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An entrainment model for the turbulent jet in a coflow

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Abstract

The entrainment hypothesis was introduced by G.I. Taylor to describe one-dimensionally the development of turbulent jets issuing into a stagnant or coflowing environment. It relates the mass flow rate of surrounding fluid entrained into the jet to the characteristic velocity difference between the jet and the coflow. A model based on this hypothesis along with axial velocity assumed to follow a realistic Gaussian distribution is presented. It possesses an implicit analytical solution, and its results are compared and shown to be fully equivalent to previously published models that are rather based on a spreading hypothesis. All of them are found to be in agreement with experimental results, on a wide range of downstream positions and for various coflow intensities. ***To cite this article: N. Enjalbert et al., C. R. Mecanique 337 (2009).***

Résumé

Un modèle d'entraînement pour le jet turbulent dans un écoulement co-courant. L'hypothèse d'entraînement a été introduite par G.I. Taylor afin de décrire unidimensionnellement le comportement des jets turbulents se déchargeant dans un environnement au repos ou co-courant. Elle consiste à relier le flux de masse du fluide environnant entraîné dans le jet à la différence de vitesse entre le jet et l'écoulement co-courant. On présente dans cet article un modèle fondé sur cette hypothèse et celle d'un profil de vitesse axiale moyenne Gaussien. Il possède une solution analytique implicite ; ses résultats sont équivalents aux modèles précédemment publiés qui s'appuient plutôt sur une hypothèse d'élargissement. Tous ces modèles sont en accord avec les résultats expérimentaux, disponibles sur un grand domaine de positions le long du jet et pour différentes intensités de l'écoulement co-courant. ***Pour citer cet article : N. Enjalbert et al., C. R. Mecanique 337 (2009).***

Keywords: Fluid mechanics; Turbulent jets; Entrainment

Mots-clés : Mécanique des fluides ; Jets turbulents ; Entraînement

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1. Introduction

Turbulent jets are encountered in countless situations, such as industrial furnaces or propulsion engines. They share similarities with other turbulent structures such as thermal plumes and pool fires, where not inertia, but buoyancy is the source of momentum. Jets involve a wide range of time and length scales: microscopic turbulent phenomena are at the core of their structure, but for an intuitive, global understanding of their overall behavior, the point of view of larger scales must be taken. In this Note only the steady state, and the region where the flow is fully developed, far from the jet origin, are considered. The potential core, a zone right downstream of the nozzle where the average velocity is uniform, is not taken into account. The jet is only characterized by its initial mass and momentum flow rates.

The jets considered here are assumed of constant density and non-reacting. They constitute a very interesting textbook case because, although the problem is quite simplified by these restrictive assumptions, theory and experiments provide tools to easily extend it to non-constant density [1] or reacting situation [2–4].

Observation shows that as a jet develops, its width increases and its velocity decreases. The core phenomenon of this evolution is the entrainment, which is the incorporation of fluid from the surroundings into the jet by turbulent eddies generated by the shear existing between the two regions. As is thoroughly presented by Morton et al. in [5], the entrainment hypothesis applied to a jet released into a still atmosphere states that the total inflow across the edge of the jet is proportional to a characteristic velocity of the jet at the level of inflow, and depends additionally on the geometry of the considered problem.

The purpose of this Note is to present a one-dimensional model of the behavior of turbulent jets based on this hypothesis. Its results are studied and linked with those of different models, recently published and relying on different hypotheses, as well as with experimental results. The case of a free round jet is first discussed, as an illustration for introducing the entrainment hypothesis. Then the hypothesis is extended to the problem of a non-zero coflow velocity and a new model formulated. In a concluding section, its results are compared with those of previous models and with experimental results.

2. The entrainment hypothesis applied to the free jet case

We consider an incompressible jet issuing through a circular nozzle into a stagnant atmosphere of the same fluid whose spatial extent is infinite. The jet is characterized by its initial mass flow rate Q_0 and momentum flow rate M_0. Characteristic initial velocity and radius can be assessed, respectively as M_0/Q_0 and $Q_0/\sqrt{\pi\rho M_0}$. The Reynolds number $\mathrm{Re} = (M_0/\pi\rho\nu^2)^{1/2}$ is assumed large enough for the flow to be fully turbulent at the considered distances from the nozzle. In its expression ρ denotes the fluid density and ν the kinematic viscosity. The average velocity profile at the injection is assumed to be axisymmetric. A cylindrical coordinate system may then be used, with x the axial position, originating from the nozzle, and r the radial distance from the nozzle axis. The quantities considered in this paper are average quantities. The entrainment model involves the intrinsic characteristics of the jet, mass and momentum fluxes through successive constant-x sections of the jet. Given the constant-density assumption, one may remove ρ from their expression and manipulate the specific fluxes, denoted by $q(x)$ and $m(x)$:

$$q(x) = 2\pi \int_0^\infty u(x,r) r \,\mathrm{d}r, \qquad m(x) = 2\pi \int_0^\infty u(x,r)^2 r \,\mathrm{d}r \tag{1}$$

The corresponding quantities $q(x, R)$ and $m(x, R)$ integrated over the finite interval $[0; R]$ instead of $[0; \infty)$ are also introduced; they verify $q(x, R) \to q(x)$ and $m(x, R) \to m(x)$ when $R \to \infty$.

As mentioned above, we study the region far from the nozzle, $x \gg q_0/\sqrt{\pi m_0}$. A key element of the modelling of the jet is to assume that, there, the average velocity field is self-similar. This is widely observed in experiments [6,7] and therefore a usual assumption. The axial average velocity profile may then be written $u(x,r) = u_m(x) f(r/b(x))$, where $u_m(x)$ and $b(x)$ are respectively the average centerline velocity and a characteristic transverse dimension of the jet at abscissa x. Here, f is the profile function which only depends on the self-similar variable $\eta = r/b(x)$. It is possible to determine f theoretically, through assumptions for a closure of the RANS transport equations [7]. But here the experimentally observed Gaussian profile is a priori assumed: $f(\eta) = \exp(-\eta^2)$.

The model system equations are now derived. The integration of the Gaussian profiles leads to the following limit expressions of the flow rates when $r \to \infty$, in terms of u_m and b:

$$q(x,r) \to q(x) = \pi b(x)^2 u_m(x), \qquad m(x,r) \to m(x) = \frac{1}{2}\pi b(x)^2 u_m(x)^2 \tag{2}$$

The equations verified by $q(x)$ and $m(x)$ remain to be established. Momentum and mass balances are carried out on a cylindrical set $[x; x + \mathrm{d}x] \times [0; R]$. As far as the mass balance is concerned, one describes the entrainment of fluid into the jet by introducing $\varepsilon(x, R)$ the entrained flow rate per unit length at a given abscissa x into an x-oriented cylinder of radius R. It is defined in such a way that the mass balance reads $\mathrm{d}q(x, R)/\mathrm{d}x = \varepsilon(x, R)$ and tends to $\mathrm{d}q(x)/\mathrm{d}x = \varepsilon(x)$ when $R \to \infty$. The entrainment increases with the jet velocity, and the entrainment hypothesis is formulated at this stage by assuming that $\varepsilon(x)$ is proportional to u_m (as historically introduced in [5]). Dimensionally, a length has to be involved in the expression of ε, and a natural choice for it is a characteristic perimeter of the jet, $2\pi b$. Finally, the entrainment assumption may be written

$$\frac{\mathrm{d}q(x)}{\mathrm{d}x} = \frac{\mathrm{d}(\pi b(x)^2 u_m(x))}{\mathrm{d}x} = 2\alpha\pi b(x) u_m(x) \tag{3}$$

where α is a coefficient assumed to be constant, called the 'entrainment coefficient'.

In turn, the momentum conservation written on the same set reads $\mathrm{d}m(x, R)/\mathrm{d}x = u(x, R)\,\mathrm{d}q(x, R)/\mathrm{d}x$, where the rhs is the lateral input of momentum due to the entrainment. As the terms of the rhs product tend respectively to zero and the finite value $\varepsilon(x)$, making R tend to infinity leads to

$$\frac{\mathrm{d}m(x)}{\mathrm{d}x} = \frac{\mathrm{d}(\pi b(x)^2 u_m(x)^2)}{\mathrm{d}x} = 0 \tag{4}$$

This expresses the overall conservation of momentum of the jet along its course.

The solution for system (3)–(4) reads

$$b(x) = 2\alpha(x - x_0), \qquad u_m(x) = \frac{\sqrt{m_0/\pi}}{2\alpha(x - x_0)} \tag{5}$$

where x_0 is the abscissa of a 'virtual origin' of the jet [8]. The well-known behavior of the turbulent jet far from its origin, with a linear spreading $b \sim x$ and a velocity decreasing as $1/x$, is found. The constant α was reliably estimated by Ricou and Spalding [1] through a study of the mass flow rate evolution; they give a relation which in terms of the initial momentum flow rate reads $\mathrm{d}q/\mathrm{d}x \sim 0.16\sqrt{\pi m_0}$: applied to (3) and (5) in the limit $x \to \infty$, it yields $\alpha \simeq 0.056$.

The entrainment hypothesis has been presented and applied to a simple jet case. It enables the writing of a model which stays close to the core characteristics of the jet, namely the integral fluxes of mass and momentum. Only one constant is needed to parametrize the system and make it fit the experiments.

3. Coflowing environment

This section applies the entrainment hypothesis to the more complex case of a jet in a coflow. The former problem hypotheses are retained, except that the environment flow is now taken of uniform non-zero average velocity, denoted by u_∞, and assumed small compared to the initial jet velocity: $u_\infty \ll m_0/q_0$.

Dilution by entrainment causes the jet velocity to tend towards u_∞. As long as $u_m \gg u_\infty$, the jet does not heed the environment and behaves as a free round jet. The velocity decreases as $1/x$; according to (5), $u_m \simeq u_\infty$ at a transition distance of the order $l_m = \sqrt{m_0/\pi}/u_\infty$. Beyond, a so-called 'wake regime' is observed, where $u_m - u_\infty \to 0$ as $x^{-2/3}$, and the jet width increases as $x^{1/3}$. Such experimental results have been gathered by Wang [6] on a wide range of inlet and coflow conditions, at far-reaching downstream positions. They are plotted in Fig. 1 in the normalized space $u^* = u_m/u_\infty$, $b^* = bu_\infty\sqrt{\pi/m_{e,0}} \simeq b/l_m$ and $x^* = xu_\infty\sqrt{\pi/m_{e,0}} \simeq x/l_m$, where $m_{e,0}$ is the initial excess of momentum flux, defined later in this section. This non-dimensionalization allows a description of the jet behavior independent of the coflow velocity: indeed, all measurements by Wang collapse in this way with a 15% precision onto unique velocity and half-width curves. The asymptotes of these curves, describing the jet regime in the close region and the wake regime in the distal region, are considered in the following as the reference for the experimental results.

In this case it is relevant and supported by experimental results [6] to lay the self-similarity assumption upon the *excess velocity* $u - u_\infty$ instead of the velocity itself. The same Gaussian similarity function f as earlier is taken.

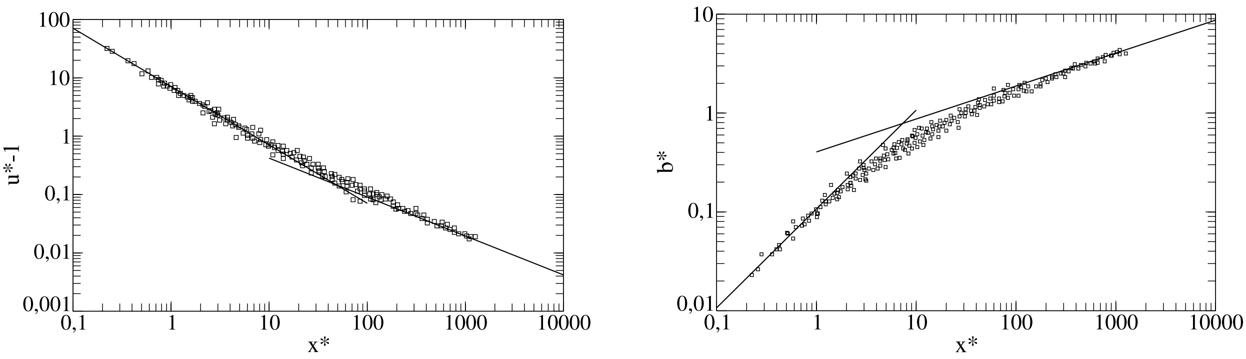

Fig. 1. Experimental measurements of the normalized average centreline velocity and normalized half-width of a jet in a coflowing environment. After [6].

Fig. 1. Mesures expérimentales de la vitesse moyenne au centre normalisée et de la demi-largeur normalisée d'un jet dans un écoulement co-courant. D'après [6].

Here, $u(x,r) \to u_\infty$ when $r \to \infty$, so that $q(x,r)$ and $m(x,r)$ do not tend to finite values anymore and the mass and momentum balances must be carried out with more care.

The mass flow rate through the surface $r \in [0; R]$ may be decomposed as $q(x,R) = \pi[R^2 - b(x)^2]u_\infty + \delta q(x,R) + q_j(x,R)$, such that $\delta q(x,R) \to 0$ and $q_j(x,R) \to q_j(x) = \pi b(x)^2 u_m(x)$ when $R \to \infty$. The first term of this sum may be understood as the contribution of the coflow to the total mass flow rate q, therefore q_j may be considered as the characteristic intrinsic mass flow rate of the jet. That way, the balance equation can be written $\mathrm{d}q_j(x)/\mathrm{d}x = \varepsilon(x)$. One frames an entrainment hypothesis similar to the former case, except that it now involves the velocity difference, in order to account for the fact that it is the shear that generates the entrainment phenomenon: $\varepsilon = 2\pi\alpha b(u_m - u_\infty)$. Assuming that α is a constant, continuously relevant from the coflow jet to the free jet (where $u_\infty = 0$) leads to the same $\alpha \simeq 0.056$. Finally, it holds that

$$\frac{\mathrm{d}q_j(x)}{\mathrm{d}x} = \frac{\mathrm{d}[\pi b(x)^2 u_m(x)]}{\mathrm{d}x} = 2\alpha\pi b(x)\big[u_m(x) - u_\infty\big] \tag{6}$$

As far as the momentum balance is concerned, the relation $\mathrm{d}m(x,R)/\mathrm{d}x = \varepsilon(x,R)u(x,R)$ may be obtained. This leads, as $R \to \infty$, to the result that not $m(x)$, but $m_e(x) = m(x) - q_j(x)u_\infty$ is conserved. This quantity, of integral formulation $\int_0^\infty 2\pi u(u - u_\infty)r\,\mathrm{d}r$, may be referred to as the *excess of momentum* flow rate. With the consistent definition $m_{e,0} = m_0 - u_\infty q_0 \simeq m_0$, its conservation equation reads:

$$\pi b(x)^2\big[u_m(x) + u_\infty\big]\big[u_m(x) - u_\infty\big] = 2m_{e,0} \tag{7}$$

Eqs. (6) and (7), in terms of the non-dimensional variables, read:

$$\frac{\mathrm{d}(u^* b^{*2})}{\mathrm{d}x^*} = 2\alpha b^*(u^* - 1), \qquad b^{*2}(u^* + 1)(u^* - 1) = 2 \tag{8}$$

It may be noted that the solution of this system has an implicit analytic expression, which may be obtained by defining an initial condition $u^*(0)$, and then reads:

$$\left[\frac{-(v^{*2} - 4v^* + 1)}{3(v^* + 1)^{3/2}(v^* - 1)^{5/2}}\right]_{u^*(0)}^{u^*} = 2\alpha x^* \tag{9}$$

The initial condition is established by setting the initial mass and excess of momentum flows to the inlet values, from which comes, for the velocity, $u^*(0) = 2(m_0/q_0)/u_\infty$.

The entrainment hypothesis, in the coflow case, leads as previously to a model whose formulation is tightly linked to the physical behavior of the jet: the entrained mass flow rate is proportional to the shear between the jet and its environment. It was seen that the jet regime behavior, similar to the free jet case, is well reproduced by it; the next section shows that the wake behavior can also be described successfully.

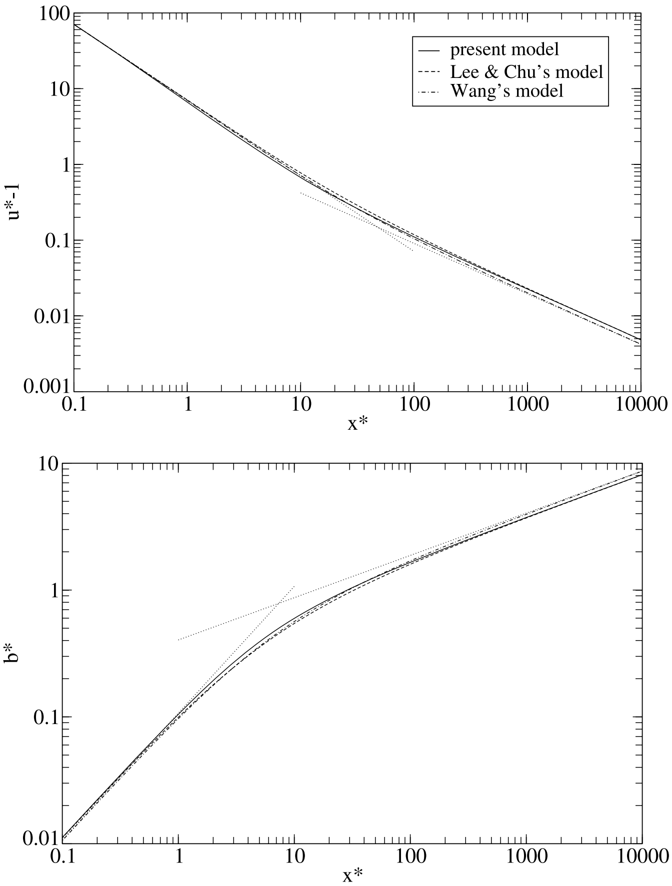

Fig. 2. Comparison between the different model results and the experimental asymptotes, in the normalized space, of velocity excess $u^* - 1$ and half-width b^*.

Fig. 2. Comparaison entre les résultats de différents modèles et les asymptotes expérimentales, dans l'espace normalisé, de l'excès de vitesse $u^* - 1$ et de la demi-largeur b^*.

4. Comparison with experimental results and other models

A comparison of the present model with the experiments and other models is conducted, through the study of the asymptotic behaviors in the jet and wake regimes. The solution of (8) displays the following asymptotic behaviors:

$$\begin{array}{lll} \text{Jet regime } (u^* \gg 1,\ x^* \gg 1): & u^* \sim (2\alpha x^*)^{-1}, & b^* \sim 2\alpha x^* \\ \text{Wake regime } (u^* \simeq 1): & u^* - 1 \sim (3\alpha x^*)^{-2/3}, & b^* \sim (3\alpha x^*)^{1/3} \end{array} \tag{10}$$

First of all, the power laws agree with the experimental results, in both the close and far regions. Moreover, the value $\alpha = 0.056$ mentioned in Section 2 leads to a very close fitting of the model with the measurements: the discrepancy is almost zero in the jet regime for velocity and of 5% for the half-width, and respectively of 14% and 6.5% for $u^* - 1$ and b^* in the wake regime. This is illustrated in Fig. 2.

These results can be compared with those of a different model, established by Lee and Chu [8]. It is based on a different approach, called a *spreading hypothesis*, in which an assumption is laid upon the half-width b^* itself, leading to:

$$u^* \frac{\mathrm{d}b^*}{\mathrm{d}x} = \alpha'(u^* - 1) \tag{11}$$

Along with a momentum conservation equation, this model leads to the correct power laws in the limit regimes, but the prefactors can only be accurately predicted in both the jet and the wake regimes if one assumes a top-hat profile contradictory to experimental observations. Although the same precision as our model is reached, the grounds of the spreading model may be considered as stronger hypotheses: in a jet regime, the equation on the half-width comes down to the assumption $b^* \sim x$, which is directly the solution one seeks. Another model, also based on a spreading hypothesis, is provided by Wang [6]: a perfect collapse of its results upon the experimental asymptotes is achieved, through the use of three parametrizing constants. Fig. 2 presents the results of these two alternative models, too.

5. Conclusion

In this Note, we have presented an application of the entrainment hypothesis to the case of a jet in a coflow. It presents two advantages: it is based on only one constant, α, which is numerically well determined. It makes it a model as simple in terms of parametrization as Lee and Chu's spreading model, and still leads to a good precision, that, to date, only the addition of more constants to tune the solution can improve (as Wang's model does). The entrainment hypothesis also leads to a solution that is directly correct, once a Gaussian profile is assumed, in both the jet and wake regimes. Also, the assumptions formulated by the presented model involve only the intrinsic characteristics of the jet, namely the mass and momentum fluxes. Parameters such as the width or axial velocity are merely used as the output of the model, to make comparisons with experiments and other models possible. This formulation is quite robust and may be inserted into more complex jet modelling, for e.g. the jet in a crossflow, interacting jets, or reacting jets.

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