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ScienceDirect
C. R. Mecanique 337 (2009) 585–590
COMPTES RENDUS
MECANIQUE

New asymptotic effects for the spectrum of problems on concentrated masses near the boundary ☆

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Received 26 June 2009; accepted 16 July 2009

Available online 12 August 2009

Presented by Evariste Sanchez-Palencia

Abstract

The Dirichlet and Neumann spectral problems for the Laplace operator in a bounded domain $\Omega \subset \mathbb{R}^2$ are considered. We assume that Ω has a piecewise smooth boundary $\partial\Omega$ and the density function is equal to $1 + \varepsilon^{-m}\chi_\varepsilon$ in Ω, where $\varepsilon > 0$ is a small parameter, $m \in \mathbb{R}$ and χ_ε is the characteristic function of the union $\omega_\varepsilon^0 \cup \cdots \cup \omega_\varepsilon^{J-1}$ of small sets (the concentrated masses) distributed periodically near a straight segment $\Gamma \subset \partial\Omega$. We describe asymptotics for the eigenelements of both problems as $\varepsilon \to 0$. ***To cite this article: S.A. Nazarov, E. Pérez, C. R. Mecanique 337 (2009).***

Résumé

Nouveaux effets asymptotiques pour le spectre des problèmes avec des masses concentrées près de la frontière. On considére des problèmes spectraux pour l'opérateur de Laplace dans un domaine bornée $\Omega \subset \mathbb{R}^2$ avec des conditions de Dirichlet et Neumann respectivement sur la frontière. On suppose que la frontière $\partial\Omega$ est réguliere par morceaux tandis que la fonction densité prend la valeur $1 + \varepsilon^{-m}\chi_\varepsilon$ dans Ω, oú $\varepsilon > 0$ est un petit paramètre, $m \in \mathbb{R}$, et χ_ε est la fonction caractéristique de l'union des petites ensembles $\omega_\varepsilon^0 \cup \cdots \cup \omega_\varepsilon^{J-1}$ (les masses concentrés), qui sont répartis périodiquement prés d'un segment droite Γ de la frontière, $\Gamma \subset \partial\Omega$. Nous décrivons le comportement asymptotique des valeurs propres de ces deux problèmes lorsque $\varepsilon \to 0$. ***Pour citer cet article : S.A. Nazarov, E. Pérez, C. R. Mecanique 337 (2009).***

Keywords: Boundary homogenization; Spectral analysis; Concentrated masses; Asymptotic expansions

Mots-clés : Homogénéisation des frontières ; Analyse spectrale ; Masses concentrées ; Développements asymptotiques

☆ The first author acknowledges the support by RFFI, grant 09-01-00759. The second author acknowledges the support by the Spanish MEC, MTM2005-07720. The work has also been partially supported by the MEC, SAB2005-0175.

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1631-0721/$ – see front matter
doi:10.1016/j.crme.2009.07.002

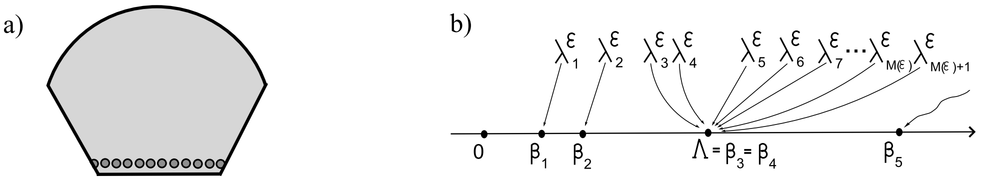

Fig. 1. (a) The domain Ω. (b) A sketch of the eigenvalues convergence in the case 6° when $N = \varkappa = 2$.

1. Introduction and setting of the problems

Let $\Omega \subset \mathbb{R}^2$ be a domain bounded by a piecewise smooth boundary $\partial\Omega$. Let $\partial\Omega$ contains the segment $\Gamma = \{x = (x_1, x_2) / x_1 \in (0, a),\ x_2 = 0\}$, with $a > 0$. To define *concentrated masses* near the boundary segment Γ (see Fig. 1a), we introduce a large integer $J \in \mathbb{N}$ and the small parameter $\varepsilon = a/J$. Furthermore, let Π be the half-band $\Pi = \{\xi \in \mathbb{R}^2 / \xi_1 \in (0, 1),\ \xi_2 > 0\}$ and let ω be a domain with a smooth boundary $\partial\omega$ such that the compact closure $\overline{\omega} = \omega \cup \partial\omega \subset \Pi$. We introduce the sets $\omega_\varepsilon^0 = \{x \in \mathbb{R}^2 / \xi := \varepsilon^{-1}x \in \omega\}$, $\omega_\varepsilon^j = \{x\colon\ (x_1 - j\varepsilon, x_2) \in \omega_\varepsilon^0\}$, $j \in \mathbb{N} = \{1, 2, \ldots\}$, and the characteristic function χ_ε of the union of sets $\omega_\varepsilon^0 \cup \cdots \cup \omega_\varepsilon^{J-1} \subset \overline{\Omega}$. We consider the equation

$$-\Delta_x u^\varepsilon(x) = \lambda^\varepsilon\big(1 + \varepsilon^{-m}\chi_\varepsilon(x)\big)u^\varepsilon(x), \quad \text{for } x \in \Omega \tag{1}$$

with either the Dirichlet condition $u^\varepsilon = 0$, or the Neumann condition $\partial_n u^\varepsilon = 0$, on $\partial\Omega$ (∂_n is the derivative along the outer normal). Here, ε and m are two parameters, $\varepsilon > 0$, $m \in \mathbb{R}$, and λ^ε denotes the spectral parameter. Throughout the paper, we indicate these spectral problems as (D) and (N), respectively. Both problems have discrete spectrum in the Sobolev spaces $H_0^1(\Omega)$ and $H^1(\Omega)$, respectively. Problem (D) has the eigenvalue sequence

$$0 < \lambda_1^\varepsilon \leqslant \lambda_2^\varepsilon \leqslant \cdots \leqslant \lambda_k^\varepsilon \leqslant \cdots \to +\infty \quad \text{as } k \to \infty \tag{2}$$

where λ_1^ε is simple. For problem (N), the sequence (2) must be augmented with the eigenvalue $\lambda_0^\varepsilon = 0$. In this paper, we describe asymptotics for the eigenelements $\{\lambda^\varepsilon, u^\varepsilon\}$ of both problems as $\varepsilon \to 0$.

Using the min–max principle and different estimates for the integrals on $\omega_\varepsilon^0 \cup \cdots \cup \omega_\varepsilon^{J-1}$, we get bounds (3) and (4) for the eigenvalues of (D) and (N). That is, for fixed k, and sufficiently small ε, we have

$$\text{for } (D)\text{:}\quad C_1 \leqslant \lambda_k^\varepsilon \leqslant C_{1k} \text{ when } m \leqslant 2, \quad \text{and} \quad C_2\varepsilon^{m-2} \leqslant \lambda_k^\varepsilon \leqslant C_{2i}\varepsilon^{m-2} \text{ when } m > 2 \tag{3}$$

$$\text{for } (N)\text{:}\quad C_3 \leqslant \lambda_k^\varepsilon \leqslant C_{3k} \text{ when } m \leqslant 1, \quad \text{and} \quad C_4\varepsilon^{m-1} \leqslant \lambda_k^\varepsilon \leqslant C_{4k}\varepsilon^{m-1} \text{ when } m > 1 \tag{4}$$

where the constants C_j and C_{jk} ($j = 1, 2, 3, 4$) do not depend on ε.

In Section 2, using (4) and the normalization of eigenfunctions u_k^ε in $H^1(\Omega)$, we can establish the convergence of the eigenelements of (N) towards those of the standard spectral Neumann problem (6) when $m < 1$; towards those of the Steklov problem (8) when $m > 1$ with the spectral parameter in the boundary conditions on Γ (once we have re-scaled the eigenvalues); and, towards those of problem (8) when $m = 1$ and the spectral parameter in both, Ω and Γ. We refer to the papers [1–5] for the technique to conclude estimates (4) and the convergence results (cf. [6] for further references).

In Section 3 for (D) with $m < 2$, a similar method works and shows that $\{\lambda_k^\varepsilon, u_k^\varepsilon\}$ converge to the eigenelements $\{\beta_k, u_k\}$ of the standard Dirichlet problem (9). However, in the case $m > 2$ the convergence analysis becomes much more complicated because all re-scaled eigenvalues $\varepsilon^{2-m}\lambda_i^\varepsilon$ approach the first eigenvalue Λ of the *local problem* (10) (see [7,8] for related problems). We now apply the technique in [9] to conclude the convergence which is based on a factorizations $u^\varepsilon(x) = U(\varepsilon^{-1}x)W^\varepsilon(x)$ where U is the eigenfunction of (10) corresponding to Λ while W^ε is an eigenfunction associated with the spectral parameter $\varrho^\varepsilon = \lambda^\varepsilon - \Lambda\varepsilon^{-1}$ of a new problem in Ω. In Section 4, we outline composite asymptotic expansions (cf. [10–12]) in the most complicated case $m = 2$ in (D), which is the only one where the justification of the analysis remains to be performed (cf. Remark 1).

Asymptotics for vibrating systems with many concentrated masses near the boundary have been widely studied. The case of concentrated masses, placed at Γ "sparsely" (that is, the period being much larger than ε), with either alternating boundary conditions, or the Neumann condition, has been studied in [1,6,13,14] for different exponent m and dimension n of the space $\mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$; both, low and high frequencies have been considered. We also mention [15] with $m < 2$ and $n = 2$, and [16] with $m < 2$, aperiodic distribution of masses and various boundary conditions. For comb-shaped junctions with heavy rectangular "teeth", we refer to [17].

The concentration of mass along a curve $\Upsilon \subset \Omega$ and $\Upsilon = \partial\Omega$ has been treated in [2] and [3,4], respectively. Many concentrated masses sparsely placed is considered in [7] which contains the nearest problems and results in the literature to those in this Note for $m > 2$. See Remark 1 to compare the results in this paper with previous ones and Remark 2 for asymptotics of high frequencies.

2. Asymptotics for the Neumann problem

Let us formulate the convergence results for positive eigenvalues of (N).

1° For $m < 1$, we have $\lambda_k^\varepsilon \to \beta_k$, $k \in \mathbb{N}$, where

$$0 < \beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_k \leqslant \cdots \to +\infty \tag{5}$$

is the sequence of positive eigenvalues in the standard Neumann problem

$$-\Delta_x v = \beta v \quad \text{in } \Omega; \qquad \partial_n v = 0 \quad \text{on } \partial\Omega \tag{6}$$

2° For $m \geqslant 1$, we have $\varepsilon^{1-m}\lambda_k^\varepsilon \to \mu_k$, $k \in \mathbb{N}$, where

$$0 < \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_k \leqslant \cdots \to +\infty \tag{7}$$

is the sequence of positive eigenvalues of the Steklov spectral problem

$$-\Delta_x v = \delta_{m,1}\mu v \quad \text{in } \Omega; \qquad \partial_n v = 0 \quad \text{on } \partial\Omega \setminus \Gamma; \qquad \partial_n v = \mu|\omega| v \quad \text{on } \Gamma \tag{8}$$

where $\delta_{m,1}$ is defined as $\delta_{1,1} = 1$ and $\delta_{m,1} = 0$ for $m \neq 1$, and, $|\omega|$ denotes the area of ω.

3. Asymptotics for the Dirichlet problem

Let us state the convergence results for the eigenvalues of (D) for different ranges of m.

3° For $m < 2$, we have $\lambda_k^\varepsilon \to \beta_k$, $k \in \mathbb{N}$, where (5) now denotes the eigenvalue sequence in the standard Dirichlet problem

$$-\Delta_x v = \beta v \quad \text{in } \Omega; \qquad v = 0 \quad \text{on } \partial\Omega \tag{9}$$

For further asymptotic descriptions, we need the first eigenvalue Λ and the corresponding eigenfunction U of the auxiliary problem in the half-band Π with the periodicity conditions on its lateral sides

$$\begin{aligned} &-\Delta_\xi U(\xi) = \Lambda \chi_\omega U(\xi), \quad \xi \in \Pi; \qquad U(\xi_1, 0) = 0, \quad \xi_1 \in (0,1) \\ &U(0,\xi_2) = U(1,\xi_2), \quad \frac{\partial U}{\partial \xi_1}(0,\xi_2) = \frac{\partial U}{\partial \xi_1}(1,\xi_2), \quad \xi_2 > 0 \end{aligned} \tag{10}$$

Problem (10) has a discrete spectrum in the space of periodic functions in ξ_1 with the norm $\|\nabla_\xi U; L^2(\Pi)\| + \|U; L^2(\omega)\|$. The first eigenvalue is simple, and the associated eigenfunction U can be chosen as positive in Π and $\|U; L^2(\omega)\| = 1$. Moreover, the Fourier method (cf. [10, Ch. 1] and [18, Ch. 2]) leads us to assert that

$$\left|U(\xi) - B\right| \leqslant c \exp\left(-(2\pi)^{-1}\xi_2\right) \tag{11}$$

where the constant $B \in \mathbb{R}$ is defined as an integral characteristic of the domain $\omega \subset \Pi$, namely

$$B = \Lambda \int_\omega \xi_2 U(\xi)\, \mathrm{d}\xi > 0 \tag{12}$$

For this first eigenvalue Λ of (10) we also introduce the Steklov spectral problem in Ω:

$$-\Delta_x v = \delta_{m,2}\Lambda v \quad \text{in } \Omega; \qquad v = 0 \quad \text{on } \partial\Omega \setminus \Gamma; \qquad \partial_n v = \mu B^{-2} v \quad \text{on } \Gamma \tag{13}$$

where $\delta_{m,2}$ is defined as $\delta_{2,2} = 1$ and $\delta_{m,2} = 0$ for $m \neq 2$.

4° For $m > 2$, we have $\varepsilon^{1-m}(\lambda_k^\varepsilon - \varepsilon^{m-2}\Lambda) \to \mu_k$, $k \in \mathbb{N}$, where (7) now stands for the eigenvalue sequence of (13) where $\delta_{m,2} \equiv 0$. Consequently, all the eigenvalues μ_k are positive.

For $m=2$, we distinguish between two options. Firstly, when Λ is not an eigenvalue of problem (9), i.e., for a certain N, $N \geqslant 0$, we have

$$0 < \beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_N < \Lambda < \beta_{N+1} \tag{14}$$

5° For $m=2$ with (14), we have $\lambda_k^\varepsilon \to \beta_k$ for $k=1,\ldots,N$, and $\varepsilon^{-1}(\lambda_{N+j}^\varepsilon - \Lambda) \to \mu_j$ for $j \in \mathbb{N}$, where (7) now denotes the sequence of eigenvalues of problem (13) at $m=2$, and, depending on the value of Λ, a finite number of eigenvalues $\mu_j = \mu_j(\Lambda)$ can be negative.

Secondly, let Λ be an eigenvalue of problem (9) with multiplicity $\varkappa$, i.e.,

$$0 < \beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_N = \cdots = \beta_{N+\varkappa-1} = \Lambda < \beta_{N+\varkappa} \tag{15}$$

while the corresponding eigenfunctions $v_N, \ldots, v_{N+\varkappa-1}$ satisfy the conditions

$$\int_\Omega v_p(x) v_q(x)\,\mathrm{d}x = \delta_{p,q}, \quad p,q = N,\ldots,N+\varkappa-1 \tag{16}$$

The derivatives $\partial v_N/\partial x_2, \ldots, \partial v_{N+\varkappa-1}/\partial x_2$ are linear independent in $L^2(\Gamma)$ and, therefore, the $\varkappa \times \varkappa$-matrix M with entries

$$M_{pq} = B^2 \int_\Gamma \frac{\partial v_p}{\partial x_2}(x_1,0)\frac{\partial v_q}{\partial x_2}(x_1,0)\,\mathrm{d}x_1, \quad p,q = N,\ldots,N+\varkappa-1 \tag{17}$$

is a symmetric and positive definite and has the eigenvalues

$$0 < \gamma_0 \leqslant \cdots \leqslant \gamma_{\varkappa-1} \tag{18}$$

6° For $m=2$ and (15), we have $\lambda_k^\varepsilon \to \beta_k$ for $k=1,\ldots,N-1$, $\varepsilon^{-\frac{1}{2}}(\lambda_{N+p}^\varepsilon - \Lambda) \to -\gamma_p^{\frac{1}{2}}$ for $p=0,\ldots,\varkappa-1$, and $\varepsilon^{-1}(\lambda_{N+\varkappa-1+q}^\varepsilon - \Lambda) \to \mu_q$ for $q \in \mathbb{N}$ (cf. Fig. 1b, when $N=\varkappa=2$).

4. Asymptotic ansätze

We provide the formal asymptotic technique for the most complicated case 6°. If $k=1,\ldots,N-1$, the asymptotic ansätze read

$$\lambda_k^\varepsilon = \beta_k + \cdots, \qquad u_k^\varepsilon(x) = v_k(x) + \varepsilon w_k^1(\xi,x_1) + \varepsilon v_k^1(x) + \cdots \tag{19}$$

where $\{\beta_k, v_k\}$ is an eigenelement of problem (9) and w_k^1 is a solution of the problem

$$\begin{aligned} &-\Delta_\xi w_k^1(\xi,x_1) = \beta_k \chi_\omega(\xi)\left(w_k^1(\xi,x_1) + v_k^1(x_1,0) + \xi_2 \frac{\partial v_k}{\partial x_2}(x_1,0)\right), \quad \xi \in \Pi \\ &w_k^1(\xi_1,0,x_1) = -v_k^1(x_1,0), \quad \xi_1 \in (0,1) \end{aligned} \tag{20}$$

with the periodicity conditions on the lateral sides of the half-band Π. Since $\beta_k < \Lambda$, problem (20) has a unique solution w_k^1 with the finite Dirichlet integral. A proper choice of the regular type term $v_k^1(x_1,0)$ gives the exponential decay of $w_k^1(\xi,x_1)$ as $\xi_2 \to \infty$ and w_k^1 gets the natural property of a boundary layer.

For $k = j + N + \varkappa - 1$, $j \in \mathbb{N}$, the asymptotic ansätze become the following:

$$\lambda_k^\varepsilon = \Lambda + \varepsilon\mu_j + \cdots, \qquad u_k^\varepsilon(x) = w_j^0(\xi,x_1) + v_j(x) + \varepsilon w_j^1(\xi,x_1) + \varepsilon v_j^1(x) + \cdots \tag{21}$$

The main term of the boundary layer

$$w_j^0(\xi,x_1) = v_j(x_1,0)B^{-1}\big(U(\xi) - B\big) \tag{22}$$

has the exponential decay due to (11). The second term is periodic in ξ_1 and satisfies the problem

$$\begin{aligned} -\Delta_\xi w_j^1(\xi,x_1) = {} & \Lambda \chi_\omega(\xi)\left(w_j^1(\xi,x_1) + v_j^1(x_1,0) + \xi_2 \frac{\partial v_j}{\partial x_2}(x_1,0)\right) + \mu_j \chi_\omega(\xi)\big(w_j^0(\xi,x_1) + v_j(x_1,0)\big) \\ & + 2B\frac{\partial U}{\partial \xi_1}(\xi)\frac{\partial v_j}{\partial x_1}(x_1,0), \quad \xi \in \Pi; \qquad w_j^1(\xi_1,0,x_1) = -v_j^1(x_1,0), \quad \xi_1 \in (0,1) \end{aligned} \tag{23}$$

Since Λ is a simple eigenvalue, problem (23) requires only one compatibility condition which, by (12), turns into the boundary condition on Γ in (13). The other equations in (13) are self-understood.

Ansätze for $\lambda_N^\varepsilon, \dots, \lambda_{N+\varkappa-1}^\varepsilon$ are intricate and we employ asymptotic procedures from [11] and [4]:

$$\lambda_{N+p}^\varepsilon = \Lambda \pm \varepsilon^{\frac{1}{2}} \gamma_p^{\frac{1}{2}} + \cdots, \qquad u_{N+p}^\varepsilon(x) = v_p^0(x) \pm \varepsilon^{\frac{1}{2}} w_p^1(\xi, x_1) \pm \varepsilon^{\frac{1}{2}} v_p^1(x) + \varepsilon w_p^2(\xi, x_1) + \varepsilon v_p^2(x) + \cdots \tag{24}$$

where the further terms w_p^2 and v_p^2 depend on the sign $\pm$. Also, we set

$$v_p^0(x) = a_N^p v_N(x) + \cdots + a_{N+\varkappa-1}^p v_{N+\varkappa-1}(x) \tag{25}$$

where $v_N, \dots, v_{N+\varkappa-1}$ are eigenfunctions of problem (9) corresponding to the eigenvalue Λ (cf. (15)) and the columns $\bar{a}^p = (a_N^p, \dots, a_{N+\varkappa-1}^p)$ have to be found. Then, the first term w_p^1 of the boundary layer type takes the form (22) with $v_j \equiv v_p^0$ on the right hand side. The regular type term v_p^1 satisfies

$$-\Delta_x v_p^1(x) = \Lambda v_p^1(x) + \gamma_p^{\frac{1}{2}} v_p^0(x), \quad x \in \Omega, \qquad v_p^1(x) = 0, \quad x \in \partial\Omega \setminus \Gamma \tag{26}$$

while the boundary condition on Γ is obtained from the compatibility condition in the problem

$$\begin{aligned} -\Delta_\xi w_p^2(\xi, x_1) = {} & \Lambda \chi_\omega(\xi) \left(w_p^2(\xi, x_1) + v_p^2(x_1, 0) + \xi_2 \frac{\partial v_p^0}{\partial x_2}(x_1, 0) \right) \\ & + \gamma_p^{\frac{1}{2}} \chi_\omega(\xi) \big(w_p^1(\xi, x_1) + v_p^1(x_1, 0) \big), \quad \xi \in \Pi; \qquad w_p^2(\xi_1, 0, x_1) = -v_p^2(x_1, 0), \quad \xi_1 \in (0, 1) \end{aligned}$$

with the usual periodicity conditions. From (12), we derive

$$v_p^1(x_1, 0) = -B^2 \gamma_p^{-\frac{1}{2}} \frac{\partial v_p^0}{\partial x_2}(x_1, 0), \quad x_1 \in (0, a) \quad (\text{on } \Gamma) \tag{27}$$

Now, according to (16), $\varkappa$ compatibility conditions in problem (26), (27) provide the algebraic system $M\bar{a}^p = \gamma_p \bar{a}^p$ which delivers the eigenvalues (18) while the eigenvectors $\bar{a}^0, \dots, \bar{a}^{\varkappa-1} \in \mathbb{R}^\varkappa$ can be chosen to compose a unitary matrix. Thus, the asymptotic ansätze (24) are completed.

Remark 1. The analysis of (N) looks very similar to that of mass concentration along a curve $\Upsilon \subset \Omega$. In the case $m < 2$, the similarity (no influence of the masses) can be observed in (D) as well. On the other hand, eigenelements of the Dirichlet problem (D) with $m > 2$ in the low frequency range behave rather similar to eigenelements of the problem consisting in the Laplace equation with alternating Dirichlet and Steklov spectral boundary conditions on Γ: the local problem (10) and the Steklov problem (13) arise to describe the asymptotics for the eigenelements. Finally, (D) with $m = 2$ implies the most complicated situation where the Dirichlet problem (9) in Ω, the local problem (10) in Π and the Steklov problem (13) in Ω, together with the matrix M (17), are involved.

Remark 2. The technique in Section 4 allows all the eigenvalues of the Dirichlet problem (9), the rest of eigenvalues of (10) and the set of positive $\gamma_p^{1/2}$ in (18), to be obtained as limiting points of eigenvalues λ_k^ε of (D) (suitable re-scaled shifted eigenvalues, respectively), but now we cannot preserve the index k any more, since it should depend on the parameter ε. If $m > 2$, the same can be said for the rest of the eigenvalues of the local problem (10) and re-scaled sequences $\varepsilon^{2-m}\lambda_{k(\varepsilon)}^\varepsilon$ for $k(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

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