Available online at www.sciencedirect.com

ScienceDirect

C. R. Mecanique 337 (2009) 610–615

Comptes Rendus Mecanique

A novel approach for detecting trapped surface waves in a canal with periodic underwater topography ☆

Sergey A. Nazarov

Institute of Mechanical Engineering Problems, V.O., Bol'shoi pr., 61, 199178, St.-Petersburg, Russia

Received 8 May 2009; accepted after revision 22 June 2009

Available online 17 July 2009

Presented by Jean-Baptiste Leblond

Abstract

A new approach is proposed to detect trapped modes in periodic canals. The obtained sufficient condition is new even for a straight canal with a body piercing the water surface as well. ***To cite this article: S.A. Nazarov, C. R. Mecanique 337 (2009).***

Résumé

Une nouvelle approche pour la détection d'ondes de surfaces piégées dans un canal à topographie souterraine périodique. Une nouvelle approche est proposée pour détecter les modes piégés dans les canaux périodiques. La condition suffisante obtenue est nouvelle également dans le cas d'un canal droit avec un corps traversant la surface de l'eau. ***Pour citer cet article : S.A. Nazarov, C. R. Mecanique 337 (2009).***

Keywords: Waves; Surface water-waves; Trapped modes; Localized solutions

Mots-clés : Ondes ; Ondes de surface marines ; Modes piégés ; Solutions localisées

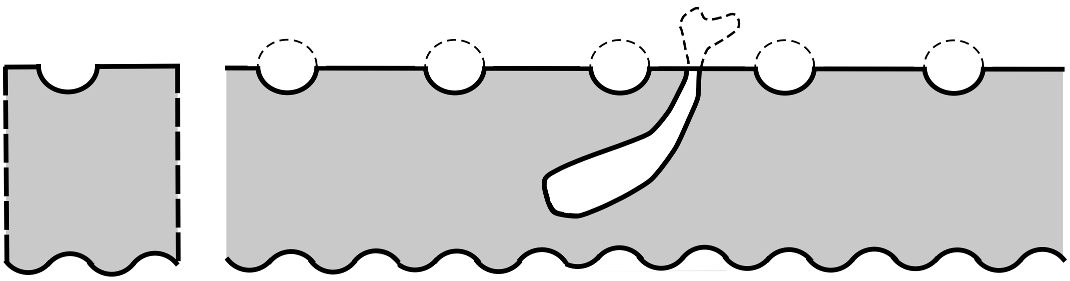
1. Different formulations of the water-wave problem

Let $\varpi \subset \mathbb{R}^2$ be an open set in the rectangle $(0,1) \times (-H,0)$, $H > 0$, while $\gamma \neq \emptyset$ is the union of open intervals belonging to the boundary $\partial\varpi$ and to the horizontal line $\mathbb{R} \times \{0\}$ as well (Fig. 1(a)). We determine the periodic set Π, a quasi-cylinder, Fig. 1(b), as the interior of

$$\overline{\Pi} = \bigcup_{j \in \mathbb{Z}} \overline{\varpi_j} \tag{1}$$

☆ The author gratefully acknowledges the support by N.W.O., the Netherlands Organization for Scientific Research.
E-mail address: srgnazarov@yahoo.co.uk.

1631-0721/$ – see front matter
doi:10.1016/j.crme.2009.06.029



(a) (b)

Fig. 1. The periodicity cell and the periodic canal with a body.

where $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ and $\varpi_j = \{(y, z)\colon\ (y - j, z) \in \varpi\}$ is a shift of the periodicity cell ϖ along the y-axis. We also fix a compact set $\Theta \subset \overline{\Pi}$. Assuming $\Omega = \Pi \setminus \Theta$ to be a Lipschitz domain, we consider the linear spectral boundary value problem in the theory of surface water-waves (cf. [1,2])

$$-\Delta\varphi + \kappa^2\varphi = 0 \quad \text{in } \Omega, \qquad \partial_\nu\varphi = 0 \quad \text{on } \Sigma, \qquad \partial_z\varphi = \lambda\varphi \quad \text{on } \Gamma \tag{2}$$

Here, φ is the velocity potential, κ the wave number in the x-direction, perpendicular to the plane $\mathbb{R}^2 \ni (y, z)$, $\lambda = \omega^2/g$ stands for a spectral parameter and ω is the frequency, $g > 0$ the acceleration due to gravity. The boundary $\partial\Omega$ is divided into two parts

$$\Sigma = \big\{(y, z) \in \partial\Omega\colon\ z < 0\big\}, \qquad \Gamma = \partial\Omega \setminus \overline{\Sigma} = \Gamma_{per} \cup \theta \tag{3}$$

where $\Gamma_{per} = \bigcup \gamma_j$ in accord with (1) and $\theta = \{(y, z) \in \partial\Theta\colon\ z = 0\}$ is the horizontal part of the obstacle boundary at the water surface level. Our assumption that Ω is a Lipschitz domain, provides the following: Ω is a connected set, the derivative ∂_ν along the outward normal ν is defined almost everywhere on Σ, and cuspidal irregularities of the boundary are forbidden which could lead to the continuous spectrum irrelative to wave processes at infinity [3]. Clearly, $\partial_\nu = \partial_z = \partial/\partial z$ on Γ.

Since the period in (1) is one, rescaling had been made and all coordinates have become dimensionless. We assume $\kappa > 0$, that is the wave deviation angle with respect to the y-axis is acute.

Owing to the possible boundary irregularities, the velocity field $\nabla\varphi = \operatorname{grad}\varphi$ may get singularities and problem (2) ought to be understood as the integral identity [4]

$$(\nabla\varphi, \nabla\psi)_\Omega + \kappa^2(\varphi, \psi)_\Omega = \lambda(\varphi, \psi)_\Gamma, \quad \psi \in H^1(\Omega) \tag{4}$$

where $(\varphi, \psi)_\Omega$ is the natural scalar product in the Lebesgue space $L^2(\Omega)$ and $H^1(\Omega)$ the Sobolev space supplied with the specific scalar product $\langle\varphi, \psi\rangle$ on the left of (4). Introducing a positive, continuous and symmetric, therefore, self-adjoint operator T in $H^1(\Omega)$

$$\langle T\phi, \psi\rangle = (\phi, \psi)_\Gamma, \quad \phi, \psi \in H^1(\Omega) \tag{5}$$

we reduce the variational problem (4) to the abstract equation

$$T\varphi = \mu\varphi \quad \text{in } H^1(\Omega) \tag{6}$$

with the new spectral parameter

$$\mu = 1/\lambda \tag{7}$$

As well known (see, e.g., [5]), the spectrum s of T is situated in the segment $[0, t]$ of the real axis $\mathbb{R}$ in the complex plane $\mathbb{C}$, where t denotes the norm of the operator,

$$t = \sup_{\phi \in H^1(\Omega)\setminus\{0\}} \frac{\langle T\phi, \phi\rangle}{\langle\phi, \phi\rangle} \tag{8}$$

The point $\mu = 0$ is an eigenvalue with the infinite multiplicity with an eigenspace composed from functions in $H^1(\Omega)$ vanishing at Γ. Since the embedding $H^1(\Omega) \subset L^2(\Gamma)$ is not compact in the unbounded domain Ω, the essential spectrum s_e of T does not reduce to the point $\mu = 0$ and its upper bound $\mu_\dagger = 1/\lambda_\dagger$ is positive.

In the sequel we employ a device, proposed in [6] and adapted here to periodic structures, in order to derive a geometrical sufficient condition for the inequality

$$t > \mu_\dagger \tag{9}$$

which ensures that $\mu_1 = t$ is an eigenvalue in the discrete spectrum $s_d = s \setminus s_e \neq \emptyset$. At the same time, $\lambda_1 = 1/\mu_1$, by the relationship (7), becomes the smallest eigenvalue below the lower bound $\lambda_\dagger$ of the essential spectrum S_e of problem (4) = (2). The corresponding eigenfunction $\varphi_1 \in H^1(\Omega)$ readily implies a trapped mode (cf. [7–11] and others).

2. The band–gap structure of the essential spectrum

The Gel'fand transform [12] (see, e.g., [13–15] for details) associates the problem (2) in the quasi-cylinder Π with the following problem in the periodicity cell:

$$-\Delta\Phi + \kappa^2\Phi = 0 \quad \text{in } \varpi, \qquad \partial_\nu \Phi = 0 \quad \text{on } \sigma, \qquad \partial_z \Phi = \Lambda\Phi \quad \text{on } \gamma \tag{10}$$

$$\Phi|_{y=0} = e^{i\eta}\Phi|_{y=1}, \qquad \partial\Phi|_{y=0} = e^{i\eta}\partial\Phi|_{y=1} \tag{11}$$

The quasi-periodicity conditions (11) which involve the transform parameter $\eta \in [0, 2\pi)$ and are posed over the lateral faces τ_0 and τ_1 of the cell marked with dotted lines in Fig. 1(a). Notice that $\sigma = \partial\varpi \setminus (\overline{\gamma \cup \tau_0 \cup \tau_1})$. The integral identity

$$(\nabla\Phi, \nabla\Psi)_\varpi + \kappa^2(\Phi, \Psi)_\varpi = \Lambda(\Phi, \Psi)_\gamma, \quad \Psi \in H^1(\varpi; \eta) \tag{12}$$

serving for problem (10), (11), refers to the subspace $H^1(\varpi;\eta)$ of functions in $H^1(\varpi)$ meeting the first condition in (11). The sesquilinear form on the left of (12) is Hermitian positive definite and closed in $H^1(\varpi;\eta)$. Furthermore, the embedding $H^1(\varpi) \subset L^2(\varpi)$ is compact in the bounded domain ϖ (see, e.g., [4, §1.6]) and, hence, eigenvalues of problem (12) = (10) + (11) compose the monotone positive unbounded sequence

$$0 < \Lambda_1(\eta) \leqslant \Lambda_2(\eta) \leqslant \cdots \leqslant \Lambda_p(\eta) \leqslant \cdots \to +\infty \tag{13}$$

where they are listed while counting multiplicity.

Fixing $\Lambda \in \mathbb{C}$, one may regard (12) as a holomorphic pencil $\mathbb{C} \ni \eta \mapsto P_\Lambda(\eta)$ [16, Ch. 1]. Let F be a linear functional in the space $H^1(\Omega)$. The operator in $H^1(\Omega)$ of the inhomogeneous problem (4)

$$(\nabla\phi, \nabla\psi)_\Omega + \kappa^2(\phi,\psi)_\Omega - \lambda(\phi,\psi)_\Gamma = F(\psi), \quad \psi \in H^1(\Omega)$$

is known (see [17], [14, Ch. 3] and cf. [13,15]) to be Fredholm if and only if the segment $[0, 2\pi) \subset \mathbb{R} \subset \mathbb{C}$ is free of the η-spectrum of the pencil P_Λ. It is straightforward to derive from this fact the following formula for the essential spectrum of problem (4) (cf. [13,15]):

$$S_e = \bigcup_{p=1}^{\infty} \Upsilon_p, \quad \Upsilon_p = \{\lambda = \Lambda_p(\eta) \mid \eta \in [0, 2\pi)\} \tag{14}$$

Since the functions $\mathbb{R} \ni \eta \mapsto \Lambda_p(\eta)$ are continuous [18, Ch. 9] and 2π-periodic (the change $\eta \mapsto \eta \pm 2\pi i$ does not effect on (11)), the bands Υ_p in (14) are closed connected segments. The segmental structure of S_e allows for gaps, i.e., intervals in $\mathbb{R}_+$ which may include points of the discrete spectrum S_d only but have their endpoints in S_e.

The lower bound of S_e takes the form

$$\lambda_\dagger = \min\{\Lambda_1(\eta) \mid \eta \in [0, 2\pi)\} > 0 \tag{15}$$

Let the point $\eta_\dagger$ constitute the minimum in (15) and let $\Phi_\dagger$ denote the corresponding eigenfunction. The function $\varphi_\dagger$,

$$\varphi_\dagger(y, z) = e^{i\eta j}\Phi_\dagger(y - j, z), \quad (y, z) \in \varpi_j, \quad j \in \mathbb{Z} \tag{16}$$

belongs to $H^1_{loc}(\overline{\Pi})$ by virtue of the quasi-periodicity conditions (11) and solves the problem (2) in Π but only formally because it falls out from $H^1(\Pi)$ due to the oscillation at infinity. We emphasize that (12) ensures the relation

$$I_\varpi := \langle\varphi_\dagger, \varphi_\dagger\rangle = \lambda_\dagger \|\varphi_\dagger; L^2(\gamma)\|^2 =: I_\gamma \tag{17}$$

Formula (7) passes the band–gap structure to the essential spectrum of the operator T

$$s_e = \bigcup_{p=1}^{\infty} \upsilon_p, \quad \upsilon_p = \{\mu = \Lambda_p(\eta)^{-1} \mid \eta \in [0, 2\pi)\}$$

3. Deriving the sufficient condition for trapped modes

With any $\varepsilon \in (0, 1]$, we insert the function $\phi_\dagger^\varepsilon = e^{-\varepsilon|y|}\varphi_\dagger \in H^1(\Pi) \subset H^1(\Omega)$ into formula (8) for the norm t. The following calculation is the crucial issue for our further consideration:

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} e^{-\varepsilon|j|} &= \int_{\mathbb{R}} e^{-\varepsilon|y|}\,\mathrm{d}y + \sum_{j\in\mathbb{Z}} \int_j^{j+1} \left(e^{-\varepsilon|j|} - e^{-\varepsilon|y|}\right)\mathrm{d}y \\
&= \frac{1}{\varepsilon} + 2\varepsilon \sum_{j\in\mathbb{Z}} e^{-\varepsilon|j|}\left(\int_j^{j+1} \left(|y| - |j|\right)\mathrm{d}y + O(\varepsilon)\right)\mathrm{d}y \\
&= \frac{1}{\varepsilon} + 2\varepsilon \sum_{j\in\mathbb{Z}} e^{-\varepsilon|j|}\left(\frac{1}{2}\operatorname{sign}(j) + O(\varepsilon)\right) \\
&= \frac{1}{\varepsilon} + O(\varepsilon)
\end{aligned}$$

Processing integrals over Π and Γ_{per} in a similar manner and taking into account that $e^{-2\varepsilon|y|} = 1 + O(\varepsilon)$ in the compact set Θ, we obtain

$$\begin{aligned}
\langle T\phi_\dagger^\varepsilon, \phi_\dagger^\varepsilon\rangle &= \|\phi_\dagger^\varepsilon; L^2(\Gamma_{per})\|^2 - \|\phi_\dagger^\varepsilon; L^2(\theta)\|^2 = \varepsilon^{-1} I_\gamma - I_\theta + O(\varepsilon) \\
\langle \phi_\dagger^\varepsilon, \phi_\dagger^\varepsilon\rangle &= \|\nabla\phi_\dagger^\varepsilon; L^2(\Pi)\|^2 + \kappa^2\|\phi_\dagger^\varepsilon; L^2(\Pi)\|^2 - \|\nabla\phi_\dagger^\varepsilon; L^2(\Theta)\|^2 - \kappa^2\|\phi_\dagger^\varepsilon; L^2(\Theta)\|^2 \\
&= \varepsilon^{-1} I_\varpi - I_\Theta + O(\varepsilon) = \varepsilon^{-1}\lambda_\dagger I_\gamma - I_\Theta + O(\varepsilon)
\end{aligned}$$

where we recall (17) and put

$$I_\Theta = \int_\Theta \left(|\nabla\varphi_\dagger(y,z)|^2 + \kappa^2|\varphi_\dagger(y,z)|^2\right)\mathrm{d}y\,\mathrm{d}z, \qquad I_\theta = \int_\theta |\varphi_\dagger(y,0)|^2\,\mathrm{d}z \tag{18}$$

Note that in the case $\Theta \subset \overline{\varpi}$ function (16) can be replaced by $\Phi_\dagger$ in the above integrands. We now have

$$t \geqslant \frac{\langle T\phi_\dagger^\varepsilon, \phi_\dagger^\varepsilon\rangle}{\langle \phi_\dagger^\varepsilon, \phi_\dagger^\varepsilon\rangle} \geqslant \frac{\varepsilon^{-1} I_\gamma - I_\theta - c\varepsilon}{\varepsilon^{-1} I_\varpi - I_\Theta + c\varepsilon} \geqslant \mu_\dagger\left(1 + \frac{\varepsilon}{I_\gamma}\mu_\dagger(I_\Theta - \lambda_\dagger I_\theta) - C\varepsilon^2\right) \tag{19}$$

Under the condition

$$I_\Theta > \lambda_\dagger I_\theta \tag{20}$$

which just relates integrals (18), the parameter $\varepsilon > 0$ can be chosen such that the right-hand side of (19) exceeds $\mu_\dagger$. In other words, inequality (9) is valid and, thus, $s_d \cap (\mu_\dagger, +\infty) \neq \emptyset$ and $S_d \cap (0, \lambda_\dagger) \neq \emptyset$. In other words, (20) implies a sufficient condition for trapped modes.

4. Discussion

Let Π be the strip $\mathbb{R} \times (-H, 0)$. Solutions of the spectral problem (10), (11) in the cell $\varpi = (0, 1) \times (-H, 0)$ get an explicit form, in particular,

$$\begin{aligned}
&\Lambda_1(\eta) = \zeta\left(1 + e^{-2H\zeta}\right)^{-1}\left(1 + e^{-2H\zeta}\right), \qquad \Phi_1(y, z; \eta) = e^{-i\eta y}\left(e^{z\zeta} + e^{-(2H+z)\zeta}\right) \\
&\zeta = \left(\eta^2 + \kappa^2\right)^{1/2}, \qquad \lambda_\dagger = \kappa\left(1 + e^{-2H\kappa}\right)^{-1}\left(1 + e^{-2H\kappa}\right)
\end{aligned} \tag{21}$$

Denoting $|\theta|$ the length of θ, the condition (20) reads:

$$\frac{1}{\kappa}\frac{1+e^{-2H\kappa}}{1+e^{-2H\kappa}}\int\limits_{\Theta}\left(e^{2z\kappa}+e^{-2(2H+z)\kappa}\right)\mathrm{d}y\,\mathrm{d}z>|\theta| \tag{22}$$

If the body Θ is submerged, i.e. $\Theta \subset \Pi$, and therefore, $\theta = \emptyset$, $I_\theta = 0$, then both inequalities (20) and (22) are automatically satisfied that provides trapped modes. For the strip $\Pi = \mathbb{R} \times (-H, 0)$ this result is known (see [7–9]), however the proof in (19) is much simpler. For a body piercing the water surface, the sufficient condition derived is new in both the cases, straight and periodic.

Let us comment on the comparison principles (cf. [8,9,19]). We consider a compact set $\Theta_\flat$ such that $\Theta \subset \Theta^\flat$, $\Theta \neq \Theta^\flat$ but $\theta = \theta^\flat = \{(y,z) \in \partial\Theta^\flat\colon z = 0\}$ and $\Gamma = \Gamma^\flat$, while $\Omega^\flat = \Pi \setminus \Theta^\flat$ is still a Lipschitz domain. According to (5) we introduce the operator $T^\flat$ in $H^1(\Omega^\flat)$ with the norm $t^\flat \geqslant \mu_\dagger$ and observe that $t^\flat > t$ because the integration domain $\Omega^\flat$ in the denominator on the right-hand side of the relation (8) for $t^\flat$ is smaller than Ω. Thus, the inclusion $t = \mu_1 \in s_d$ assures that $t^\flat = \mu_1^\flat \in s_d^\flat$. Moreover, in the same way the max–min principle provides that the total multiplicity of the part of the discrete spectrum $s_d^\flat$ located in $(\mu_\dagger, +\infty]$ is bigger than the one of $s_d \cap (\mu_\dagger, +\infty]$.

In the above-cited papers the comparison principles were established and applied in a straight canal. The approach presented here, assists for adopting some of these results to periodic canals.

5. Open questions

If the segment Υ_p in (14) shrinks up to a single point $\lambda^{(p)}$, then $\lambda^{(p)}$ becomes an eigenvalue with infinite multiplicity of the problem (4) in the quasi-cylinder Π. Is the formula $\Upsilon_p = \{\lambda^{(p)}\}$ possible? What does it happen in this case with the spectrum of the problem (4) in Ω?

The peculiar formulas (21) demonstrate that $\eta_\dagger = 0$ for a straight canal. It is not difficult to verify that the point $\eta = 0$ constitutes a local strict minimum of the function $\eta \mapsto \Lambda_1(\eta)$. Is this minimum global and, thus, $\eta_\dagger = 0$ always?

For any $\delta > 0$, $N = 1, 2, \ldots$ and arbitrary canal Π, method [20] provides a body Θ such that problem (4) has at least N eigenvalues in the interval $(0, \delta)$. An example of a domain $\Omega = \Pi \setminus \Theta$ with an eigenvalue in the band of the essential spectrum or inside a gap is not known yet.

References

[1] J.J. Stoker, Water Waves: The Mathematical Theory with Applications, John Willey & Sons, Inc., New York, 1992. Reprint of the 1957 original.

[2] N. Kuznetsov, V. Maz'ya, B. Vainberg, Linear Water Waves, Cambridge University Press, Cambridge, 2002.

[3] S.A. Nazarov, J. Taskinen, On the spectrum of the Steklov problem in a domain with a peak, Vestnik St. Petersburg Univ. Math. 41 (1) (2008) 45–52.

[4] O.A. Ladyzhenskaya, The Boundary Value Problems of Mathematical Physics, Applied Mathematical Sciences, vol. 49, Springer-Verlag, New York, 1985.

[5] J. Sanchez-Hubert, E. Sanchez-Palencia, Vibration and Coupling of Continuous Systems: Asymptotic Methods, Springer-Verlag, Heidelberg, 1989.

[6] I.V. Kamotskii, S.A. Nazarov, Elastic waves localized near periodic sets of flaws, Dokl. Ross. Akad. Nauk 368 (6) (1999) 771–773; English transl.: Dokl. Phys. 44 (10) (1999) 715–717.

[7] R.M. Garipov, On the linear theory of gravity waves: the theorem of existence and uniqueness, Arch. Rational Mech. Anal. 24 (1967) 352–362.

[8] F. Ursell, Mathematical aspects of trapping modes in the theory of surface waves, J. Fluid Mech. 183 (1987) 421–437.

[9] A.-S. Bonnet-Ben Dhia, P. Joly, Mathematical analysis of guided water-waves, SIAM J. Appl. Math. 53 (1993) 1507–1550.

[10] N. Kuznetsov, R. Porter, D.V. Evans, M.J. Simon, Uniqueness and trapped modes for surface-piercing cylinders in oblique waves, J. Fluid Mech. 365 (1998) 351–368.

[11] C.M. Linton, P. McIver, Embedded trapped modes in water waves and acoustics, Wave Motion 45 (2007) 16–29.

[12] I.M. Gel'fand, Expansions in eigenfunctions of an equation with periodic coefficients, Dokl. Acad. Nauk SSSR 73 (1950) 1117–1120.

[13] P.A. Kuchment, Floquet theory for partial differential equations [in Russian], Uspekhi Mat. Nauk 37 (4) (1982) 3–52; English transl.: Russian Math. Surveys 37 (4) (1982) 1–60.

[14] S.A. Nazarov, B.A. Plamenevsky, Elliptic Problems in Domains with Piecewise Smooth Boundaries, Walter de Gruyter, Berlin, New York, 1994.

[15] P. Kuchment, Floquet Theory for Partial Differential Equations, Birkhäuser, Basel, 1993.

[16] I.C. Gohberg, M.G. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators, Amer. Math. Soc., Providence, 1969.

[17] S.A. Nazarov, Elliptic boundary value problems with periodic coefficients in a cylinder, Izv. Akad. Nauk SSSR Ser. Mat. 45 (1) (1981) 101–112; English transl.: Math. USSR Izv. 18 (1) (1982) 89–98.

[18] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, Heidelberg, 1966.
[19] O.V. Motygin, On trapping of surface water waves by cylindrical bodies in a channel, Wave Motion 45 (2008) 940–951.
[20] S.A. Nazarov, Concentration of trapped modes in problems of the linearized theory of water waves, Mat. Sb. 199 (12) (2008) 53–78; English transl.: Sb. Math. 199 (12) (2008) 1783–1807.