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Overall ultimate yield strength of a quasi-periodic masonry

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Abstract

The purpose of this Note is the determination of the in-plane homogenized strength domain of a "quasi-periodic" masonry under the assumption of infinitely resistant blocks connected by cohesionless Mohr–Coulomb interfaces. This masonry is obtained by introducing a random perturbation on the horizontal width of the blocks of a periodic running bond masonry. It is found that in some non-trivial cases the strength domain coincides exactly with that of the initial periodic masonry. ***To cite this article: K. Sab, C. R. Mecanique 337 (2009).***

Résumé

Domaine de résistance macroscopique d'une maçonnerie quasi-périodique. L'objet de cette Note est la détermination du domaine de résistance homogénéisé dans le plan d'une maçonnerie quasi-périodique constituée de blocs infiniment résistants en contact à travers des interfaces de Mohr–Coulomb sans cohésion. Cette maçonnerie est obtenue en perturbant aléatoirement la dimension horizontale des blocs à partir d'une maçonnerie périodique. On trouve que, dans certains cas non triviaux, le domaine de résistance coincide exactement avec celui de la maçonnerie périodique initiale. ***Pour citer cet article : K. Sab, C. R. Mecanique 337 (2009).***

Keywords: Solids and structures; Homogenization; Strength domain; Random microstructure; Bounds; Masonry; Mohr–Coulomb

Mots-clés : Solides et structures ; Homogénéisation ; Domaine de résistance ; Microstructure aléatoire ; Bornes ; Maçonnerie ; Mohr–Coulomb

1. Main results

Most of existing homogenized models for brickwork-like materials concern periodic microstructures. Reference is made to [1,2] and [3] for the determination and the experimental validation of the overall strength domain of the so-called periodic running bond (RB) masonry. On the other hand, few works are dedicated to historical non-periodic masonry. Cluni and Gusella [4] and Falsone and Lombardo [5] studied irregular masonry structures using stochastic methods for linear continuum media. Cecchi and Sab [6,7], evaluated the effect of a random variation of the horizontal block dimension of the RB-masonry on its homogenized linear behavior. Starting from the RB-masonry, a random

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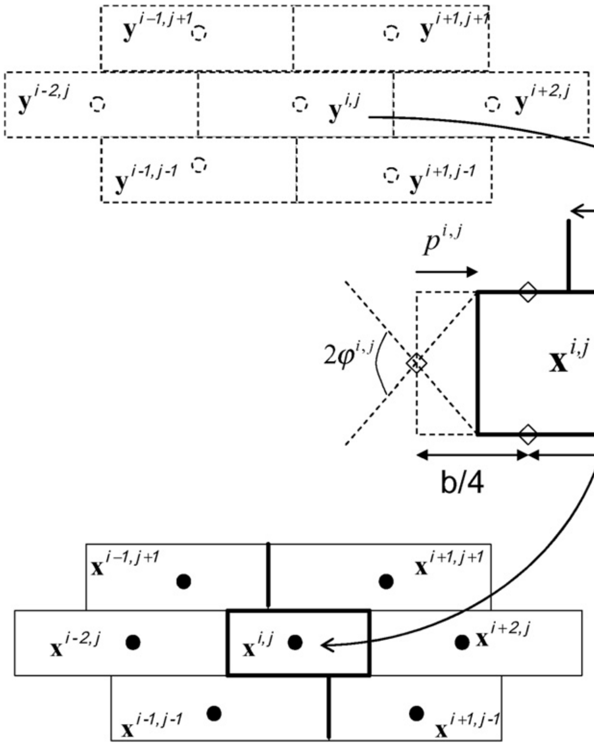

Fig. 1. Quasi-periodic masonry obtained by a random perturbation of the running bond periodic masonry.

perturbation on the horizontal positions of the vertical interfaces between the blocks which form the *infinite* masonry is introduced. In Fig. 1, $\mathbf{y}^{i,j}$ is the position of the centre of the generic block $B^{i,j}$ (height a, width b) of the initial RB-masonry; $p^{i,j}$ is the random perturbation, with average $\overline{p^{i,j}} = 0$, on the horizontal component of the vertical interface between the blocks $B^{i,j}$ and $B^{i-2,j}$; $\mathbf{x}^{i,j} = \mathbf{y}^{i,j} + \frac{p^{i,j}+p^{i+2,j}}{2}\mathbf{e}_1$ is the new position of the centre of the generic perturbed block $B_p^{i,j}$ (height a, average width b) of the so-called quasi-periodic masonry. Here $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are the base vectors and $(i, j) \in \mathcal{I}$ where j can take arbitrary integer values while i is such as $i + j$ is even. For a given parameter $0 \leqslant \rho < 1$, assuming that $p^{i,j}$ is the realization of a uniform random variable on the interval $[-\rho\frac{b}{4}, \rho\frac{b}{4}]$ guarantees that each perturbed block has exactly 6 neighboring blocks. In this case, the effective in-plane elastic moduli are not sensitive ($<10\%$) to the random perturbation [6].

The purpose of this Note is the determination of the in-plane homogenized strength domain, $\mathbf{G}^{\text{hom}}$, of the quasi-periodic masonry under the following assumptions: (a) the infinitely resistant blocks are connected by cohesionless Mohr–Coulomb interfaces (ϕ is the friction angle of the interfaces between the blocks), (b) the perturbations $p^{i,j}$, $(i, j) \in \mathcal{I}$ are statistical homogeneous and ergodic (s.h.e.) and (c) each perturbed block has exactly 6 neighboring blocks.

Recall that the homogenized strength domain of an infinite s.h.e. random medium is the *deterministic* limit strength domain obtained by imposing kinematically uniform boundary conditions to a finite volume of any realization of the infinite medium, as the volume size increases to infinity.[1] This limit can actually be determined by solving an auxiliary problem on the infinite medium involving s.h. random stress and strain rate fields [8,9]. Considering this auxiliary problem and using kinematic and static definitions of the homogenized strength domain, the following results are obtained in this Note:

$$p^{\infty} < \min\left(\frac{a}{2\tan\phi}, \frac{b}{4}\right) \Longrightarrow \mathbf{G}^{\text{hom}} = \mathbf{G}^{RB} \tag{1}$$

[1] Although the strength of a finite structure is random, however, its dispersion around the strength of the homogenized structure goes to zero as the number of heterogeneities in the structure goes to infinity [8,9].

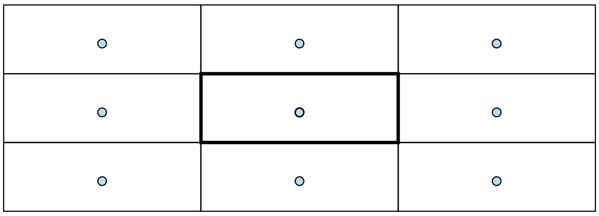

Fig. 2. Stack bond masonry.

and

$$\frac{a}{2\tan\phi} < p^\infty < \frac{b}{4} \Longrightarrow \overline{\mathbf{G}^\infty \cup \mathbf{G}^{SB}}^{co} \subset \mathbf{G}^{\text{hom}} \subset \mathbf{G}^{RB} \tag{2}$$

Here, $p^\infty \equiv \text{Sup}_{(i,j)\in\mathcal{I}} |p^{i,j}|$ is the supremum value of the $p^{i,j}$; $\mathbf{G}^{RB}$ is the homogenized strength domain of the initial RB-masonry given by de Buhan and de Felice [1]; $\mathbf{G}^\infty$ is the strength domain of the initial RB-masonry obtained after substituting $\phi^\infty = \arctan \frac{a}{2p^\infty}$ for the vertical interfaces friction angle; $\mathbf{G}^{SB}$ is the strength domain of the periodic stack bond (SB) masonry given in [10] (Fig. 2). Finally, $\overline{\mathbf{G}^\infty \cup \mathbf{G}^{SB}}^{co}$ is the convex hull of $\mathbf{G}^\infty$ and $\mathbf{G}^{SB}$. In conclusion, $\mathbf{G}^{\text{hom}}$ coincides exactly with $\mathbf{G}^{RB}$ in the case (1) of small enough variation of the horizontal width of the blocks. This result is consistent with the following *règle de l'art* for historical quasi-periodic masonries: each block must have six neighboring blocks. However, in the case of "large" p^∞ ($> \min(\frac{a}{2\tan\phi}, \frac{b}{4})$), a quantitative estimation of $\mathbf{G}^{\text{hom}}$ would necessitate the use of numerical simulations on finite representative volume elements of increasing sizes as in [11]. Of course, these simulations should be consistent with the analytical bounds (2).

2. The discrete model

The virtual motion of the infinite perturbed masonry is described by the set $(\mathbf{v}^{i,j}, \boldsymbol{\omega}^{i,j})$, $(i,j) \in \mathcal{I}$, of the in-plane virtual rigid body motions of the blocks:

$$\mathbf{v}(\mathbf{y}) = \mathbf{v}^{i,j} + \boldsymbol{\omega}^{i,j} \times (\mathbf{y} - \mathbf{x}^{i,j}), \quad \forall \mathbf{y} \in B_p^{i,j} \tag{3}$$

where $\mathbf{v}(\mathbf{y})$ is the velocity at point $\mathbf{y}$, $\mathbf{v}^{i,j} = v_1^{i,j}\mathbf{e}_1 + v_2^{i,j}\mathbf{e}_2$ is the velocity at $\mathbf{x}^{i,j}$ and $\boldsymbol{\omega}^{i,j} = \omega^{i,j}\mathbf{e}_3$ is the rotation velocity vector of the $B_p^{i,j}$ block. The analysis is restricted to the case $p^\infty < \frac{b}{4}$ which ensures that each perturbed block has exactly 6 neighboring blocks. The intersections of the six interfaces of block $B_p^{i,j}$ with the $(1,2)$-plane are denoted $S_{k_1,k_2}^{i,j}$ with

$$(k_1,k_2) \in \mathcal{K} \equiv \{(2,0), (-2,0), (+1,+1), (+1,-1), (-1,+1), (-1,-1)\}$$

Their corresponding normal vectors and centres are $\mathbf{n}_{k_1,k_2}^{i,j}$ and $\mathbf{x}_{k_1,k_2}^{i,j}$, respectively. From (3), the relative velocity between block $B_p^{i,j}$ and $B_p^{i+k_1,j+k_2}$ at $\mathbf{y} \in S_{k_1,k_2}^{i,j}$ is given by:

$$\mathbf{d}_{k_1,k_2}^{i,j}(\mathbf{y}) = \bar{\mathbf{d}}_{k_1,k_2}^{i,j} + \boldsymbol{\delta}_{k_1,k_2}^{i,j} \times (\mathbf{y} - \mathbf{x}_{k_1,k_2}^{i,j}) \tag{4}$$

where $\bar{\mathbf{d}}_{k_1,k_2}^{i,j}$ is the relative velocity at the centre $\mathbf{x}_{k_1,k_2}^{i,j}$ of $S_{k_1,k_2}^{i,j}$ and $\boldsymbol{\delta}_{k_1,k_2}^{i,j}$ is the relative rotation $\omega^{i+k_1,j+k_2} - \boldsymbol{\omega}^{i,j}$. The tractions at $\mathbf{y} \in S_{k_1,k_2}^{i,j}$ are noted $\mathbf{t}_{k_1,k_2}^{i,j}(\mathbf{y}) = t_{k_1,k_2(1)}^{i,j}\mathbf{e}_1 + t_{k_1,k_2(2)}^{i,j}\mathbf{e}_2$ and they obey to Mohr–Coulomb yield criterion. Their force resultant[2] $\mathbf{f}_{k_1,k_2}^{i,j} = f_{k_1,k_2(1)}^{i,j}\mathbf{e}_1 + f_{k_1,k_2(2)}^{i,j}\mathbf{e}_2$ and their couple resultant $\mathbf{c}_{k_1,k_2}^{i,j} = c_{k_1,k_2}^{i,j}\mathbf{e}_3$ at $\mathbf{x}_{k_1,k_2}^{i,j}$ are given by:

$$\mathbf{f}_{k_1,k_2}^{i,j} = \int_{S_{k_1,k_2}^{i,j}} \mathbf{t}_{k_1,k_2}^{i,j}\,\mathrm{d}S, \qquad \mathbf{c}_{k_1,k_2}^{i,j} = \int_{S_{k_1,k_2}^{i,j}} (\mathbf{y} - \mathbf{x}_{k_1,k_2}^{i,j}) \times \mathbf{t}_{k_1,k_2}^{i,j}\,\mathrm{d}S \tag{5}$$

[2] Per unit thickness in the out-of-plane direction.

The cohesionless Mohr–Coulomb yield criterion at a point of a generic S is:

$$F(\mathbf{t};\mathbf{n};\phi) \equiv t_n \tan\phi + \left|\mathbf{t}^{\perp}\right| \leqslant 0 \tag{6}$$

Here, $\mathbf{t}$ is the traction vector at $\mathbf{y} \in S$, $\mathbf{n}$ is the normal to the interface, $t_n = \mathbf{t}.\mathbf{n}$ is the normal component of $\mathbf{t}$ and $\mathbf{t}^{\perp}$ is the orthogonal component of $\mathbf{t}$. The Mohr–Coulomb support π-function is [12]:

$$\pi(\mathbf{d};\mathbf{n};\phi) \equiv \sup_{\mathbf{t},\,F(\mathbf{t};\mathbf{n};\phi)\leqslant 0} \mathbf{t}.\mathbf{d} = \begin{cases} 0 & \text{if } d_n \geqslant |\mathbf{d}^{\perp}| \tan\phi \\ +\infty & \text{otherwise} \end{cases} \tag{7}$$

where $\mathbf{d} = d_n\mathbf{n} + \mathbf{d}^{\perp}$, the relative velocity given by (4), is decomposed into normal and orthogonal components. It is clear that $(\bar{\mathbf{d}}, \boldsymbol{\delta})$ is dual to $(\mathbf{f}, \mathbf{c})$ given by (5) in the sense of the virtual power:

$$\mathbf{f}.\bar{\mathbf{d}} + \mathbf{c}.\delta = \int_S \mathbf{t}(\mathbf{y}).\mathbf{d}(\mathbf{y})\,\mathrm{d}S$$

The resultant strength domain of S, G_S, is the convex set of $(\mathbf{f}, \mathbf{c})$ such that there is $\mathbf{t}$ verifying (5) and (6). The G_S support π-function is obtained by integrating (7) over S:

$$\pi_S(\bar{\mathbf{d}}, \boldsymbol{\delta}) \equiv \mathrm{Sup}_{(\mathbf{f},\mathbf{c})\in G_S}\, \mathbf{f}.\bar{\mathbf{d}} + \mathbf{c}.\delta = \begin{cases} 0 & \text{if (8)} \\ +\infty & \text{otherwise} \end{cases}$$

with

$$\bar{d}_n \geqslant \left|\bar{\mathbf{d}}^{\perp}\right| \tan\phi + \frac{|S|}{2}|\boldsymbol{\delta}| \tag{8}$$

and $|S|$ is the length of S. From the kinematic definition of G_S,

$$G_S = \left\{(\mathbf{f}, \mathbf{c}),\ \mathbf{f}.\bar{\mathbf{d}} + \mathbf{c}.\delta \leqslant \pi_S(\bar{\mathbf{d}}, \delta)\ \forall(\bar{\mathbf{d}}, \delta)\right\}$$

it is found that G_S is the set of $(\mathbf{f}, \mathbf{c})$ verifying:

$$f_n \tan\phi + \left|\mathbf{f}^{\perp}\right| \leqslant 0 \tag{9}$$

$$|\mathbf{c}| + \frac{|S|}{2} f_n \leqslant 0 \tag{10}$$

The condition (10) on the couple $\mathbf{c}$ means that the resultant force $\mathbf{f}$ is applied at a point which belongs to S.

3. The probabilistic description

The measure preserving flow concept that ensures statistical homogeneity and ergodicity of a random medium is used to describe the random masonry. With this description, the homogenized linear [13,14] and nonlinear [8,9] properties of a random medium may be determined by solving an auxiliary stochastic problem on the infinite medium. See also [15] for a review. The infinite random masonry is completely characterized by the probability space $(\mathbb{X}, \mathcal{A}, \mathbb{P})$ where:

- $\mathbb{X}$ is the sample space of all possible realizations such that each perturbed block has exactly 6 neighboring blocks:

$$\mathbf{X} = \left\{p^{m,n} \in \mathbb{R},\ (m,n) \in \mathcal{I},\ \left|p^{m,n}\right| < \frac{b}{4}\right\}$$

- $\mathcal{A}$ is the $\mathcal{I}$-product Lebesgue σ-algebra on $\mathbb{R}$; and
- $\mathbb{P}$ is a probability measure on $\mathcal{A}$ which attributes to each subset A of $\mathbb{X}$ in $\mathcal{A}$, $A \in \mathcal{A}$, its probability $0 \leqslant \mathbb{P}(A) \leqslant 1$ with $\mathbb{P}(\mathbb{X}) = 1$.

Consider a realization $\mathbf{X} \in \mathbb{X}$ of the random medium and let $\tau_{i,j}\mathbf{X}$, $(i, j) \in \mathcal{I}$, denote the realization corresponding to the translated medium by vector $\mathbf{y}^{i,j}$. Hence:

$$(\tau_{i,j}\mathbf{X})^{m,n} = p^{m+i,n+j}$$

Statistical invariance of the random masonry means that this application preserves the probability measure for all $(i,j)\in\mathcal{I}$:

$$\forall A\in\mathcal{A},\quad \mathbb{P}(A)=\mathbb{P}(\tau_{i,j}A),\quad \tau_{i,j}A=\{\mathbf{X},\tau_{-(i,j)}\mathbf{X}\in A\} \tag{11}$$

We say that $\tau_{i,j}$, $(i,j)\in\mathcal{I}$, is a measure preserving flow. A real random variable, $\mathbf{X}\to R(\mathbf{X})$, is a measurable map from $\mathbb{X}$ into $\mathbb{R}$. The set of real random variables $L^q(\mathbb{X})$, $1\leqslant q<+\infty$, is such that

$$\mathbb{E}\big(|R|^q\big)=\int_{\mathbf{X}\in\mathbb{X}}\big|R(\mathbf{X})\big|^q\,\mathrm{d}\mathbb{P}<+\infty$$

where $\mathbb{E}$ is the ensemble-average operator.

Given the real random variable R, the corresponding statistically homogeneous (s.h.) real random process is the set of the random variables $R^{i,j}$, $(i,j)\in\mathcal{I}$, defined by:

$$\forall\mathbf{X}\in\mathbb{X},\quad R^{i,j}(\mathbf{X})=R(\tau_{-(i,j)}\mathbf{X}) \tag{12}$$

The random variable $R^{i,j}$ is actually associated to the block $B_p^{i,j}$ and it is equal to the random variable $R=R^{0,0}$ associated to the block $B_p^{0,0}$ after a translation of the medium by the vector $-\mathbf{y}^{i,j}$. On the base of this definition and measure preserving property (11), the following invariance property holds true:

$$R\in L^1(\mathbb{X})\implies R^{i,j}\in L^1(\mathbb{X}),\quad \mathbb{E}\big(R^{i,j}\big)=\mathbb{E}(R)$$

It is assumed that the random variable $\mathbf{X}\to p^{0,0}$ is in $L^1(\mathbb{X})$ with $\mathbb{E}(p^{0,0})=0$. Moreover, the measure preserving flow $\tau_{i,j}$, $(i,j)\in\mathcal{I}$, is assumed to be ergodic. Roughly speaking, this means that all the statistical information about the infinite random medium is contained in each of his realizations. As a consequence, the ensemble-average of any R, $\mathbb{E}(R)$, can be identified to the limit of the arithmetic-average of the corresponding $R^{i,j}$ in a volume containing many blocks, as the volume size goes to infinity.

4. The strength domain

The homogenization method for periodic lattices [16,17] are here adapted to quasi-periodic random lattices. Let $\mathbf{D}=(D_{\alpha\beta})$ and $\boldsymbol{\Sigma}=(\Sigma_{\alpha\beta})$, $\alpha,\beta=1,2$, be a macroscopic in-plane virtual strain rate tensor and a macroscopic stress tensor, respectively. Following [6], the set of $\mathbf{D}$-kinematically compatible virtual motions is:

$$\mathcal{KC}(\mathbf{D})=\big\{\big(\mathbf{v}^{i,j}=\mathbf{D}.\mathbf{x}^{i,j}+\mathbf{w}^{i,j},\boldsymbol{\omega}^{i,j}\big),\ (\mathbf{w},\boldsymbol{\omega})\in L^2(\mathbb{X})^2\times L^2(\mathbb{X})\big\}$$

where $\mathbf{X}\to(\mathbf{w}(\mathbf{X}),\boldsymbol{\omega}(\mathbf{X}))$ is a random in-plane rigid body motion of the $B_p^{0,0}$ block. The stationary property (12) is used to generate, $(\mathbf{w}^{i,j},\boldsymbol{\omega}^{i,j})$, the random rigid body motion of the $B_p^{i,j}$ block. We say that the virtual strain rate $\mathbf{D}$ is plastically compatible if there is at least one virtual motion in $\mathcal{KC}(\mathbf{D})$ such that all the corresponding $(\bar{\mathbf{d}}^{i,j}_{k_1,k_2},\boldsymbol{\delta}^{i,j}_{k_1,k_2})$ comply to (8) almost surely. The kinematic definition of the homogenized strength domain, $\mathbf{G}^{\mathrm{hom}}$, is given by:

$$\mathbf{G}^{\mathrm{hom}}=\{\boldsymbol{\Sigma},\ \boldsymbol{\Sigma}:\ \mathbf{D}\leqslant\mathbf{0}\text{ for all plastically compatible }\mathbf{D}\}$$

The motion consisting of the deterministic (independent of $\mathbf{X}$) uniform rotations $\boldsymbol{\omega}^{i,j}(\mathbf{X})=\boldsymbol{\omega}(\mathbf{X})=\omega^*\mathbf{e}_3$ and the random translations $\mathbf{w}^{i,j}=\mathbf{D}.(\mathbf{y}^{i,j}-\mathbf{x}^{i,j})$ is in $\mathcal{KC}(\mathbf{D})$. For this motion, $\bar{\mathbf{d}}^{i,j}_{k_1,k_2}=\mathbf{F}.\mathbf{y}^{k_1,k_2}$ and $\boldsymbol{\delta}^{i,j}_{k_1,k_2}=0$ are deterministic and periodic (independent of (i,j)) where $\mathbf{F}$ is given by:

$$\mathbf{F}=\begin{pmatrix}D_{11} & D_{12}-\omega^*\\ D_{12}+\omega^* & D_{22}\end{pmatrix}$$

Optimizing over $\mathbf{D}$ and ω^* under the constraint (8) gives the upper bound $\mathbf{G}^{\mathrm{hom}}\subset\mathbf{G}^{RB}$.

According to [6], $\mathcal{KC}(\mathbf{D})$ is dual to $\mathcal{SC}(\Sigma)$, the set of Σ-statically compatible interaction forces and couples $(\mathbf{f}^{i,j}_{k_1,k_2},\mathbf{c}^{i,j}_{k_1,k_2})$ of the infinite perturbed masonry verifying:

- The stationarity condition:

$$\left(\mathbf{f}^{i,j}_{k_1,k_2}, \mathbf{c}^{i,j}_{k_1,k_2}\right)(\mathbf{X}) = (\mathbf{f}_{k_1,k_2}, \mathbf{c}_{k_1,k_2})(\tau_{-(i,j)}\mathbf{X}), \quad \forall \mathbf{X} \in \mathbb{X}$$

where $(\mathbf{f}_{k_1,k_2}, \mathbf{c}_{k_1,k_2}) \in L^2(\mathbb{X})^2 \times L^2(\mathbb{X})$ is the random interaction force and couple of the $S^{0,0}_{k_1,k_2}$ interface.
- The consistency condition:

$$(\mathbf{f}_{k_1,k_2}, \mathbf{c}_{k_1,k_2})(\tau_{k_1,k_2}\mathbf{X}) = -(\mathbf{f}_{-(k_1,k_2)}, \mathbf{c}_{-(k_1,k_2)})(\mathbf{X})$$

- The balance equations:

$$\sum_{(k_1,k_2)\in\mathcal{K}} \mathbf{f}_{k_1,k_2} = 0, \qquad \sum_{(k_1,k_2)\in\mathcal{K}} \mathbf{m}_{k_1,k_2} = 0$$

where

$$\mathbf{m}_{k_1,k_2} = \mathbf{c}_{k_1,k_2} + \left(\mathbf{x}^{0,0}_{k_1,k_2} - \mathbf{x}^{0,0}\right) \times \mathbf{f}_{k_1,k_2}$$

is the resultant moment that the $B_p^{k_1,k_2}$ block applies to the $B_p^{0,0}$ block at its centre.
- And

$$\Sigma = \frac{1}{2V} \sum_{(k_1,k_2)\in\mathcal{K}} \mathbb{E}(\mathbf{f}_{k_1,k_2}) \otimes^s \mathbf{y}^{k_1,k_2}$$

where $\otimes^s$ is the symmetric part of the dyadic product of two vectors. It should be noted that $\mathbf{y}^{k_1,k_2} = \mathbf{y}^{i+k_1,j+k_2} - \mathbf{y}^{i,j}$ is the branch vector of the $S^{i,j}_{k_1,k_2}$ interface in the initial periodic running bond microstructure.

The static definition of $\mathbf{G}^{\mathrm{hom}}$ is as follows: it is the set of $\boldsymbol{\Sigma}$ such that there are forces and couples in $\mathcal{SC}(\boldsymbol{\Sigma})$ which comply to (9) and (10) for all (i, j) and (k_1, k_2) with probability one. For the deterministic periodic case, $\mathbf{G}^{RB}$ is obtained by restricting the static analysis to periodic (deterministic) forces in $\mathcal{SC}(\boldsymbol{\Sigma})$, $\mathbf{f}^{i,j}_{k_1,k_2} = \mathbf{f}^{per}_{k_1,k_2} = -\mathbf{f}^{per}_{-(k_1,k_2)}$, applied at $\mathbf{y}^{i,j}_{k_1,k_2}$, the centres of the interfaces in the non-perturbed configuration, and which comply to (9). For the random case, the idea is to use these periodic resultant forces and their corresponding couples $\mathbf{c}^{i,j}_{k_1,k_2} = (\mathbf{y}^{i,j}_{k_1,k_2} - \mathbf{x}^{i,j}_{k_1,k_2}) \times \mathbf{f}^{per}_{k_1,k_2}$ at $\mathbf{x}^{i,j}_{k_1,k_2}$, the centres of the interfaces in the perturbed configuration. The additional condition (10) on these couples is equivalent to $\mathbf{y}^{i,j}_{k_1,k_2} \in S^{i,j}_{k_1,k_2}$ for the horizontal interfaces, and to (9) where $\varphi^{i,j} = \arctan \frac{a}{2|p^{i,j}|}$ is substituted for ϕ for the vertical interfaces (Fig. 1). Hence, optimizing over the periodic forces in $\mathcal{SC}(\Sigma)$ gives the following lower bounds for $\mathbf{G}^{\mathrm{hom}}$: $\mathbf{G}^{\infty}$ if $\frac{a}{2\tan\phi} < p^{\infty} < \frac{b}{4}$ and $\mathbf{G}^{RB}$ if $p^{\infty} < \min(\frac{a}{2\tan\phi}, \frac{b}{4})$. Finally, a Reuss lower bound is obtained by considering forces and couples in $\mathcal{SC}(\Sigma)$ which derive from uniform tractions $\mathbf{t}^{i,j}_{k_1,k_2} = \boldsymbol{\Sigma}.\mathbf{n}^{i,j}_{k_1,k_2}$ generated by the uniform stress field $\boldsymbol{\Sigma}$. From (6), we find that $\mathbf{G}^{SB}$, the strength domain of the periodic stack bond (SB) masonry given in [10], is a lower bound for $\mathbf{G}^{\mathrm{hom}}$. All the obtained bounds summarize in (1) and (2).

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