e}_i$ acting on the upper and lower faces $\Gamma^{\pm,\varepsilon} = \omega \times \partial Y_\pm^\varepsilon$ of the plate then their components are functions $h_i^\varepsilon : \omega \times (\partial Y_-^\varepsilon \cup \partial Y_+^\varepsilon) \to \mathbb{R}$. The plate is clamped on its lateral boundary Γ^ε. When subjected to the given loading, the plate undergoes the displacement field $u_i^\varepsilon(\mathbf{x}^\varepsilon,\ \mathbf{y}')\mathbf{e}_i$. The three functions $u_i^\varepsilon : \overline{\Omega^\varepsilon} \times \overline{Y'} \to \mathbb{R}$ are the components of the displacement field $u_i^\varepsilon(\mathbf{x}^\varepsilon,\ \mathbf{y}')\mathbf{e}_i$ with $\mathbf{x}^\varepsilon = (x_1, x_2, x_3^\varepsilon)$. The components of the displacement field are defined by functions $u_i^\varepsilon : \bar{\omega} \times \overline{Y^\varepsilon} \to \mathbb{R}$. Next we denote the vector field $\mathbf{u}^\varepsilon = (u_i^\varepsilon) : \bar{\omega} \times \overline{Y^\varepsilon} \to \mathbb{R}^3$. The deformation gradient is defined by $\mathbf{F}^\varepsilon(\mathbf{u}^\varepsilon) = \mathbf{I} + \operatorname{grad}\mathbf{u}^\varepsilon$ and $\operatorname{grad}\mathbf{u}^\varepsilon$ is the gradient of the displacement field $\mathbf{u}^\varepsilon$ defined by $(\operatorname{grad}\mathbf{u}^\varepsilon)_{i\alpha} = \partial_{x_\alpha} u_i^\varepsilon + \frac{1}{\varepsilon}\partial_{y_\alpha} u_i^\varepsilon$, $(\operatorname{grad}\mathbf{u}^\varepsilon)_{i3} = \partial_{x_3^\varepsilon} u_i^\varepsilon$. The plate is made of hyperelastic material of which the polyconvex stored energy function is defined by (Ciarlet and Geymonat [5], see also Ciarlet [7])

$$W^\varepsilon(\mathbf{F}^\varepsilon) = a|\mathbf{F}^\varepsilon|^2 + b|\operatorname{Cof}\mathbf{F}^\varepsilon|^2 + c\det(\mathbf{F}^{\varepsilon T}\mathbf{F}^\varepsilon) - d\ln(\det(\mathbf{F}^{\varepsilon T}\mathbf{F}^\varepsilon)) + e$$

where a, b, c, d and e are real positive functions independent of ε. We have $\lim_{\det(\mathbf{F}^\varepsilon)\to 0^+} W(\mathbf{F}^\varepsilon) = \infty$ which means that an infinite amount of energy is necessary to annihilate a volume. $C^\infty_{per}(\overline{\omega} \times Y^\varepsilon)$ is the set of functions of $\mathbf{C}^\infty(\overline{\omega} \times \mathbb{R}^2 \times [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}])$, Y' periodic in $\mathbf{y}'$. $W^{1,2}_{per}(\omega \times Y^\varepsilon)$ is the completion of the space $C^\infty_{per}(\overline{\omega} \times Y^\varepsilon)$ with respect to the norm of $W^{1,2}(\omega \times Y^\varepsilon)$ and $\mathbf{W}^{1,2}_{per}(\omega \times Y^\varepsilon) = (W^{1,2}_{per}(\omega \times Y^\varepsilon))^3$.

3. The two-scale three-dimensional minimization problem

The equilibrium state of the plate satisfies the minimization problem $P^\varepsilon(\omega \times Y^\varepsilon)$

$$\text{Find } \mathbf{v}^\varepsilon \in \mathbf{V}^\varepsilon(\omega \times Y^\varepsilon) \text{ such that } J^\varepsilon(\mathbf{v}^\varepsilon) = \inf_{\mathbf{u}^\varepsilon \in \mathbf{V}^\varepsilon(\omega \times Y^\varepsilon)} J^\varepsilon(\mathbf{u}^\varepsilon)$$

The total energy functional J^ε takes the form $J^\varepsilon(\mathbf{u}^\varepsilon) = I^\varepsilon(\mathbf{u}^\varepsilon) - l^\varepsilon(\mathbf{u}^\varepsilon)$, where

$$I^\varepsilon(\mathbf{u}^\varepsilon) = \int_{\omega \times Y^\varepsilon} W^\varepsilon(\mathbf{F}^\varepsilon(\mathbf{u}^\varepsilon))\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{y}^\varepsilon \quad \text{and} \quad l^\varepsilon(\mathbf{u}^\varepsilon) = \int_{\omega \times Y^\varepsilon} f_i^\varepsilon u_i^\varepsilon\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{y}^\varepsilon + \int_{\omega \times \partial Y^\varepsilon_\pm} h_i^\varepsilon u_i^\varepsilon\,\mathrm{d}a$$

measure, respectively, the internal energy of the plate, and the work of the external forces. f_i^ε and h_i^ε $(i = 1, 3)$ belong to $L^2(\omega \times Y^\varepsilon)$ and $L^2(\omega \times (\partial Y^\varepsilon_- \cup \partial Y^\varepsilon_+))$ respectively. The set of admissible displacement field is defined by

$$\mathbf{V}^\varepsilon(\omega \times Y^\varepsilon) = \{\mathbf{u}^\varepsilon \in \mathbf{W}^{1,2}_{per}(\omega \times Y^\varepsilon)\text{: } \det(\mathbf{F}^\varepsilon(\mathbf{u}^\varepsilon)) > 0 \text{ and } \mathbf{u}^\varepsilon = 0 \text{ on } \partial\omega \times Y^\varepsilon\}$$

Now we suppress the difficulty originating from the dependence on the small parameter of the plate by defining a rescaling operator by $(\pi_\varepsilon g)(\mathbf{x}', \mathbf{y}) = g(\mathbf{x}', \mathbf{y}', x_3^\varepsilon)$. Then for all functions g^ε and functionals G^ε, we set $g(\varepsilon) = \pi_\varepsilon^{-1} g^\varepsilon$ and $G(\varepsilon)(\psi) = \pi_\varepsilon^{-1} G^\varepsilon(\psi)$. Thus the domain Y^ε becomes the domain $Y = Y' \times (-\frac{1}{2}, \frac{1}{2})$. The boundaries ∂Y^ε, $\partial Y^\varepsilon_\pm$, ∂Y^ε_L become ∂Y, $\partial Y_\pm = Y' \times (\{-\frac{1}{2}\} \cup \{\frac{1}{2}\})$, $\partial Y_L = \partial Y' \times [-\frac{1}{2}, \frac{1}{2}]$ respectively. Scaled in this fashion, the minimization problem $P^\varepsilon(\omega \times Y^\varepsilon)$ becomes problem $P(\varepsilon)(\omega \times Y)$

$$\text{Find } \mathbf{v}(\varepsilon) \in \mathbf{V}(\varepsilon, \omega \times Y) \text{ such that } J(\varepsilon)(\mathbf{v}(\varepsilon)) = \inf_{\mathbf{u}(\varepsilon) \in \mathbf{V}(\varepsilon, \omega \times Y)} J(\varepsilon)(\mathbf{u}(\varepsilon))$$

The space of admissible deformation is now defined as:

$$\mathbf{V}(\varepsilon, \omega \times Y) = \{\mathbf{u}(\varepsilon) \in \mathbf{W}^{1,2}_{per}(\omega \times Y)\text{: } \det(\mathbf{F}(\varepsilon)(\mathbf{u})) > 0 \text{ and } \mathbf{u}(\varepsilon) = 0 \text{ on } \partial\omega \times Y\}$$

The formal asymptotic two-scale method is based on the hypothesis that the unknown vector field $\mathbf{u}(\varepsilon)$, the data $\mathbf{f}(\varepsilon)$ and $\mathbf{h}(\varepsilon)$ can be expanded in powers of the small parameter ε as

$$\sum_{n=0}^{\infty} \varepsilon^n \mathbf{u}^n, \quad \sum_{n=0}^{\infty} \varepsilon^n \mathbf{f}^n \quad \text{and} \quad \sum_{n=0}^{\infty} \varepsilon^n \mathbf{h}^n$$

respectively. These assumptions imply expansions on the deformation gradient

$$\mathbf{F}(\varepsilon)(\mathbf{u}(\varepsilon)) = \sum_{n=-1}^{\infty} \varepsilon^n F^n(\mathbf{u})$$

the stored energy function

$$W(\varepsilon)(\mathbf{F}(\varepsilon)(\mathbf{u}(\varepsilon))) = \sum_{n=-6}^{\infty} \varepsilon^n W^n(\mathbf{u})$$

and the total energy functional

$$J(\varepsilon)(\mathbf{u}(\varepsilon)) = \sum_{n=-5}^{\infty} \varepsilon^n J^n(\mathbf{u})$$

where $\mathbf{u}$ is the sequence $(\mathbf{u}^0, \mathbf{u}^1, \mathbf{u}^2, \ldots)$.

The next proposition shows that the solution $\mathbf{v}(\varepsilon)$ of the problem $P(\varepsilon;\omega\times Y)$ can be obtained by solving a sequence of partial variational problems. This idea was introduced by O. Pantz [9] in the case of homogeneous plates. The solution of problem $P(\varepsilon)(\omega\times Y)$ is such that

$$\mathbf{v}=(\mathbf{v}^0,\mathbf{v}^1,\mathbf{v}^2,\ldots)\in\bigcap_{n=-5}^{\infty}\mathbf{V}_n(\omega\times Y),\quad \text{where}$$
$$\mathbf{V}_{n+1}(\omega\times Y)=\Big\{\mathbf{v}\in\mathbf{V}_n(\omega\times Y):\ J^n(\mathbf{v})=\inf_{\mathbf{u}\in\mathbf{V}_n(\omega\times Y)}J^n(\mathbf{u})\Big\}$$
$$\mathbf{V}_{-5}(\omega\times Y)=\Big\{\mathbf{v}\in\mathbf{W}^{1,2}_{per}(\omega\times Y)^{\mathbb{N}}:\ \det\big(\mathbf{F}(\varepsilon)\big(\mathbf{v}(\varepsilon)\big)\big)>0 \text{ and } \mathbf{v}=0 \text{ on } \partial\omega\times Y\Big\}$$

Let $P_n(\omega\times Y)$ be the boundary-value problem: finding $\mathbf{v}\in\mathbf{V}_n(\omega\times Y)$ such that $J^n(\mathbf{v})=\inf_{\mathbf{u}\in\mathbf{V}_n(\omega\times Y)}J^n(\mathbf{u})$. By solving the first variational problems $P_n(\omega\times Y)$ for $n\leqslant 0$, we see that $\mathbf{v}^0$ does not depend on the microscopic scale $\mathbf{y}$ and $\int_{\partial Y_\pm}h^0\,\mathrm{d}a^*=0$; $\mathrm{d}a^*$ denotes the area element along the boundary $\partial Y_\pm$. Thus we assume the stronger condition, $h^0=0$ on $\omega\times\partial Y_\pm$.

4. A nonlinear membrane plate theory

We show that nonlinear membrane plate theory arises as problem $P_1(\omega\times Y)$. If $\mathbf{v}=(\mathbf{v}^0,\mathbf{v}^1,\ldots)$ is a solution of the problem $P_1(\omega\times Y)$ then the first term $\mathbf{v}^0$ in the asymptotic expansion of the displacement field solves the global minimization problem

$$J_m(\mathbf{v}^0)=\inf_{\mathbf{u}^0\in\mathbf{U}^0(\omega)}J_m(\mathbf{u}^0)$$

in which the set of displacement field

$$\mathbf{U}^0(\omega)=\big\{\mathbf{u}^0\in\mathbf{W}^{1,2}(\omega),\ \mathbf{u}^0=0 \text{ on } \partial\omega\big\}$$

and the membrane energy is defined by

$$J_m(\mathbf{u}^0)=\int_\omega W_m\big(\mathrm{grad}_{\mathbf{x}'}(\mathbf{u}^0)\big)\,\mathrm{d}\mathbf{x}'-\int_\omega\bigg(\int_Y f_i^0\,\mathrm{d}\mathbf{y}+\int_{\partial Y_\pm}h_i^1\,\mathrm{d}\mathbf{y}'\bigg)u_i^0\,\mathrm{d}\mathbf{x}'$$

where W_m is the membrane elastic energy obtained by computing the following local minimization problem

$$W_m\big(\mathrm{grad}_{\mathbf{x}'}(\mathbf{u}^0)\big)=\inf_{\bar{\mathbf{u}}^1\in\mathbf{U}^1(\omega\times Y)}\int_Y W^0(\mathbf{u}^0,\bar{\mathbf{u}}^1)\,\mathrm{d}\mathbf{y}\tag{1}$$

in which

$$W^0(\mathbf{u}^0,\bar{\mathbf{u}}^1)=a\big|\mathbf{F}^0(\mathbf{u}^0,\bar{\mathbf{u}}^1)\big|^2+b\big|\mathrm{Cof}\,\mathbf{F}^0(\mathbf{u}^0,\bar{\mathbf{u}}^1)\big|^2+c\det\big(\mathbf{C}^0(\mathbf{u}^0,\bar{\mathbf{u}}^1)\big)-d\ln\big(\det\big(\mathbf{C}^0(\mathbf{u}^0,\bar{\mathbf{u}}^1)\big)\big)+e$$
$$\mathbf{C}^0(\mathbf{u}^0,\bar{\mathbf{u}}^1)=\mathbf{F}^0(\mathbf{u}^0,\bar{\mathbf{u}}^1)^T\mathbf{F}^0(\mathbf{u}^0,\bar{\mathbf{u}}^1),\quad \mathbf{F}^0(\mathbf{u}^0,\bar{\mathbf{u}}^1)=\mathbf{I}+\mathrm{grad}_{\mathbf{x}'}(\mathbf{u}^0)+\mathrm{grad}_{\mathbf{y}}(\bar{\mathbf{u}}^1)$$

and the set $\mathbf{U}^1(\omega\times Y)$ is defined by

$$\mathbf{U}^1(\omega\times Y)=\bigg\{\bar{\mathbf{u}}^1\in\mathbf{W}^{1,2}_{per}(\omega\times Y),\ \det\mathbf{F}^0(\mathbf{u}^0,\bar{\mathbf{u}}^1)>0,\ \int_Y\bar{\mathbf{u}}^1\,\mathrm{d}\mathbf{y}=0\bigg\}$$

This membrane model generalizes the membrane one previously proposed by Pruchnicki [4] for Saint Venant–Kirchhoff materials. We can show that $W_m(\mathrm{grad}_{\mathbf{x}'}(\mathbf{u}^0))\to\infty$ as

$$\det\big(\widetilde{\mathbf{F}}^{0T}\widetilde{\mathbf{F}}^0\big)=\big|\widetilde{\mathbf{F}}^0_1\wedge\widetilde{\mathbf{F}}^0_2\big|^2\to 0^+\quad\big(\widetilde{\mathbf{F}}^0_{i\alpha}=\big(\widetilde{\mathbf{F}}^0_i\big)_\alpha=\delta_{i\alpha}+\partial_{x_\alpha}u_i^0\big)$$

As a consequence, the orientation-preserving condition is naturally imposed and the two-dimensional model precludes singular folds of the midsurface of the plate. This membrane energy inherits the material frame indifference and isotropy properties of each material of the plate. By considering the rigorous Γ-convergence argument of

S. Muller [16], it can be shown that membrane energy is obtained by minimizing the local energy on k basic cell (Y becomes kY in formula (1)) and then on k. Nevertheless as shown by G. Geymonat et al. [17] and J.C. Michel et al. [18], for specific deformation gradient $\mathbf{F}^0$, the one cell homogenized energy is the correct one. For homogeneous material the analytical expression of function W_m is continuous (K. Trabelsi [19]). However for heterogeneous material, no mathematical results would allow to show the continuity of W_m (G. Geymonat et al., Section 5.2 [17]).

5. A nonlinear membrane inextensional-bending plate theory

To study higher order model, we choose vanishing membrane loading; $\mathbf{f}^0 = \mathbf{h}^1 = 0$. Then a solution of problem $P_1(\omega \times Y)$ can be written in the following form $\mathbf{u}^1(\mathbf{x}', y_3) = \widetilde{\mathbf{u}}^1(\mathbf{x}') + \bar{\mathbf{u}}^1(x', y_3)$ in which $\bar{u}_i^1(x', y_3) = n_i(\mathbf{x}')\, y_3 - y_3\delta_{i3}$ and $\mathbf{n} = \widetilde{\mathbf{F}}_1^0 \wedge \widetilde{\mathbf{F}}_2^0$ is the unit normal to the deformed mid-surface of the plate. Thus the leading order term of the displacement field $\mathbf{u}^0$ inevitably belongs to the space of inextensional displacement field

$$\mathbf{U}^{0\,\mathrm{iso}}(\omega, \mathbb{R}^3) = \left\{\mathbf{v} \in \mathbf{W}^{2,2}(\boldsymbol{\omega}),\, \mathbf{v} = 0 \text{ on } \partial\omega,\ E_{\alpha\beta}(\mathbf{v}) = \frac{1}{2}\big(\big(\mathbf{I} + \mathrm{grad}_{\mathbf{x}'}(\mathbf{v})\big)_{k\alpha}\big(\mathbf{I} + \mathrm{grad}_{\mathbf{x}'}(\mathbf{v})\big)_{k\beta} - \delta_{\alpha\beta}\big) = 0\right\}$$

which is assumed to be different of $\{0\}$. Then we observe that problem $P_2(\omega \times Y)$ is without internal energy and becomes trivial if we choose

$$\mathbf{g}^1 = \int_Y \mathbf{f}^1\,\mathrm{d}\mathbf{y} + \int_{\partial Y_\pm} \mathbf{h}^2\,\mathrm{d}\mathbf{y}' = 0$$

To avoid boundary-layer region around the neighborhoods of the clamped part of the plates, we replace this boundary condition by its average on the unit cell. We assume that external loading satisfies $\mathbf{f}^0 = 0$, $\mathbf{h}^1 = 0$ and $\mathbf{g}^1 = 0$. If $\mathbf{v} = (\mathbf{v}^0, \mathbf{v}^1, \ldots)$ is a solution of the problem $P_3(\omega \times Y)$ then both the leading term $\mathbf{v}^0$ and the macroscopic first order term $\widetilde{\mathbf{v}}^1$ in the asymptotic expansion of the displacement field solve the global minimization problem

$$J_{bm}\big(\mathbf{v}^0, \widetilde{\mathbf{v}}^1\big) = \inf_{\mathbf{u}^0 \in \mathbf{U}^{0\,\mathrm{iso}}(\omega), \widetilde{\mathbf{u}}^1 \in \mathbf{U}^0(\omega)} J_{bm}\big(\mathbf{u}^0, \widetilde{\mathbf{u}}^1\big)$$

where the membrane inextensional-bending energy is defined by

$$J_{bm}\big(\mathbf{u}^0, \widetilde{\mathbf{u}}^1\big) = \int_\omega W_{bm}\big(\mathrm{grad}_{\mathbf{x}'}(\mathbf{u}^0), \mathrm{grad}_{\mathbf{x}'}(\widetilde{\mathbf{u}}^1)\big)\,\mathrm{d}\mathbf{x}' - \int_\omega \bigg(\int_Y f_i^2\,\mathrm{d}\mathbf{y} + \int_{\partial Y_\pm} h_i^3\,\mathrm{d}\mathbf{y}'\bigg) u_i^0\,\mathrm{d}\mathbf{x}'$$
$$- \int_\omega \bigg(\int_Y f_i^1 y_3\,\mathrm{d}\mathbf{y} + \frac{1}{2}\bigg(\int_{\partial Y_+} h_i^2\,\mathrm{d}\mathbf{y}' - \int_{\partial Y_-} h_i^2\,\mathrm{d}\mathbf{y}'\bigg)\bigg) n_i\,\mathrm{d}\mathbf{x}'$$

W_{bm} is the membrane inextensional-bending elastic energy obtained by computing the following local minimization problem

$$W_{bm}\big(\mathrm{grad}_{\mathbf{x}'}(\mathbf{u}^0), \mathrm{grad}_{\mathbf{x}'}(\widetilde{\mathbf{u}}^1)\big) = \inf_{\bar{\mathbf{u}}^2 \in \mathbf{U}^2(\omega \times Y, \mathbb{R}^3)} \int_Y W^2\big(\mathbf{u}^0, \widetilde{\mathbf{u}}^1, \bar{\mathbf{u}}^2\big)\,\mathrm{d}\mathbf{y}$$

in which we set

$$W^2\big(\mathbf{u}^0, \widetilde{\mathbf{u}}^1, \bar{\mathbf{u}}^2\big) = (a+b)\Big\{\big(\widetilde{\mathbf{F}}_{i1}^0\mathbf{F}_{i2}^1 + \widetilde{\mathbf{F}}_{i2}^0\mathbf{F}_{i1}^1\big)^2 + \big(\widetilde{\mathbf{F}}_{i\alpha}^0\mathbf{F}_{i3}^1 + n_i\mathbf{F}_{i\alpha}^1\big)\big(\widetilde{\mathbf{F}}_{j\alpha}^0\mathbf{F}_{j3}^1 + n_j\mathbf{F}_{j\alpha}^1\big)\Big\}$$
$$- 4(a+b)\big(\widetilde{\mathbf{F}}_{i1}^0\mathbf{F}_{i1}^1\widetilde{\mathbf{F}}_{j2}^0\mathbf{F}_{j2}^1 + \widetilde{\mathbf{F}}_{i\alpha}^0\mathbf{F}_{i\alpha}^1 n_j\mathbf{F}_{j3}^1\big) + 2d\big(\widetilde{\mathbf{F}}_{i\alpha}^0\mathbf{F}_{i\alpha}^1 + n_j\mathbf{F}_{j3}^1\big)^2$$

where $\mathbf{F}_{i\alpha}^1 = \partial_{x_\alpha}\widetilde{u}_i^1 + y_3\partial_{x_\alpha}n_i + \partial_{y_\alpha}\bar{u}_i^2$, $\mathbf{F}_{i3}^1 = \partial_{y_3}\bar{u}_i^2$ and the set $\mathbf{U}^2(\omega \times Y)$ is defined by

$$\mathbf{U}^2(\omega \times Y) = \left\{\bar{\mathbf{u}}^2 \in \mathbf{W}_{per}^{1,2}(\omega \times Y),\ \int_Y \bar{\mathbf{u}}^2\,\mathrm{d}\mathbf{y} = 0\right\}$$

For homogeneous plate, this model becomes a bending one which is rigorously obtained by Γ-convergence argument (G. Friesecke et al. [20]).

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