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Existence of a directional Stokes drift in asymmetrical three-dimensional travelling gravity waves

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Abstract

We consider periodic travelling gravity waves at the surface of an infinitely deep perfect fluid. The pattern is non-symmetric with respect to the propagation direction of the waves and we consider a general non-resonant situation. Defining a couple of amplitudes $\varepsilon_1, \varepsilon_2$ along the basis of wave vectors which satisfy the dispersion relation, following Iooss and Plotnikov (2009), travelling waves exist with an asymptotic expansion in powers of $\varepsilon_1, \varepsilon_2$, for nearly all pair of angles made by the basic wave vectors with the critical propagation direction, and for values of the couple $(\varepsilon_1^2, \varepsilon_2^2)$ in a subset of the plane, with asymptotic full measure at the origin. We prove the remarkable property that on the free surface, observed in the moving frame, the propagation direction of the waves differs from the asymptotic direction taken by fluid particles, by a small angle which is computed. ***To cite this article: G. Iooss, P. Plotnikov, C. R. Mecanique 337 (2009).***

Résumé

Existence d'une dérive directionnelle de Stokes dans les vagues de gravité tri-dimensionnelles asymétriques. On considère les vagues périodiques à la surface d'une couche de fluide parfait, de profondeur infinie, soumise à la seule gravité. Le réseau bidimensionnel des périodes est pris non symétrique par rapport à la direction de propagation et on suppose ne pas être dans un cas résonant. On définit le couple d'amplitudes $\varepsilon_1, \varepsilon_2$ le long des deux vecteurs d'onde de base qui vérifient l'équation de dispersion. D'après Iooss et Plotnikov (2009), les vagues asymétriques existent et possèdent un développement asymptotique en puissances de $(\varepsilon_1, \varepsilon_2)$, pour presque tous les angles faits par les vecteurs d'onde de base avec la direction critique de propagation, et pour des valeurs de $(\varepsilon_1^2, \varepsilon_2^2)$ dans un sous-ensemble du quadrant ayant une mesure asymptotiquement pleine à l'origine. Nous montrons la propriété remarquable dans le référentiel relatif, qu'à la surface libre, la direction de propagation des ondes diffère de la direction asymptotique prise par les trajectoires des particules de fluide, d'un petit angle qu'on calcule. ***Pour citer cet article : G. Iooss, P. Plotnikov, C. R. Mecanique 337 (2009).***

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1. Introduction

We consider small-amplitude three-dimensional doubly periodic travelling *gravity waves* on the free surface of a perfect fluid. These waves are steady in a frame moving with the velocity of the wave ($-\mathbf{c}$ in the absolute reference frame). The fluid layer is supposed to be infinitely deep, and the flow is irrotational only subjected to gravity. The bifurcation parameters are the horizontal phase velocity, and the direction of propagation of the travelling waves.

In 1847 Stokes [1] gave a nonlinear theory of *two-dimensional* travelling gravity waves, computing formally the flow up to the cubic order of the amplitude of the waves. He showed that, in the moving frame, the average velocity of particles in the horizontal direction is slightly different (at second order in the amplitude of the waves) from the propagation velocity of the waves. In this note we show an analogous result for the common asymptotic direction taken by the trajectories of fluid particles in the moving frame (which we define as a "directional Stokes drift"), in the cases when the periodic pattern is asymmetric with respect to the propagation direction of the travelling waves. Formal calculations of solutions in power series of the amplitudes ($\varepsilon_1, \varepsilon_2$) have a long history in literature (since the 1950s for 3D travelling waves), see [2] for a recent review on water waves. However quite surprisingly, it seems that the phenomenon of directional Stokes drift is not mentioned yet, even for gravity-capillary travelling waves.

One serious theoretical difficulty here, is that *we assume the absence of surface tension*. Indeed the surface tension plays a major role in all existing proofs for three-dimensional travelling gravity-capillary waves (see [3–7]), and when the surface tension is very small, which is the case in many usual situations, this implies a reduced domain of validity of results strongly dependent on the existence of a non-small surface tension. In our previous work [8] on three-dimensional travelling gravity waves, there is indeed no surface tension, but we restricted the study to the existence of diamond waves: the periodic lattice is a diamond lattice, and there are equal amplitudes at the leading order associated with the two basic wave vectors K_1, K_2 symmetric with respect to the propagation direction of the waves. We proved the existence of *bifurcating diamond gravity waves*, symmetric with respect to the propagation direction of the waves. In such a case the horizontal projection of fluid particles trajectories is asymptotically in the direction of propagation velocity, which is the bissectrix of the angle made by K_1 and K_2.

In the present work, *we consider asymmetrical travelling gravity waves*, which implies that the basic wave vectors K_1 and K_2 have not the same length, and given the two amplitudes ($\varepsilon_1, \varepsilon_2$) on the basic modes, *the direction of propagation* $\mathbf{u}$ *of the waves is part of the unknown*. As in the symmetrical case, there is a small divisor problem which leads to the use of the Nash–Moser implicit function theorem. The existence of asymmetrical gravity waves is proved in [9] (see Theorem 2.2 below) for nearly all choices of angles θ_1, θ_2 made by the non-symmetric wave vectors K_1, K_2 with the direction $\mathbf{u}_0$ of the critical velocity, and for values of the square of the amplitudes ($\varepsilon_1^2, \varepsilon_2^2$) on these wave vectors, in a set for which 0 is a Lebesgue point (0 corresponds to the critical value $\mathbf{c}_c$ of the velocity vector $\mathbf{c}$). Then Lemma 3.1 establishes the existence of a small angle between the propagation direction of the waves and the asymptotic direction of fluid particle trajectories in the frame moving with the velocity of the waves.

2. Formulation of the problem

Let us denote by φ the potential defined by

$$\varphi = \phi - \mathbf{c} \cdot X$$

where ϕ is the usual velocity potential in the moving frame (the rest state is then $\varphi = 0$), $X = (x_1, x_2)$ is the 2-dim horizontal coordinate, x_3 is the vertical coordinate, and the fluid region is

$$\Omega = \big\{ (X, x_3) \colon -\infty < x_3 < \eta(X) \big\}$$

which is bounded by the free surface Σ defined by

$$\Sigma = \big\{ (X, x_3) \colon x_3 = \eta(X) \big\}$$

We choose $|\mathbf{c}|$ for the velocity scale, and L for a length scale (to be chosen below), and we still denote by (X, x_3) the new coordinates, and by φ, η the unknown functions. Now defining the parameter $\mu = \frac{gL}{c^2}$, where g denotes the acceleration of gravity, and $\mathbf{u}$ the unit vector in the direction of $\mathbf{c}$, the system reads

$$\Delta\varphi = 0 \quad \text{in } \Omega$$

$$\nabla_X \eta \cdot (\mathbf{u} + \nabla_X \varphi) - \frac{\partial\varphi}{\partial x_3} = 0 \quad \text{on } \Sigma$$

$$\mathbf{u} \cdot \nabla_X \varphi + \frac{(\nabla\varphi)^2}{2} + \mu\eta = 0 \quad \text{on } \Sigma$$

$$\nabla\varphi \to 0 \quad \text{as } x_3 \to -\infty$$

We specialise our study to *spatially periodic three-dimensional travelling waves*, i.e. the solutions η and φ are *periodic in X*. There are two independent wave vectors $K_1, K_2 \in \mathbb{R}^2$ generating a lattice

$$\Gamma = \{\mathbf{k} = n_1 K_1 + n_2 K_2 \colon n_j \in \mathbb{Z}, \ j = 1, 2\}$$

and we consider in the further analysis, a lattice Γ generated by the wave vectors

$$K_1 = (1, \tau_1), \qquad K_2 = \lambda(1, -\tau_2)$$

This means that we have chosen the length scale L for having 1 in the first component of K_1, and τ_1, τ_2 measure the angles between the wave vectors K_1, K_2 and the x_1 axis. We can reduce the above system for (φ, η) to a system of two scalar equations in choosing the new unknown function (following V.E. Zakharov)

$$\psi(X) = \varphi\big(X, \eta(X)\big)$$

and we define the Dirichlet–Neumann operator $\mathcal{G}_\eta$ by

$$\mathcal{G}_\eta \psi = \sqrt{1 + (\nabla\eta)^2} \frac{d\varphi}{dn}\bigg|_{x_3 = \eta(X)} = \frac{\partial\varphi}{\partial x_3}\bigg|_{x_3 = \eta(X)} - \nabla\eta \cdot \nabla_X \varphi \tag{1}$$

where n is normal to Σ, exterior to Ω, and φ is the solution of the η-dependent Dirichlet problem

$$\Delta\varphi = 0, \quad x_3 < \eta(X)$$

$$\varphi = \psi, \quad x_3 = \eta(X)$$

$$\nabla\varphi \to 0 \quad \text{as } x_3 \to -\infty$$

This definition of $\mathcal{G}_\eta$ follows [10] and insures the selfadjointness and positivity of this linear operator in $L^2(\mathbb{R}^2/\Gamma)$ (see [8]). Now the system to solve reads

$$\mathcal{F}(U, \mu, \mathbf{u}) = 0, \quad \mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) \tag{2}$$

where $U = (\psi, \eta)$, and

$$\mathcal{F}_1(U, \mathbf{u}) =: \mathcal{G}_\eta(\psi) - \mathbf{u} \cdot \nabla\eta \tag{3}$$

$$\mathcal{F}_2(U, \mu, \mathbf{u}) =: \mathbf{u} \cdot \nabla\psi + \mu\eta + \frac{(\nabla\psi)^2}{2} - \frac{1}{2(1 + (\nabla\eta)^2)} \big\{\nabla\eta \cdot (\nabla\psi + \mathbf{u})\big\}^2 \tag{4}$$

The mapping $\mathcal{F}(\cdot, \mu, \mathbf{u})$ is equivariant under translations of the plane:

$$\mathcal{T}_\mathbf{v} \mathcal{F}(U, \mu, \mathbf{u}) = \mathcal{F}(\mathcal{T}_\mathbf{v} U, \mu, \mathbf{u})$$

where

$$\mathcal{T}_\mathbf{v} U(X) = U(X + \mathbf{v})$$

and it is also equivariant with respect of the symmetry $\mathcal{S}_0$ defined by the following representation of the symmetry with respect to 0

$$(\mathcal{S}_0 U)(X) = \big(-\psi(-X), \eta(-X)\big)$$

The first result proved in [9], is the following:

Lemma 2.1. *For any fixed integer $N \geqslant 3$, and $\boldsymbol{\tau} = (\tau_1, \tau_2) \in \mathbb{R}^{+2} \backslash \{0\}$, we define*

$$\mu_c = \left(1 + \tau_1^2\right)^{-1/2}, \qquad \lambda = \left(\frac{1 + \tau_2^2}{1 + \tau_1^2}\right)^{1/2}$$

and assume (*non-resonance condition*) *that the* dispersion equation

$$\mu_c |\mathbf{k}| - (\mathbf{k} \cdot \mathbf{u}_0)^2 = 0$$

where $\mathbf{u}_0 = (1, 0)$, has only the solutions $\mathbf{k} = \pm K_1, \pm K_2, \mathbf{0}$, where $K_1 = (1, \tau_1)$, $K_2 = \lambda(1, -\tau_2)$, and $\mathbf{k} \in \Gamma$. Then, for any $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^{+2}$, there is a family of approximate asymmetric travelling waves $\mathcal{T}_{\mathbf{v}} U_N(\boldsymbol{\varepsilon})$, $\mathbf{v} \in \mathbb{R}^2$, where $U_N(\boldsymbol{\varepsilon})$ is given by the expansion

$$U_N(\boldsymbol{\varepsilon}) = \sum_{1 \leqslant p+q \leqslant N} \varepsilon_1^p \varepsilon_2^q U_{pq}$$

$$\mu_N(\boldsymbol{\varepsilon}) = \mu_c + \sum_{1 \leqslant p+q \leqslant N/2} \varepsilon_1^{2p} \varepsilon_2^{2q} \mu_{pq} \in \mathbb{R}^+$$

$$\mathbf{u}_N(\boldsymbol{\varepsilon}) = \mathbf{u}_0 + \sum_{1 \leqslant p+q \leqslant N/2} \varepsilon_1^{2p} \varepsilon_2^{2q} \mathbf{u}_{pq} \in \mathbb{S}_1$$

with $X \mapsto U_{pq}(X)$ periodic and $\mathcal{C}^\infty$, such that $U_{pq} = (\psi_{pq}, \eta_{pq})$ satisfies $\psi_{pq}(-X) = -\psi_{pq}(X)$, $\eta_{pq}(-X) = \eta_{pq}(X)$ and

$$U_{10} = \left(\sin(K_1 \cdot X), \frac{-1}{\mu_c} \cos(K_1 \cdot X)\right)$$

$$U_{01} = \left(\sin(K_2 \cdot X), \frac{-\lambda}{\mu_c} \cos(K_2 \cdot X)\right)$$

$$\mu_{10} = \alpha_1, \qquad \mu_{01} = \alpha_2, \qquad \mathbf{u}_{10} = (0, \beta_1), \qquad \mathbf{u}_{01} = (0, \beta_2)$$

the coefficients α_j and β_j being explicit analytic functions of (τ_1, τ_2) (explicitly given in [9]). Moreover

$$\mathcal{F}\big(U_N(\boldsymbol{\varepsilon}), \mu_N(\boldsymbol{\varepsilon}), \mathbf{u}_N(\boldsymbol{\varepsilon})\big) = O\big(|\boldsymbol{\varepsilon}|^{N+1}\big)$$

We notice that the non-resonance condition is in fact a condition on (τ_1, τ_2). It is indeed satisfied for a full measure set in $\mathbb{R}^{+2}$. Notice that if we make $\tau_1 = \tau_2$, and $\varepsilon_1 = \varepsilon_2$ we recover the result on diamond waves [8]. This lemma may be proved by using a standard formal Lyapunov–Schmidt process, taking care of the four-dimensional kernel of the linearised operator and in taking benefit of the symmetries $\mathcal{T}_{\mathbf{v}}$ and $\mathcal{S}_0$.

The following theorem is the main result of the paper [9]:

Theorem 2.2. *Let us choose arbitrary integers $l \geqslant 34$, $N \geqslant 8$ and a real number $0 < \delta < 1$. Then, there is a full measure subset $\mathcal{T}$ of $\mathbb{R}^{+2}$ such that for any $\boldsymbol{\tau} = (\tau_1, \tau_2) \in \mathcal{T}$ there exists a subset $\mathcal{E}(\boldsymbol{\tau})$ of the quadrant $\{(\varepsilon_1^2, \varepsilon_2^2) \in \mathbb{R}^{+2}\}$ for which 0 is a Lebesgue point, i.e.*

$$\frac{2}{\epsilon^2} \operatorname{meas}\big(\mathcal{E}(\boldsymbol{\tau}) \cap \{\varepsilon_1^2 + \varepsilon_2^2 < \epsilon\}\big) \to 1 \quad as \; \epsilon \to 0$$

Moreover, for $\delta < \varepsilon_1/\varepsilon_2 < 1/\delta$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2) \in \mathcal{E}(\boldsymbol{\tau})$, the system (2)–(4) has a unique solution of the form

$$U = U_N(\boldsymbol{\varepsilon}) + |\boldsymbol{\varepsilon}|^{N/2} \breve{U}(\boldsymbol{\varepsilon}) \in \mathbb{H}_{(S)}^l\big(\mathbb{R}^2/\Gamma\big)$$

$$\mu = \mu_N(\boldsymbol{\varepsilon}) + |\boldsymbol{\varepsilon}|^{N/2} \breve{\mu}(\boldsymbol{\varepsilon}) \in \mathbb{R}^+$$

$$\mathbf{u} = \mathbf{u}_N(\boldsymbol{\varepsilon}) + |\boldsymbol{\varepsilon}|^{N/2} \breve{\mathbf{u}}(\boldsymbol{\varepsilon}) \in \mathbb{S}_1$$

where $(\breve{U}(\boldsymbol{\varepsilon}), \breve{\mu}(\boldsymbol{\varepsilon}), \breve{\mathbf{u}}(\boldsymbol{\varepsilon})) \in \mathbb{H}_{(S)}^l(\mathbb{R}^2/\Gamma) \times \mathbb{R} \times \mathbb{R}^2$, and where $\mathbb{H}_{(S)}^l(\mathbb{R}^2/\Gamma)$ means that ψ and η belong to a Sobolev class of periodic functions of order l, ψ and η being respectively odd and even in X.

The proof of this theorem is quite technical due to the occurrence of a well known *small divisor problem* occurring in deriving the expansion given in Lemma 2.1, and follows the same structure as the proof of the main theorem in [11] and in [8]. As in [8], the above result expressed in terms of bifurcation analysis, provides a branch of solutions bifurcating from a *non-isolated eigenvalue* 0 in the spectrum of the linearised operator at the origin. The *additional difficulty here is due to the asymmetry* of solutions, which precisely induces the phenomenon of *Stokes directional drift* explained below. The key point here is that, for proving the above theorem, we use a diffeomorphism of the torus which transforms the horizontal projection of the fluid particles trajectories (integral curves of a velocity field V), into straight lines. For being able to do this we need to assume that the rotation number ρ of the horizontal velocity field V satisfies a Diophantine condition of the form

$$|k_1 + \rho k_2| \geqslant \frac{c}{|\mathbf{k}|^{1+\alpha}}$$

where $0 < \alpha < 1/2$ is fixed, and for any $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2 \backslash \{\mathbf{0}\}$.

3. Directional Stokes drift

We now prove the following:

Lemma 3.1 (*Directional Stokes drift*). *In the frame moving with the velocity of the waves, the horizontal projection of the asymptotic direction taken by fluid particles makes an angle* $\sim \mu_c^{-2}\{-\tau_1\varepsilon_1^2 + \lambda^4\tau_2\varepsilon_2^2\}$ *with the direction of propagation of the waves. There is a special value of the ratio* $\varepsilon_1^2/\varepsilon_2^2 \sim \lambda^4\tau_2/\tau_1$ *for which both directions are identical.*

The idea is to identify the slope p giving the asymptotic direction of the integral curves of the velocity field V, just knowing that this common asymptotic direction exists thanks to Theorem 2.2.

The velocity field V is the horizontal projection of the velocity of fluid particles $\mathbf{u} + \nabla_X\varphi$, given by

$$V = \mathbf{u} + \nabla\psi - \mathfrak{b}\nabla\eta$$
$$\mathfrak{b} = \frac{1}{1 + (\nabla\eta)^2}\{\nabla\eta \cdot (\mathbf{u} + \nabla\psi)\}$$

where $\mathfrak{b}$ denotes $\frac{\partial\varphi}{\partial x_3}|_{x_3=\eta(X)}$. From the expressions given in Lemma 2.1, we notice that

$$\mathbf{u} = \left(1, \beta_1\varepsilon_1^2 + \beta_2\varepsilon_2^2\right) + O\left(|\boldsymbol{\varepsilon}|^4\right)$$

and we find

$$V_1 = 1 + \varepsilon_1\cos(X \cdot K_1) + \lambda\varepsilon_2\cos(X \cdot K_2) - \frac{\varepsilon_1^2}{2\mu_c^2} - \frac{\varepsilon_2^2\lambda^4}{2\mu_c^2} + O'_e\left(|\boldsymbol{\varepsilon}|^2\right)$$

$$V_2 = \tau_1\varepsilon_1\cos(X \cdot K_1) - \lambda\tau_2\varepsilon_2\cos(X \cdot K_2) + \left(\beta_1 - \frac{\tau_1}{2\mu_c^2}\right)\varepsilon_1^2 + \left(\beta_2 + \frac{\tau_2\lambda^4}{2\mu_c^2}\right)\varepsilon_2^2 + O'_e\left(|\boldsymbol{\varepsilon}|^2\right)$$

where the terms $O'_e(|\boldsymbol{\varepsilon}|^2)$ are even in X, and represent a sum of terms of order $|\boldsymbol{\varepsilon}|^2$ with 0 average, and higher order terms. Now we have for any integral line of the vector field V in the plane $X = (x_1, x_2)$:

$$\frac{dx_2}{dx_1} = \frac{V_2}{V_1} = \tau_1\varepsilon_1\cos(X \cdot K_1) - \lambda\tau_2\varepsilon_2\cos(X \cdot K_2)$$

$$+ \left(\beta_1 - \frac{\tau_1}{2\mu_c^2} - \frac{\tau_1}{2}\right)\varepsilon_1^2 + \left(\beta_2 + \frac{\tau_2\lambda^4}{2\mu_c^2} + \frac{\tau_2\lambda^2}{2}\right)\varepsilon_2^2 + O'_e\left(|\boldsymbol{\varepsilon}|^2\right) \tag{5}$$

On the other hand, thanks to the existence of the diffeomorphism with a Diophantine rotation number, we know that the integral lines take the form

$$x_2 = px_1 + q + \mathcal{Y}(X \cdot K_1, X \cdot K_2)$$

where $\mathcal{Y}$ is 2π-periodic in its arguments and is of order $O(|\boldsymbol{\varepsilon}|)$ and p is independent of the integral curve. Denoting by $x'_1 = X \cdot K_1$, $x'_2 = X \cdot K_2$, we have to identify the expression of dx_2/dx_1 given by (5) with the expression given by

$$(1 - \tau_1 \partial_{x_1'} \mathcal{Y} + \lambda \tau_2 \partial_{x_2'} \mathcal{Y}) \frac{dx_2}{dx_1} = p + \partial_{x_1'} \mathcal{Y} + \lambda \partial_{x_2'} \mathcal{Y}$$

It results that

$$\mathcal{Y} = \tau_1 \varepsilon_1 \sin(X \cdot K_1) - \tau_2 \varepsilon_2 \sin(X \cdot K_2) + O_o(|\boldsymbol{\varepsilon}|^2)$$

where the term $O_o(|\boldsymbol{\varepsilon}|^2)$ is odd in X, and represents terms of order $|\boldsymbol{\varepsilon}|^2$. Next order averaged gives

$$p = \left(\beta_1 - \frac{\tau_1}{2\mu_c^2} - \frac{\tau_1}{2}(1 + \tau_1^2) \right) \varepsilon_1^2 + \left(\beta_2 + \frac{\tau_2 \lambda^4}{2\mu_c^2} + \frac{\tau_2 \lambda^2}{2}(1 + \tau_2^2) \right) \varepsilon_2^2 + O(|\boldsymbol{\varepsilon}|^4)$$

$$= \left(\beta_1 - \frac{\tau_1}{\mu_c^2} \right) \varepsilon_1^2 + \left(\beta_2 + \frac{\tau_2 \lambda^4}{\mu_c^2} \right) \varepsilon_2^2 + O(|\boldsymbol{\varepsilon}|^4)$$

Comparing this slope with the slope $\beta_1 \varepsilon_1^2 + \beta_2 \varepsilon_2^2 + O(|\boldsymbol{\varepsilon}|^4)$ of the propagation direction **u**, this leads to the result of the lemma. It is a remarkable fact that there is a neutral value of the ratio between amplitudes $\varepsilon_1 / \varepsilon_2$ for which both directions are identical.

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