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Energy balance for undular bores

Balance énergétique pour un mascaret

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ABSTRACT

The energy loss in the shallow-water theory for an undular bore is thought to be due to oscillations that carry away the energy lost at the front of the bore. Using a higher-order dispersive model equation, this expectation is confirmed through a quantitative study which shows that there is no energy loss if dispersion is accounted for.

RÉSUMÉ

La perte d'énergie dans la théorie des mascarets en eau peu profonde est considérée comme dûe aux oscillations qui transportent l'énergie à l'avant du mascaret. En utilisant une équation modèle d'ordre supérieur pour la dispersion, nous confirmons cette assertion par une analyse quantitative qui montre qu'il n'y a pas de perte d'énergie lorsqu'on prend en compte la dispersion.

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1. Introduction

Consideration is given to energy conservation properties of different model equations describing the evolution of an undular bore. In its simplest description, a bore is a transition between two uniform flows with different flow depths. River bores are generally created by tidal forcing if the surrounding conditions such as the bathymetry, the river mouth and the tidal flows are favorable. Bores occur regularly in many rivers around the globe, and some of the better known bores appear in the Severn river in England, the Dordogne river in France, and the Qiantang river in China.

According to experimental studies by Favre [1] and Binnie and Orkney [2], bores appear in two types. If the ratio between the difference in flow depth to the undisturbed water depth is smaller than 0.28, then the bore tends to exhibit an undular character, in other words, the bore will feature oscillations in the downstream part. If this ratio is greater than approximately 0.75, a so-called turbulent bore may be seen. If the ratio is between 0.28 and 0.75, the bore will be turbulent, but also feature some oscillations.

Quite commonly, the bore is studied in the context of shallow-water theory. In this case, a shock-wave solution may be given in exact form, and an analysis using conservation of mass and momentum shows that energy must be lost at the front of the bore. In the case of the turbulent bore, the energy appears to dissipate through turbulent motion at the front of the

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bore. In an undular bore, the energy is thought to be disseminated through an increasing number of oscillations behind the bore, though dissipation may also have an effect here [3].

In this Note, it is shown that if dispersion is included into the model equations, then the energy loss experienced by an undular bore can be accounted for. Indeed, monitoring the energy E_{disp} in the dispersive theory shows that the rate of change of E_{disp} is exactly equal to the net influx of energy. Thus it may be said that the part of the net influx of energy which is lost at the bore front in the shallow-water theory is carried away by the oscillations developing behind the front.

In previous works [3,4], a steady version of the Korteweg–de Vries equation was used to study the same problem. While taking account of dispersion, it is clear that these studies focused on the time-independent problem, thus neglecting dynamical effects. Moreover, the Korteweg–de Vries equation used in these works is limited to unidirectional propagation, and therefore only includes waves propagating in the same direction as the bore. As a result, it is not possible to set boundary conditions for the surface excursion and the horizontal velocity independently, and a comparison with the shallow-water theory appears problematic. In contrast, we use a time-dependent system of equations allowing for counterpropagating waves. A similar system was studied in [5], but no information on the development of the energy was provided.

2. Shallow-water theory

In the case of a purely undular bore, the transition from low to high surface elevation is rather gentle, so that a long-wave (shallow-water) approximation is justified [6]. Therefore, the problem may be studied using the Saint-Venant system of equations given by

$$\left.\begin{aligned} &\eta_t + h_0 u_x + (u\eta)_x = 0 \\ &u_t + g\eta_x + uu_x = 0 \end{aligned}\right\} \tag{1}$$

Here η is the deflection of the free surface from its rest position, h_0 is the undisturbed water depth, and u represents the horizontal flow velocity. The constant g denotes the gravitational acceleration, and the width of the fluid domain and density of the fluid are assumed to be unity. Eqs. (1) describe the conservation of mass and momentum, and the assumption is made that the horizontal velocity u is independent of the depth.

Smooth solutions of this system conserve mass, momentum and energy of an initial state, as well as an infinite number of higher-order integrals that do not have an obvious physical interpretation. Let h be the total depth of the water, given by $h(x,t) = h_0 + \eta(x,t)$. The conservation equations for mass and horizontal momentum in a section $x_1 < x < x_2$ are

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} h(x,t)\,\mathrm{d}x = u(x_1,t)h(x_1,t) - u(x_2,t)h(x_2,t) \tag{2}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} u(x,t)h(x,t)\,\mathrm{d}x = u^2(x_1,t)h(x_1,t) - u^2(x_2,t)h(x_2,t) + \frac{1}{2}gh^2(x_1,t) - \frac{1}{2}gh^2(x_2,t) \tag{3}$$

respectively. The mechanical energy associated to the shallow-water approximation is given by the integral

$$E_{sw} = \frac{1}{2}\int_{x_1}^{x_2} \left\{u^2(x,t)h(x,t) + gh^2(x,t)\right\}\mathrm{d}x \tag{4}$$

while the energy flux at x_i is given by

$$F_i = \frac{1}{2}u^3(x_i,t)h(x_i,t) + gu(x_i,t)h^2(x_i,t) \tag{5}$$

for $i = 1, 2$. To model a bore in the context of the shallow-water theory, one may find a shock-wave solution of (1) taking prescribed values upstream and downstream, and with a discontinuity fitted in at the bore front. As indicated in Fig. 1, one may choose a reference frame in which the transition is from the undisturbed water level h_0 upstream of the bore to a prescribed water level $h_0 + a_0$ downstream of the bore, and such that the upstream velocity is zero. Since river bores generally move upstream, we take upstream to mean in the direction of increasing values of x. The conservation equations (2) and (3) give the following shock conditions [6]:

$$-U[h]_{x_1}^{x_2} + [hu]_{x_1}^{x_2} = 0, \qquad -U[uh]_{x_1}^{x_2} + \left[hu^2 + \frac{1}{2}gh^2\right]_{x_1}^{x_2} = 0 \tag{6}$$

It is well known that these equations define the downstream velocity u_0 and the velocity of the front of the bore U. Moreover, it is common knowledge that this discontinuous solution cannot conserve the energy, and the loss of energy

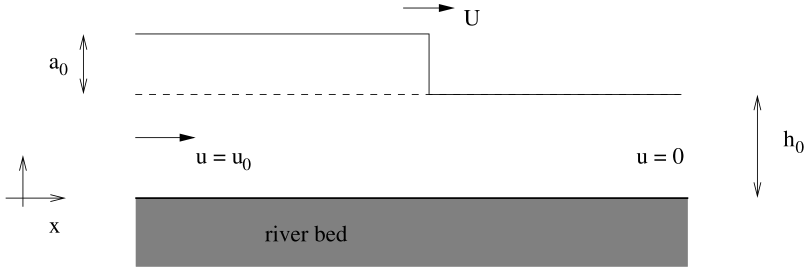

Fig. 1. Schematic of a discontinuous solution of (1). The velocity u jumps from 0 to u_0, and the flow depth jumps from h_0 to $h_0 + a_0$. The front of the bore moves upstream at a velocity U.

across the bore may be computed exactly. Denoting the downstream and upstream energy fluxes by F_1 and F_2, respectively, we find the equation

$$-\frac{\mathrm{d}E_{sw}}{\mathrm{d}t} + (F_1 - F_2) = \frac{a_0^3}{4}\sqrt{\frac{1}{2}g^3\left(\frac{1}{h_0} + \frac{1}{a_0 + h_0}\right)} \tag{7}$$

The right-hand side of this equation represents the energy lost due the approximate nature of the governing equations (1), and the discontinuous solution shown in Fig. 1.

3. Dispersive theory

To incorporate dispersion into the equation, we drop the assumption that the horizontal velocity is uniform across the depth. Let $w(x,t)$ denote the horizontal velocity at a height $\sqrt{\frac{2}{3}}h_0$. Now instead of η and u, we use η and w as the primary dependent variables of the flow. If this is done, the system of dispersive evolutionary partial differential equations

$$\left.\begin{aligned} &\eta_t + h_0 w_x + (w\eta)_x + \frac{h_0^3}{6} w_{xxx} = 0 \\ &w_t + g\eta_x + w w_x + \frac{g h_0^2}{6}\eta_{xxx} = 0 \end{aligned}\right\} \tag{8}$$

appears [7]. The system (8) is a variant of the original Boussinesq system [8]. An additional assumption necessary for the derivation of this system is that the amplitude of the waves is small when compared to the depth of the fluid. In the context of bores, the natural ratio to consider is $\frac{a_0}{h_0}$, and we see that this is indeed small for undular bores. It is also observed that for very long waves, the terms w_{xxx} and η_{xxx} are nearly zero, so that the system reduces to (1). The associated mechanical energy which can be found by the same asymptotic analysis which yields the system (8), is given by the expression

$$E_{disp} = \frac{1}{2}\int_{x_1}^{x_2} \left\{ (h_0 + \eta)w^2 + g\big(h_0^2 + 2h_0\eta + \eta^2\big) + \frac{h_0^3}{3} w w_{xx} + \frac{h_0^3}{3} w_x^2 \right\} \mathrm{d}x \tag{9}$$

Also here, it is evident that for very long waves the last two terms in the integral disappear, and the expression reduces to E_{sw}, the energy associated with the Saint-Venant system (1). Note that E_{disp} is not the Hamiltonian function for (8), and there is no exact conservation law connected with E_{disp}. Indeed, in the same way as solutions of (8) approximate solutions of the full surface water-wave problem, so does E_{disp} approximate the energy of the full problem.

To study the bore development, solutions of (8) are approximated numerically. The numerical treatment is based on a finite-difference approximation, and will not be detailed here. It suffices to say that we have done some convergence and stability studies, and are confident that our numerical approximation is sound. In all the numerical experiments the depth h_0 is taken to be equal to 1 m and the gravitational acceleration is $g = 9.8\ \mathrm{m\,s^{-2}}$. The initial bore front is taken to be at the origin, and we take $x_1 \ll 0$, and $x_2 \gg 0$. The initial water surface and the initial velocity are given by

$$\eta(x,0) = \frac{1}{2}a_0\big[1 - \tanh(kx)\big], \qquad w(x,0) = \frac{1}{2}u_0\big[1 - \tanh(kx)\big] \tag{10}$$

where the parameter k denotes the steepness of the initial bore slope, and u_0 is given from the shock conditions (6). A typical wave profile is shown in Fig. 2 (left), which shows the creation of a wavetrain following the bore front. The net energy flux for a situation in which $a_0 = 0.25$ m is shown in Fig. 2 (right). As can be seen, the rate of change of E_{disp} is equal to the net influx of energy, while the rate of change of E_{sw} is different by the amount given in (7). The parameter k does not seem to have a strong influence on the data shown in Fig. 2 (right). Computations with increasing steepness were done, and it was found that the results were all approximately equal.

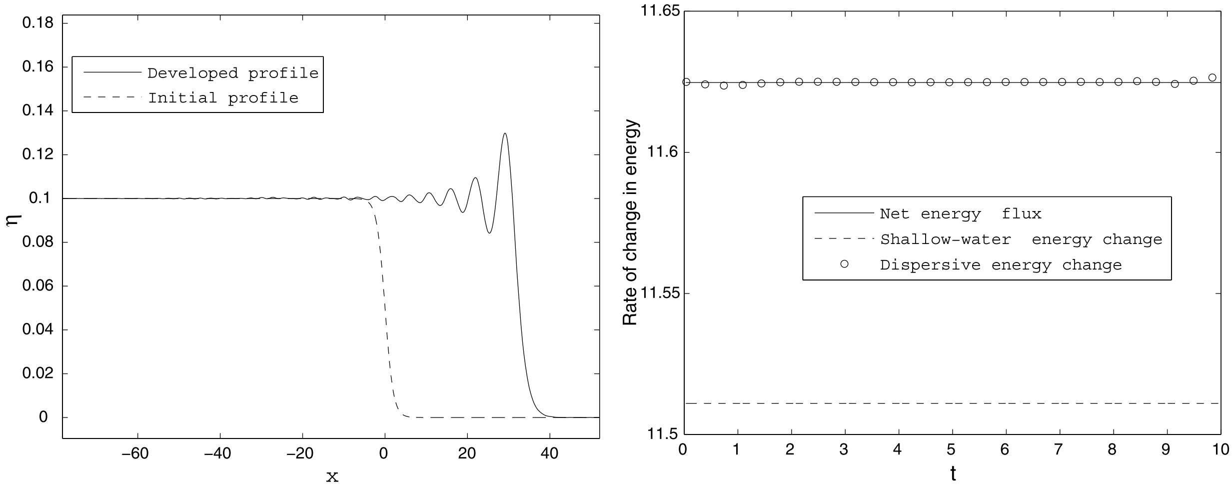

Fig. 2. Left: Development of an undular bore for total time $T = 14$ s, with initial amplitude $a_0 = 0.1$ m. Right: A comparison of the rate of change of energy in the dispersive and the shallow-water system, vs. the net energy flux through the boundaries at x_1 and x_2. The initial bore amplitude was $a_0 = 0.25$ m.

Table 1
Column 1 shows the bore amplitude a_0 in m. The net energy flux is shown in column 2. Column 3 displays the rate of change in the energy in the shallow-water theory. Column 4 shows the percentage difference when compared to the net energy flux. Columns 5 and 6 show the rate of change in the energy in the dispersive theory and the percentage difference, respectively. The dispersive theory gives the correct result to within less than 0.1% error.

a_0	$F_1 - F_2$	$\frac{dE_{sw}}{dt}$	% Difference	$\frac{dE_{disp}}{dt}$	% Difference
0.1	3.64	3.635	0.2	3.64	0.00
0.2	8.59	8.53	0.7	8.58	0.00
0.3	15.08	14.88	1.3	15.08	0.06
0.4	23.36	22.90	1.9	23.36	0.04
0.5	33.69	32.82	2.6	33.69	0.00
0.6	46.36	44.87	3.2	46.35	0.00
0.7	61.64	59.29	3.8	61.63	0.00
0.8	79.83	76.36	4.3	79.84	0.03
0.9	101.24	96.36	4.8	101.30	0.06
1.0	126.20	119.56	5.3	126.28	0.06

Table 1 shows the calculation of the energy rates for undular bores with increasing initial amplitudes, including some that strictly speaking fall outside of the range of purely undular bores. Nevertheless, the agreement between the net energy flux and the change in E_{disp} is striking. While the energy loss experienced by the shallow-water theory is not dramatic in relative terms, it is strongly dependent on the bore amplitude. The dispersive theory on the other hand can easily handle cases up to a ratio $a_0/h_0 \sim 1$. In conclusion, it appears that the model (8) successfully captures the energy loss incurred by the shock-wave solution of the Saint-Venant system.

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