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ScienceDirect
C. R. Mecanique 337 (2009) 260–267
Comptes Rendus Mecanique

Closed-form solutions for the hollow sphere model with Coulomb and Drucker–Prager materials under isotropic loadings

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Received 2 March 2009; accepted after revision 8 June 2009

Available online 29 July 2009

Presented by Jean-Baptiste Leblond

Abstract

Though the solution to the limit analysis problem of the hollow sphere model—with a von Mises matrix and under spherical symmetry—is well known, it is not available, to our knowledge, for both isotropic loadings (tension and compression) in the case of a Coulomb matrix and partially for a Drucker–Prager matrix. In the present Note, we establish in a unified framework, for this class of materials, closed-form solutions for stress and strain fields in a hollow sphere under external isotropic tension and compression. These analytical results not only give useful reference solutions, but can also be considered as a part of a trial velocity field in the hollow sphere submitted to an arbitrary loading. Comparisons with 3D finite element-based limit analysis approaches and with recent results in the literature are provided. In addition to the established analytical results, we present a rigorous evaluation of a recent Gurson-type macroscopic criterion corresponding to the Drucker–Prager hollow sphere under an arbitrary loading, by means of the previous 3D limit analysis codes. ***To cite this article: Ph. Thoré et al., C. R. Mecanique 337 (2009).***

Résumé

Solutions exactes du modèle de la sphère creuse à matrice de Coulomb et de Drucker–Prager sous chargements isotropes. Alors que la solution du problème d'analyse limite du modèle de la sphère creuse—à matrice de von Mises et en symétrie sphérique—est bien connue, elle n'est pas, à notre connaissance, disponible dans les deux cas de chargement isotrope (traction, compression) pour une matrice solide de Coulomb et partiellement pour une matrice de Drucker–Prager. Dans la présente Note, nous établissons dans un cadre unifié et pour cette classe de matériaux, les solutions exactes (champs de contraintes, de déformation) pour une sphère creuse soumise à une traction ou compression isotrope externe. Ces résultats analytiques sont non seulement utiles comme solutions de référence, mais elles peuvent aussi être considérées comme partie de champ d'essai en vitesse pour le modèle de la sphère creuse soumise à un chargement arbitraire. On fournit une comparaison avec des approches 3D par éléments finis du problème d'analyse limite, ainsi qu'avec de récents résultats dans la littérature. Outre les solutions analytiques établies, nous présentons une évaluation d'un récent critère de plasticité macroscopique correspondant à la sphère creuse de Drucker–Prager sous chargement quelconque, ceci à l'aide des précédents codes d'analyse limite 3D. ***Pour citer cet article : Ph. Thoré et al., C. R. Mecanique 337 (2009).***

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1631-0721/$ – see front matter
doi:10.1016/j.crme.2009.06.030

Keywords: Porous media; Gurson model; Limit analysis; Exact solution; Isotropic loading; Drucker–Prager and Coulomb materials

Mots-clés : Milieux poreux ; Modèle de Gurson ; Analyse limite ; Solution exacte ; Drucker–Prager ; Coulomb

1. Introduction

In the famous paper [1], Gurson proposed an upper bound limit analysis approach of a hollow sphere and a hollow cylinder having a von Mises solid matrix. The computation has been performed under uniform boundary strain rate conditions and led to a macroscopic yield function of the "Porous von Mises"-type materials. Gurson's analysis consists in the use of a kinematic approach of limit analysis (LA) of the hollow sphere in order to obtain an exterior approach. It is based on the consideration of a two-part velocity field, the first part being the solution to the problem of the hollow sphere under isotropic stress or strain rate loading. It was later shown that such an exterior LA method provides an upper bound of the macroscopic criterion of the spherically porous material (at least in the sense of Composite Sphere Assemblage of Hashin) subjected to an arbitrary loading on its external boundary.

Several extensions of the Gurson's model have been further proposed in the literature, the probably most important developments being those accounting for voids shape effects [2–4]. Mention can also be made of models taking into account plastic anisotropy [5,6]. More recent extensions concern the consideration of plastic compressibility of the matrix with Drucker–Prager—not for Coulomb—criterion, as it is the case for polymers or cohesive geomaterials [7–9], and [10]. For validation purpose of these recent models, and for reference solutions in the Coulomb case, there is still a strong need of numerical [11] or analytical solutions.

The main objective of the present Note is to derive the complete closed-form solutions to the problem of a hollow sphere under external pressure, with a Coulomb or a Drucker–Prager type solid matrix: the limit load and the associated stress, velocity and strain rate fields, which could be considered as a part of trial velocity fields in the hollow sphere under arbitrary loadings, will be also given. We present a comparison of the derived results with our previous and recent finite element data, and with partial results recently available in the literature. Finally, taking advantage of our 3D finite element-based limit analysis bounds, we provide a rigorous evaluation of a Gurson-type macroscopic criterion recently established by [10] in the case of a hollow sphere with Drucker–Prager matrix under an arbitrary loading.

2. Unified statement of the problem

The considered hollow sphere is made up of a single spherical cavity embedded in a homothetic cell of a rigid-plastic isotropic and homogeneous Drucker–Prager or Coulomb material. Let us denote by r_0 and R the internal and the external radii of this hollow sphere, respectively. The hollow sphere is submitted at its external boundary to a uniform pressure P, so that the mechanical problem complies with the spherical symmetry. Obviously, this is equivalent to the problem of the same hollow sphere submitted at its external boundary to a uniform isotropic strain rate (or equivalently to a radial, constant velocity).

In the following, we make use of the appropriate spherical coordinates (r, θ, φ), r being the radius, θ the inclination angle and φ the azimuth angle. The objective in this section is to provide a unified statement which allows to consider both Drucker–Prager and Coulomb criteria.

Let us note that here the stress field $\underline{\underline{\sigma}}$ has only three non-zero components: σ_{rr}, $\sigma_{\theta\theta} = \sigma_{\varphi\varphi}$, and that the equilibrium equations reduce in absence of body forces to:

$$\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r} + 2\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + f_r = 0 \tag{1}$$

Due to the spherical symmetry, the two plasticity criteria considered in the present Note can be put into the form:

$$\varepsilon(\sigma_{\theta\theta} - \sigma_{rr}) + A\sigma_{\theta\theta} + B\sigma_{rr} - C \leqslant 0 \tag{2}$$

with the following convention:

$\varepsilon = +1$ in *tension* with $0 < \sigma_{rr} < \sigma_{\theta\theta}$,
$\varepsilon = -1$ in *compression* with $\sigma_{\theta\theta} < \sigma_{rr} < 0$.

– *Coulomb criterion*:
As detailed in Section 3.2, from the spherical symmetry, the Coulomb criterion can be written as:

$$f_{r\theta}(\underline{\underline{\sigma}}) = |\sigma_{\theta\theta} - \sigma_{rr}| + (\sigma_{rr} + \sigma_{\theta\theta})\sin\phi - 2c\cos\phi \tag{3a}$$

$$f_{r\varphi}(\underline{\underline{\sigma}}) = |\sigma_{\varphi\varphi} - \sigma_{rr}| + (\sigma_{rr} + \sigma_{\varphi\varphi})\sin\phi - 2c\cos\phi \tag{3b}$$

In the following static approach, as no derivatives of the criterion are needed, we can directly consider $\sigma_{\theta\theta} = \sigma_{\varphi\varphi}$, and the Coulomb criterion can be written as:

$$|\sigma_{\theta\theta} - \sigma_{rr}| + (\sigma_{\theta\theta} + \sigma_{rr})\sin\phi - 2c\cos\phi \leqslant 0 \tag{4}$$

which leads to: $A = B = \sin\phi$ and $C = 2c\cos\phi$ in (2).

– *Drucker–Prager criterion*:
The Drucker–Prager criterion reads:

$$\sqrt{J_2} + \alpha\,\mathrm{tr}(\underline{\underline{\sigma}}) - k \leqslant 0 \tag{5}$$

where $J_2 = \frac{1}{6}[(\sigma_{rr} - \sigma_{\theta\theta})^2 + (\sigma_{\theta\theta} - \sigma_{\varphi\varphi})^2 + (\sigma_{\varphi\varphi} - \sigma_{rr})^2] + \sigma_{r\theta}^2 + \sigma_{\theta\varphi}^2 + \sigma_{\varphi r}^2$ and $\alpha = \frac{\sin\phi}{\sqrt{3(3+\sin^2\phi)}}$; $H = c\cot\phi$; $k = 3\alpha H = 3\alpha c\cot\phi$ and $\phi \in [0; \frac{\pi}{2}]$; $\alpha \in [0; \frac{\sqrt{3}}{6}]$.
Considering again $\sigma_{\theta\theta} = \sigma_{\varphi\varphi}$, it follows that $J_2 = \frac{1}{3}(\sigma_{rr} - \sigma_{\theta\theta})^2$, and then the Drucker–Prager criterion (5) takes the form:

$$|\sigma_{\theta\theta} - \sigma_{rr}| + \alpha\sqrt{3}(2\sigma_{\theta\theta} + \sigma_{rr}) - k\sqrt{3} \leqslant 0 \tag{6}$$

which corresponds in (2) to:

$$A = 2\alpha\sqrt{3} = 2\sin\phi/\sqrt{3 + \sin^2\phi}; \quad B = A/2 \quad \text{and} \quad C = k\sqrt{3} = 3\alpha\sqrt{3}c\cot\phi$$

3. Solutions to the case of Coulomb matrix

3.1. Expression of the stress field

Starting as in [12] and [13,14] for an infinite Coulomb matrix, it is assumed that the whole solid matrix is yielding; both previous criteria read also:

$$2(\sigma_{rr} - \sigma_{\theta\theta}) = a\sigma_{rr} - b \tag{7}$$

in which:

$$a = 2\frac{A+B}{A+\varepsilon}; \quad b = \frac{2C}{A+\varepsilon} \quad \Rightarrow \quad \frac{b}{a} = \frac{C}{A+B} \tag{8}$$

with $A = B = \sin\phi$ and $C = 2c\cos\phi$.

Taking into account (7) and integrating the equilibrium equation (1), the solution for the radial component σ_{rr} of the stress field reads:

$$\sigma_{rr} = \frac{b}{a}\left[1 - f^{\frac{a}{3}}\left(\frac{R}{r}\right)^a\right] = c\cot\phi\left\{1 - \left[f^{\frac{1}{3}}\frac{R}{r}\right]^{\frac{4\varepsilon\sin\phi}{1+\varepsilon\sin\phi}}\right\} \tag{9}$$

where f is the porosity.

It follows that the normal stress (pressure) at the external boundary $r = R$ is:

$$P_0 = \frac{b}{a}\left[1 - f^{\frac{a}{3}}\right] = c\cot\phi\left[1 - f^{\varepsilon\frac{4}{3}\frac{\sin\phi}{1+\varepsilon\sin\phi}}\right] \tag{10}$$

From (7), the other components $\sigma_{\theta\theta}$ and $\sigma_{\varphi\varphi}$ are then:

$$\sigma_{\theta\theta} = \sigma_{\varphi\varphi} = \frac{b}{a}\left[1 - \left(1 - \frac{a}{2}\right)\left(\frac{r_0}{r}\right)^a\right] = c\cot\phi\left\{1 - \frac{\varepsilon - \sin\phi}{\varepsilon + \sin\phi}\left[f^{\frac{1}{3}}\frac{R}{r}\right]^{\frac{4\tan\phi}{\cos\phi}(\varepsilon - \sin\phi)}\right\} \tag{11}$$

Since there is no boundary condition for the corresponding velocity field, this stress solution is exact if one can find an associated velocity field without any boundary condition. In this case the limit load calculated by using the kinematic theorem of LA will be the same as P_0.

3.2. Determination of the associated velocity fields

As classically, the spherical symmetry implies for the velocity vector field ($v_r(r)$, $v_\theta = v_\varphi = 0$), and then for the non-zero components of the strain rate tensor $\underline{\underline{d}}$: $d_{rr} = \frac{\partial v_r}{\partial r}$; $d_{\theta\theta} = d_{\varphi\varphi} = \frac{v_r}{r}$.

Let us recall that the Coulomb criterion (4) reads, in terms of principal stresses:

$$f(\underline{\underline{\sigma}}) = |\sigma_i - \sigma_j| - 2c\cos\phi + (\sigma_i + \sigma_j)\sin\phi \leqslant 0, \quad \text{with } i, j = 1, 2, 3$$

In the present problem σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{\varphi\varphi}$ are principal; from the isotropy of the material, d_{rr}, $d_{\theta\theta}$ and $d_{\varphi\varphi}$ are also principal, and the normality rule holds using these principal values. Under isotropic tension or compression of the hollow sphere, the apex of the cone cannot be reached at the limit load; then, only four of these six inequalities remain relevant since they can become equalities, hence possibly being concerned by the normality rule. Another consequence of the spherical symmetry ($\sigma_{\theta\theta} = \sigma_{\varphi\varphi}$) is that the point representing the stress tensor is constantly on one of the edges of this polyhedral cone. Hence let us denote the four remaining equalities as following:

$$f_{r\theta}(\underline{\underline{\sigma}}) = \varepsilon(\sigma_\theta - \sigma_r) + (\sigma_r + \sigma_\theta)\sin\phi - 2c\cos\phi \tag{12a}$$

$$f_{r\varphi}(\underline{\underline{\sigma}}) = \varepsilon(\sigma_\varphi - \sigma_r) + (\sigma_r + \sigma_\varphi)\sin\phi - 2c\cos\phi \tag{12b}$$

with the previous convention for ε (which takes the value $+1$ or -1). From the normality rule and the previous remarks, the components of the strain rate tensor are:

$$d_{rr} = \lambda_{r\theta}(r)\frac{\partial f_{r\theta}}{\partial \sigma_{rr}} + \lambda_{r\varphi}(r)\frac{\partial f_{r\varphi}}{\partial \sigma_{rr}} = \big[\lambda_{r\theta}(r) + \lambda_{r\varphi}(r)\big](\sin\phi - \varepsilon) \tag{13a}$$

$$d_{\theta\theta} = \lambda_{r\theta}(r)\frac{\partial f_{r\theta}}{\partial \sigma_{\theta\theta}} + \lambda_{r\varphi}(r)\frac{\partial f_{r\varphi}}{\partial \sigma_{\theta\theta}} = \lambda_{r\theta}(r)(\sin\phi + \varepsilon) \tag{13b}$$

$$d_{\varphi\varphi} = \lambda_{r\theta}(r)\frac{\partial f_{r\theta}}{\partial \sigma_{\varphi\varphi}} + \lambda_{r\varphi}(r)\frac{\partial f_{r\varphi}}{\partial \sigma_{\varphi\varphi}} = \lambda_{r\varphi}(r)(\sin\phi + \varepsilon) \tag{13c}$$

with $\lambda_{ij} > 0 \Rightarrow f_{ij} = 0$.

Using now the spherical symmetry ($d_{\varphi\varphi} = d_{\theta\theta}$), one obtains: $\lambda_{r\varphi}(r) = \lambda_{r\theta}(r) = \lambda(r)$. Hence, the components of the strain rate field can be expressed as:

$$d_{rr} = 2\lambda(r)(\sin\phi - \varepsilon); \qquad d_{\theta\theta} = d_{\varphi\varphi} = \lambda(r)(\sin\phi + \varepsilon) \tag{14}$$

From the relation velocity field–strain rate tensor, we obtain:

$$v_r = (\sin\phi + \varepsilon)r\lambda(r); \qquad 2\lambda(r)(\sin\phi - \varepsilon) = (\sin\phi + \varepsilon)\left[\frac{\mathrm{d}\lambda(r)}{\mathrm{d}r} + \lambda(r)\right] \tag{15}$$

Finally, with $m = \frac{3\varepsilon - \sin\phi}{\varepsilon + \sin\phi}$, $\lambda(r)$ verifies:

$$\frac{\mathrm{d}\lambda(r)}{\lambda(r)} = -m\frac{\mathrm{d}r}{r} \tag{16}$$

The general solution to (16) is given by $\lambda(r) = Kr^{-m}$ where K is an arbitrary constant, positive due to the normality law. From (15), the admissible velocity field finally becomes:

$$v_r = \varepsilon(1 + \varepsilon\sin\phi)Kr^{-2\frac{1-\varepsilon\sin\phi}{1+\varepsilon\sin\phi}}; \qquad v_\theta = v_\varphi = 0 \tag{17}$$

In addition to the computation of this velocity field, it can be easily verified that, by using the virtual power theorem, one obtains the same limit load as in (10). Indeed, as expected, the obtained velocity field complies with the PA (plastic admissibility) condition and the stress field and the velocity (and the corresponding strain rate) field are admissible and associated with each other.

4. Solutions to the case of a Drucker–Prager matrix

4.1. The stress field

From (9), (10) and (11) one obtains:

$$\sigma_{rr} = c\cot\phi\left\{1 - \left[f^{\frac{1}{3}}\frac{R}{r}\right]^{\frac{2\tan\phi}{\cos\phi}[\varepsilon\sqrt{3+\sin^2\phi}-2\sin\phi]}\right\} \tag{18}$$

Hence the normal stress P_0 at the external boundary reads in the case of a Drucker–Prager matrix:

$$P_0 = c\cot\phi\left\{1 - f^{\frac{2\tan\phi}{3\cos\phi}[\varepsilon\sqrt{3+\sin^2\phi}-2\sin\phi]}\right\} \tag{19}$$

which is the result that has been already established by [7] following a similar reasoning (see also [15] for only compressive loading).

Moreover:

$$\sigma_{\theta\theta} = \sigma_{\varphi\varphi} = c\cot\phi\left\{1 - \left[1 - \frac{\tan\phi(\varepsilon\sqrt{3+\sin^2\phi} - 2\sin\phi)}{\cos\phi}\right]\left[f^{\frac{1}{3}}\frac{R}{r}\right]^{\frac{2\tan\phi}{\cos\phi}[\varepsilon\sqrt{3+\sin^2\phi}-2\sin\phi]}\right\} \tag{20}$$

4.2. Components of the strain rate and the velocity field

Considering (5), the normality rule reads:

$$d_{rr} = \lambda(r)\frac{\partial f}{\partial\sigma_{rr}} = \lambda(r)\left[\alpha + \frac{2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi}}{6\sqrt{J_2}}\right] \tag{21}$$

and

$$d_{\theta\theta} = \lambda(r)\frac{\partial f}{\partial\sigma_{\theta\theta}} = \lambda(r)\left[\alpha + \frac{2\sigma_{\varphi\varphi} - \sigma_{rr} - \sigma_{\theta\theta}}{6\sqrt{J_2}}\right] \tag{22}$$

From the spherical symmetry $\sigma_{\theta\theta} = \sigma_{\varphi\varphi}$, $d_{\theta\theta} = d_{\varphi\varphi}$, $\sigma_{r\theta} = \sigma_{\theta\varphi} = \sigma_{\varphi r} = 0$, $J_2 = \frac{1}{3}(\sigma_{rr} - \sigma_{\theta\theta})^2$, we get:

$$d_{rr} = \lambda(r)\left[\alpha - \frac{\varepsilon}{\sqrt{3}}\right]; \qquad d_{\theta\theta} = \lambda(r)\left[\alpha + \frac{\varepsilon}{2\sqrt{3}}\right]; \qquad d_{\theta\theta} = d_{\varphi\varphi} \tag{23}$$

with the same previous convention on ε (see Section 2).

Considering the relation between the velocity field and the strain rate tensor and the criterion (23), we obtain similarly to what was done above:

$$\lambda(r) = Kr^{-\frac{3}{1+2\varepsilon\alpha\sqrt{3}}} \tag{24}$$

and finally the radial component of the velocity field:

$$v_r = \varepsilon\frac{K}{2\sqrt{3}}\left[1 + \frac{2\varepsilon\sin\phi}{\sqrt{3+\sin^2\phi}}\right]r^{-2\frac{1-\varepsilon\sin\phi/\sqrt{3+\sin^2\phi}}{1+2\varepsilon\sin\phi/\sqrt{3+\sin^2\phi}}} \tag{25}$$

This velocity field is the one already considered by Guo et al. [10] in their LA approach devoted to the derivation of a macroscopic criterion of porous media with a Drucker–Prager matrix. One can find also this velocity field in [15] (p. 162) in the compressive case, after changing $\sqrt{3/2}$ to $\sqrt{3/4}$ in the expression of the parameter α. As for the Coulomb case, by applying the kinematic theorem the same expression of the limit load as in (19) is obtained for the case of a Drucker–Prager matrix, as expected.

Table 1
Comparison of P_0/c exact values to those obtained by the finite element methods for porous Coulomb and Drucker–Prager materials.

		Porosity f	10%	10%	10%
		Friction angle ϕ	10°	20°	30°
Hollow sphere with Coulomb matrix	Compression	MEF static	−5.0416	−10.439	−33.140
		Closed-form	−5.1397	−10.805	−35.584
		MEF kinematic	−5.1848	−10.977	−36.770
	Tension	MEF static	2.0488	1.4744	1.0422
		Closed-form	2.0704	1.4911	1.1096
		MEF kinematic	2.1003	1.5016	1.1145
Hollow sphere with Drucker–Prager matrix	Compression	MEF static	−4.3116	−8.7407	−27.559
		Closed-form	−4.3961	−9.0408	−28.764
		MEF kinematic	−4.6174	−9.6553	−29.811
	Tension	MEF static	1.7806	1.2906	0.9664
		Closed-form	1.8045	1.3031	0.9704
		MEF kinematic	1.8293	1.3110	0.9728

Table 2
Comparison of our closed-form exact values and numerical MEF results to those kindly provided by [10] in the case of an isotropic loading, a Drucker–Prager matrix, a porosity $f = 5\%$ and a friction angle $\phi = 19.85757°$.

	Guo et al. (2008)	Jeong (2002)	Closed-form solution	Kinematic MEF	Static MEF
Tension	0.9787	0.9787	0.978747	0.9815	0.9747
Compression	−9.5577	−9.5577	−9.557671	−9.8631	−9.1725

5. Results and comparisons

5.1. Comparison between porous Coulomb and Drucker–Prager materials

In Table 1, we present, for the porous material with Coulomb and Drucker–Prager solid matrix, a comparison of the closed-form solutions to those obtained by the finite element method (static and kinematic approaches) partly presented in [11] and [16], for a porosity $f = 10\%$ and three values of the friction angle ($\phi = 10°$, $20°$, $30°$). It is observed that the theoretical results derived here are in remarkable agreement with the upper and lower bounds provided by our Finite Element codes.

5.2. Comparison with porous Drucker–Prager estimates of [10]

To simply extend the above comparison, the closed-form solutions (for Σ_m/σ_0), established in the present Note, are reported in Table 2 together with results given by Jeong [8] (see also the more recent paper by [10]) in the case of a Drucker–Prager matrix, with a porosity $f = 5\%$. It is worth noting that we have to consider a friction angle $\phi = 19.85757°$ as the value corresponding to the angle $\Psi_\alpha = 30°$ (from $\alpha = \tan(\Psi_\alpha)/\sqrt{27}$) used by [10] and a cohesion value $c = \sigma_0 \tan\phi/(\alpha\sqrt{27})$, with α defined in (5). As expected, the present closed-form solutions coincide with results of both authors and are between the bounds given by the previous FEM codes.

More interestingly, we aim now at evaluating the accuracy of the approximate yield surface established by Guo et al. [10] by using the kinematic approach of LA, with a two-component velocity field. The first component is the velocity field given in (25) added to an arbitrary linear velocity in cylindrical coordinates, which accounts for average strain rate boundary conditions. Nevertheless, as the plastic admissibility (PA) condition is not imposed on the total field, the result cannot be stated as a rigorous upper bound to the final criterion. Hence their Upper Bound Model, "UBM model", should only be considered as a parametric estimate of the exact macroscopic criterion on the basis of Gurson's approach. As the authors could not obtain a closed-form expression of the macroscopic criterion, they also gave an approximate estimate, found to be close to their "UBM model".

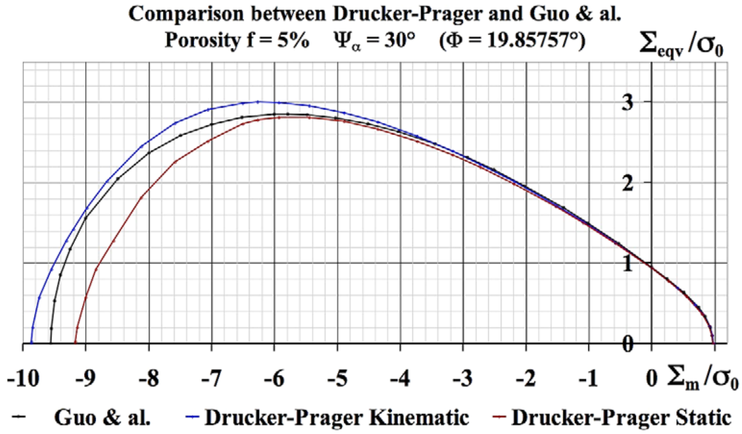

Fig. 1. Comparison between [10] results and our 3D porous Drucker–Prager MEF results; porosity: $f = 5\%$; internal friction angle: $\phi = 19.85757°$ ($\Psi_\alpha = 30°$).

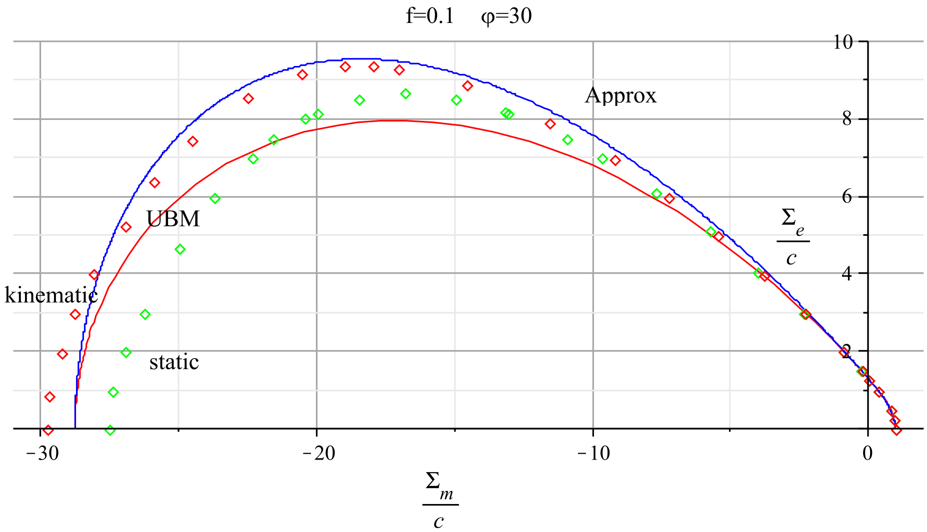

Fig. 2. Comparison between [10] results and our 3D porous Drucker–Prager MEF results; porosity: $f = 10\%$; internal friction angle: $\phi = 30°$.

To the best of our knowledge, their study is one of the most elaborate analytical attempts to obtain the criterion of a porous material with a Drucker–Prager matrix. This is an ideal case to use our above-mentioned 3D codes as validation tools. Fig. 1 gives the "UBM model" (Guo et al.) predictions together with those of the 3D codes under general loadings. As indicated above, their values on the Σ_m axis coincide with the exact closed-form values of Table 2, as expected from the fact that the velocity field of [10] becomes fully plastically admissible when its second part vanishes.

For a porosity $f = 5\%$ and $\phi = 19.85757°$ (case of Fig. 3 in [10]), the comparison (see Fig. 2) with the FEM lower bounds—never violated—suggests that the "UBM" approach is a good estimate of the exact macroscopic criterion, specially for tensile kinematical loadings, as it was the case, but more accurately, for the Gurson criterion in [17]. However, we note in Fig. 2 that for a porosity $f = 10\%$ and an internal friction angle $\phi = 30°$, which are common for geomaterials, the"UBM" approach violates the FEM lower bounds. This confirms that the "UBM" approach is not in fact a rigorous upper bound in all cases.

Acknowledgements

This study has benefited of the financial support of the program "MICROFISS (ANR-07-BLAN-0300)".

References

[1] A.L. Gurson, Continuum theory of ductile rupture by void nucleation and growth – part I: Yield criteria and flow rules for porous ductile media, Journal of Engineering Materials and Technology 99 (1977) 2–15.
[2] M. Gologanu, J.B. Leblond, G. Perrin, J. Devaux, Recent extensions of Gurson's model for porous ductile metals, in: P. Suquet (Ed.), Continuum Micromechanics, Springer-Verlag, 1997.
[3] M. Garajeu, P. Suquet, Effective properties of porous ideally plastic or viscoplastic materials containing rigid particles, Journal of the Mechanics and Physics of Solids 45 (1997) 873–902.
[4] V. Monchiet, E. Charkaluk, D. Kondo, An improvement of Gurson-type models of porous materials by using Eshelby-like trial velocity fields, Comptes Rendus Mécanique 335 (2007) 32–41.
[5] A.A. Benzerga, J. Besson, Plastic potentials for anisotropic porous solids, European Journal of Mechanics/A Solids 20 (2001) 397–434.
[6] V. Monchiet, O. Cazacu, E. Charkaluk, D. Kondo, Macroscopic yield criteria for plastic anisotropic materials containing spheroidal voids, International Journal of Plasticity 24 (2008) 1158–1189.
[7] H.Y. Jeong, J. Pan, A macroscopic constitutive law for porous solids with pressure-sensitive matrices and its implications to plastic flow localization, International Journal of Solids and Structures 32 (1995) 3669–3691.
[8] H.Y. Jeong, A new yield function and a hydrostatic stress-controlled model for porous solids with pressure-sensitive matrices, International Journal of Solids and Structures 39 (2002) 1385–1403.
[9] J.-F. Barthélémy, L. Dormieux, Détermination du critère de rupture macroscopique d'un milieu poreux par homogénéisation non linéaire, Comptes Rendus Mécanique, Acad. Sci. Paris 331 (2003) 271–276.
[10] T.F. Guo, J. Faleskog, C.F. Shih, Continuum modeling of a porous solid with pressure sensitive dilatant matrix, Journal of the Mechanics and Physics of Solids 56 (2008) 2188–2212.
[11] M. Trillat, J. Pastor, Ph. Thoré, Limit analysis and conic programming: "porous Drucker–Prager" material and Gurson's model, Comptes Rendus Mécanique, Acad. Sci. Paris 334 (2006) 599–604.
[12] P. Chadwick, The quasi-static expansion of a spherical cavity in metals and ideal soils, The Quarterly Journal of Mechanics and Applied Mathematics 12 (1) (1959) 52–71.
[13] J. Salençon, Expansion quasi-statique d'une cavité à symétrie sphérique ou cylindrique dans un milieu élastoplastique, Annales des Ponts et Chaussées III (1966) 175–187.
[14] J. Salençon, Contraction quasi-statique d'une cavité à symétrie sphérique ou cylindrique dans un milieu élastoplastique, Annales des Ponts et Chaussées IV (1969) 231–236.
[15] J.-F. Barthélémy, Approche micromécanique de la rupture et de la fissuration dans les géomatériaux, Thése de Doctorat, École Nationale des Ponts et Chaussées, Paris, 2005.
[16] J. Pastor, Ph. Thoré, C. Vandenbussche, Gurson model and conic programming for pressure-sensitive materials, Key Engineering Materials 348–349 (2007) 693–696.
[17] M. Trillat, J. Pastor, Limit analysis and Gurson's model, European Journal of Mechanics A/Solids 24 (2005) 800–819.