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A dispersive wave equation using nonlocal elasticity

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Abstract

Nonlocal continuum mechanics allows one to account for the small length scale effect that becomes significant when dealing with micro- or nano-structures. This Note investigates a model of wave propagation in a nonlocal elastic material. We show that a dispersive wave equation is obtained from a nonlocal elastic constitutive law, based on a mixture of a local and a nonlocal strain. This model comprises both the classical gradient model and the Eringen's integral model. The dynamic properties of the model are discussed, and corroborate well some recent theoretical studies published to unify both static and dynamics gradient elasticity theories. Moreover, an excellent matching of the dispersive curve of the Born–Kármán model of lattice dynamics is obtained with such nonlocal model. ***To cite this article: N. Challamel et al., C. R. Mecanique 337 (2009).***

Résumé

Une équation des ondes dispersive utilisant l'élasticité non-locale. La mécanique des milieux continus non-locaux permet de prendre en compte des effets d'échelle qui peuvent être significatifs lorsque l'on s'intéresse aux structures à faible échelle (micro ou nano-structures). Cette Note s'intéresse à un modèle de propagation d'ondes dans un milieu élastique non-local. Nous montrons qu'une équation d'ondes dispersive est obtenue à partir d'une loi constitutive non-locale, basée sur une combinaison des déformations locales et non-locales. Le modèle comprend à la fois le modèle au gradient classique et le modèle intégral d'Eringen. Les propriétés dynamiques du modèle sont discutées et corroborent des résultats récents permettant d'unifier les approches au gradient en régime statique et dynamique. De plus, ce modèle permet de décrire de manière très précise les courbes de dispersion du modèle de Born–Kármán. ***Pour citer cet article : N. Challamel et al., C. R. Mecanique 337 (2009).***

Keywords: Continuum mechanics; Nonlocal model; Gradient model; Wave equation; Elasticity; Heterogeneous material; Dispersive properties; Born–Kármán model

Mots-clés : Milieux continus ; Modèle non-local ; Modèle au gradient ; Équation des ondes ; Élasticité ; Matériau hétérogène ; Dispersion ; Modèle de Born–Kármán

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La mécanique des milieux continus non-locaux permet de prendre en compte des effets d'échelle qui peuvent être significatifs lorsque l'on s'intéresse aux structures à faible échelle (micro ou nano-structures). On distingue généralement les modèles au gradient (la contrainte est définie explicitement à partir de la déformation et de ses dérivées), et les modèles élastiques intégraux (la contrainte est obtenue implicitement à partir d'un opérateur intégral qui s'applique à la déformation). Cette Note s'intéresse à un modèle de propagation d'ondes dans un milieu élastique non-local. Nous montrons qu'une équation d'ondes dispersive est obtenue à partir d'une loi constitutive non-locale, basée sur une combinaison des déformations locale et non-locale. Le modèle comprend à la fois le modèle au gradient classique et le modèle intégral d'Eringen. Les propriétés dynamiques du modèle sont discutées et corroborent des résultats récents permettant d'unifier les approches au gradient en régime statique et dynamique.

Les équations fondamentales du modèle unidimensionnel sont présentées en Éq. (3). (σ, ε) correspondent à la contrainte uniaxiale et à la déformation uniaxiale, E est le module de Young, et (l_c, ξ) sont deux paramètres spécifiques des effets non-locaux. Ce modèle peut découler d'une approche énergétique interprétée dans un cadre micromorphique (Forest [10]). La déformation non-locale $\bar{\varepsilon}$ est reliée à la déformation locale ε par l'équation différentielle Éq. (2) si bien que l'opérateur de Green permet d'exprimer la variable $\bar{\varepsilon}$ comme une moyenne spatiale de la déformation locale ε. Le modèle non-local est donc un modèle à deux longueurs internes l_c et $|\xi| l_c$. Le modèle est basé sur une combinaison des déformations locales et non-locales à partir d'une loi de mélange paramétrée par le facteur adimensionnel ξ. Un tel modèle peut aussi se comprendre comme le modèle d'un matériau élastique non-local à deux phases (voir aussi Eringen [11]). Ce modèle comprend le modèle classique au gradient et le modèle intégral non-local d'Eringen (Eringen [7]) lorsque le paramètre ξ s'annule. Challamel et Wang [12] ont appliqué un modèle analogue à l'échelle de la poutre pour appréhender certains effets d'échelle. Des modèles similaires ont été appliqués pour contrôler la localisation induite dans les structures radoucissantes (voir par exemple Challamel et al. [9]).

Le problème de propagation d'ondes dans un milieu unidimensionnel est abordé. L'équation de Newton est donnée en Éq. (6). Les équations de la dynamique aboutissent à l'équation aux dérivées partielles Éq. (7) en introduisant la loi constitutive Éq. (5). c_0 est la célérité axiale. L'équation de propagation des ondes aux propriétés dispersives est obtenue à partir de ce modèle non-local. On reconnaît le modèle postulé par Metrikine [6] pour obtenir une solution inconditionnellement stable (sans introduire de concept de causalité). Ce modèle peut aussi découler d'une approche discrète, comme le montrent (Metrikine et Askes [14]). L'approche micromécanique développée par Metrikine et Askes [14] implique que le champ de variable continue peut être considéré comme la moyenne des déplacements de trois particules voisines. Les solutions correspondant à une propagation d'onde harmonique sont ensuite étudiées. L'expression de la pulsation ω est donnée et corrobore les résultats de Metrikine [6]. Enfin, on examine les modèles existants permettant d'aboutir à une telle équation des ondes dispersive. Alors que Georgiadis et al. [16] introduisent un terme de gradient dans l'énergie cinétique, Askes et Aifantis [4] incorporent des termes additionnels d'inertie dans la loi constitutive. Il nous semble que l'introduction dans cette note de termes additionnels à partir de la loi constitutive non-locale nous semble physiquement plus fondée. De plus, ce modèle non-local permet de décrire de manière très précise les courbes de dispersion du modèle de Born–Kármán (voir Fig. 1), à partir d'une identification de paramètres micromécaniques pertinents.

1. Introduction

Integral type or gradient nonlocal constitutive models abandon the classical assumption of locality, and admit that stress depends not only on the strain at that point. The first models of this type were applied in the 1960s for modelling elastic wave dispersion in crystals. Nonlocal field theory of mechanics has been applied to some various engineering problems, such as dispersion of phonon, Rayleigh wave, and stress concentration at the crack tip (see recently Maugin [1]; Eringen [2]; Rakotomanana [3]; Askes and Aifantis [4]; Lazar et al. [5]; Metrikine [6]). The distinction between gradient elastic models (the stress is explicitly defined from the local strain and its derivatives) and integral elastic models (the stress is implicitly obtained from an integral operator of the local strain) can be established for nonlocal elastic models. In this paper, a mixed gradient and integral nonlocal model is developed that is based on combining the local and the nonlocal strain in the constitutive elastic relation. The dynamic properties of the model are discussed, and corroborate well some recent theoretical studies published to unify both static and dynamics gradient elasticity theories. This model aims at describing wave dispersion in heterogeneous materials.

2. Fundamental model

A one-dimensional problem is investigated in this Note from the functional energy:

$$W[u]=\frac{1}{2}\int_0^L ES\left\{\varepsilon^2+(\xi-1)(\varepsilon-\bar{\varepsilon})^2+l_c^2(\xi-1)(\bar{\varepsilon}')^2\right\}\mathrm{d}x \quad \text{with} \quad \varepsilon=u'(x) \tag{1}$$

where E is the Young modulus, S is the cross section and L is the length of the one-dimensional element. The nonlocal strain $\bar{\varepsilon}$ is related to the local strain ε via a differential equation:

$$\bar{\varepsilon}-l_c^2\bar{\varepsilon}''=\varepsilon \tag{2}$$

(l_c,ξ) are two nonlocal parameters, l_c is a length parameter, and ξ a dimensionless parameter. Therefore, the model implicitly contains two length scales l_c and $|\xi|l_c$. As investigated by Eringen [7], the differential equation (2) clearly shows that the nonlocal strain variable is a spatial weighted average of the strain variable where the weighting function is the Green's function of the differential system associated to relevant boundary conditions. Some similar integral models have been used for nonlocal plasticity or damage models to control the softening-induced localization process (Peerlings et al. [8], see more recently Challamel et al. [9]). It can be shown that this functional energy can be cast in a micromorphic framework (Forest [10]) where the nonlocal strain is considered as an additional micromorphic variable. The stationarity of the functional energy leads to the nonlocal stress–strain relation:

$$\sigma=E\big((1-\xi)\bar{\varepsilon}+\xi\varepsilon\big) \tag{3}$$

with associated boundary conditions $(\bar{\varepsilon}'(0)=\bar{\varepsilon}'(L)=0)$. This nonlocal model is based on combining the local and the nonlocal strain in the constitutive elastic relation (see also Eringen [11]). Challamel and Wang [12] used the same model for the bending of a nonlocal beam. Note that a different functional postulated by Challamel and Wang [12] leads to the same constitutive law with similar boundary conditions:

$$W[u]=\frac{1}{2}\int_0^L ES\left\{\varepsilon\bar{\varepsilon}+\xi l_c^2\varepsilon'\bar{\varepsilon}'\right\}\mathrm{d}x \tag{4}$$

The nonlocal stress–strain constitutive law (3) can be also presented in the differential form (see also Aifantis [13] for the use of such a model to preclude the strain and the stress singularities in dislocation and crack problems):

$$\sigma-l_c^2\sigma''=E\big(\varepsilon-\xi l_c^2\varepsilon''\big) \tag{5}$$

This model comprises the nonlocal integral model of Eringen (Eringen [7]) when the parameter ξ is vanishing ($\xi=0$).

3. Wave equation

The dynamic equation of motion of a uniform bar of section S is obtained through Newton's law:

$$S\frac{\partial\sigma}{\partial x}=\rho S\frac{\partial^2 u}{\partial t^2} \tag{6}$$

The dynamics equations are obtained by combining the constitutive equation (5) with Eq. (6) leading to:

$$\frac{\partial^2 u}{\partial t^2}-c_0^2\frac{\partial^2 u}{\partial x^2}-l_c^2\frac{\partial^4 u}{\partial x^2\partial t^2}+\xi l_c^2c_0^2\frac{\partial^4 u}{\partial x^4}=0 \quad \text{with} \quad c_0=\sqrt{\frac{E}{\rho}} \tag{7}$$

where c_0 is the celerity of axial wave. One recognizes in Eq. (7) the model postulated by Metrikine [6] to obtain an unconditionally stable solution with relevant lower and upper bounds for the speed of energy transfer (even if this model suffers from non-causality). This model may be issued from a discrete approach, as shown by Metrikine and Askes [14]. The micromechanics approach developed by Metrikine and Askes [14] implies that the continuous field variable is considered as an average of displacement of three neighbouring particles. We show in this Note that the dispersive wave equation (7) can be directly achieved with a simple nonlocal constitutive behavior. This model can describe experimentally observed dispersive character of wave propagation in heterogeneous materials. However, the spatial attenuation phenomenon cannot be captured with such a model (the reader can refer to Rakotomanana [3] for the modelling of this specific effect). Note that the case $\xi = 0$ (Eringen's nonlocal model) is similar to the wave equation originally introduced by Boussinesq [15] for shallow channels (Boussinesq [15], see also Maugin [1]).

For harmonic wave propagation, the corresponding solutions can be written in complex form as $u(x,t) = U\exp(j\omega t - jkx)$, leading to the following dispersive equation:

$$\omega = \pm k c_0 \sqrt{\frac{1+\xi l_c^2 k^2}{1+l_c^2 k^2}} \tag{8}$$

A similar model was obtained by Georgiadis et al. [16] who introduced a gradient elasticity term in the kinetic energy, leading to the system of equations:

$$\sigma = E\left(\varepsilon - \xi l_c^2 \frac{\partial^2 \varepsilon}{\partial x^2}\right), \quad \varepsilon = \frac{\partial u}{\partial x}, \quad \text{and} \quad S\frac{\partial \sigma}{\partial x} = \rho S \frac{\partial^2 u}{\partial t^2} - \rho S l_c^2 \frac{\partial^4 u}{\partial x^2 \partial t^2} \tag{9}$$

It is easily checked that Eq. (9) leads to the dispersive wave equation (7). This is also the model postulated by Askes and Aifantis [4] by adding some higher-order inertial terms in the constitutive equation:

$$\sigma = \rho l_c^2 \frac{\partial^2 \varepsilon}{\partial t^2} + E\left(\varepsilon - \xi l_c^2 \frac{\partial^2 \varepsilon}{\partial x^2}\right), \quad \varepsilon = \frac{\partial u}{\partial x}, \quad \text{and} \quad S\frac{\partial \sigma}{\partial x} = \rho S \frac{\partial^2 u}{\partial t^2} \tag{10}$$

Introduction of these additional terms from the mixed constitutive law, as proposed in this paper, seems to be physically more readable.

4. Comparison with the Born–Kármán model of lattice dynamics

The present nonlocal model can be compared to the Born–Kármán model of lattice dynamics where the nearest-neighbour interactions are accounted. This model gives the following dispersion relation:

$$\frac{\omega a}{c_0} = 2\sin\left(\frac{ka}{2}\right) \tag{11}$$

where a is the distance between atoms. The presented nonlocal model is able to predict the Born–Kármán model for specified values of the two parameters (l_c, ξ). These parameters could be identified from the characteristic prediction of the dispersive curve of lattice dynamics:

$$\omega\left(k = \frac{\pi}{a}\right) = \frac{2c_0}{a} \quad \text{and} \quad \frac{\mathrm{d}\omega}{\mathrm{d}k}\left(k = \frac{\pi}{a}\right) = 0 \tag{12}$$

The first condition leads to the length scales equation for the nonlocal model:

$$\frac{l_c}{a} = \sqrt{\frac{1-\frac{4}{\pi^2}}{4-\xi\pi^2}} \quad \text{if } \xi < \frac{4}{\pi^2} \tag{13}$$

The well-known value $l_c \cong 0.386a$ is found in case of Eringen's nonlocal model ($\xi = 0$) (see also Eringen [11]). Injecting Eq. (13) into the second condition of Eq. (12) leads to the optimal parameter identification:

$$\xi = \frac{16}{\pi^2(8-\pi^2)} \cong -0.867 \quad \text{and} \quad \frac{l_c}{a} = \frac{\sqrt{\pi^2-8}}{2\pi} \cong 0.218 \tag{14}$$

In this case, the optimal value of $l_c = 0.218a$ is obtained. Note that this is also the optimal value obtained for the Kunin's model, based on a bi-Helmholtz type equation, leading to the dispersive equation (see also Eringen [2]; Lazar et al. [5]):

$$\omega = \pm k c_0 \sqrt{\frac{1}{1+l_c^2 k^2 + \gamma^4 k^4}} \quad \text{with} \quad \frac{l_c}{a} = \frac{\sqrt{\pi^2-8}}{2\pi} \cong 0.218 \quad \text{and} \quad \frac{\gamma}{a} = \frac{1}{\pi} \cong 0.318 \tag{15}$$

Fig. 1 shows the excellent matching of the mixed nonlocal model compared to the Born–Kármán model of lattice dynamics. Kunin's model appears to be quite less efficient.

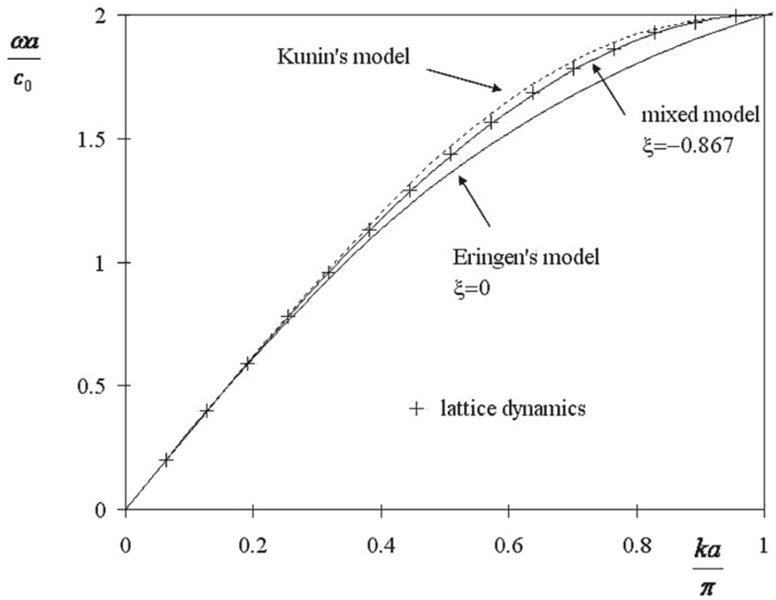

Fig. 1. Dispersion curve for the nonlocal elasticity model – Comparison with the Born–Kármán model of lattice dynamics.

5. Conclusions

In the present paper, a mixed gradient and integral nonlocal model is developed that is based on combining the local and the nonlocal strain in the constitutive elastic relation. The present constitutive model can be used in the dynamic regime, and the dispersive wave models recently postulated by Metrikine [6] or Askes and Aifantis [4] are achieved. A physically-based dispersive wave equation is obtained from this model based on a mixture of the local and the nonlocal strain. With an atomic point of view, the model takes into account more general interactions than interactions from the nearest atoms. Moreover an excellent matching of the dispersive curve of the Born–Kármán model is obtained with such a model. Therefore, both static and dynamics gradient elasticity theories can be merged with such a simple nonlocal constitutive behavior.

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