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Yield initiation of compressible material with a central void under dynamic load

Limite d'élasticité d'un matériau compressible à cavité centrale sous contrainte dynamique

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ABSTRACT

A rigorous analytical solution is presented for a compressible solid with a central void under dynamic hydrostatic tension. It is revealed that for a solid with a preexisting central void under the dynamic loading, the yielding is nucleated preferentially in the zone near the void surface rather than on the void surface. For a void forming under the dynamic load, the subsequent yielding will always initiate in the region with the depth below 1/4 longitudinal wavelength from the void surface.

RÉSUMÉ

Une solution analytique rigoureuse est présentée pour un solide compressible comportant une cavité centrale sous une tension dynamique hydrostatique. Pour un solide muni d'une cavité centrale pré-existante, la limite d'élasticité est obtenue préférentiellement dans une zone proche de la cavité plutôt qu'à sa surface. Pour une cavité formée sous charge dynamique, la limite d'élasticité résultante apparaîtra toujours dans une zone dont la distance à la cavité ne dépasse pas 1/4 de la longueur d'onde longitudinale du matériau.

1. Introduction

Mechanical responses of materials with voids have attracted great research interests [1–14]. In recent years, extensive works have been done concerning the dynamic response of materials with a central void under hydrostatic tension [15–19]. This type of load is important in dynamic loading case, as pointed out by Wright [18], since under an impact loading, the interaction of a reflection wave with other waves can generate a nearly hydrostatic tension stress at high loading rates. The hydrostatic tension stress will promote void nucleation and growth, and lead to rupture eventually. However, the previous works generally treat the solid as incompressible [2,6,20], or use the elastic–plastic boundary which is derived from the static case [9,15], assuming that the yielding always initiates on the surface of the void and the elastic–plastic boundary will

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propagate outwards when loading increases [17]. To the authors' knowledge, few attempts have been made to investigate whether or not the yielding will initiate on the void surface for a compressible solid under dynamic loading. It is plausible that the yielding under dynamic loading may also initiate in the inner compressible solid due to the interaction between the incident wave and the reflection wave from the free surface of voids.

The present Note aims to explore the yielding nucleation sites under different conditions. A rigorous analytical solution for the mechanical response of compressible solid with a central void under dynamic hydrostatic tension is obtained, and the conclusion may facilitate a controlled formation of void structures of engineering interests.

2. Theory

The representative volume element (RVE) used in this work is a sphere with radius r_0 and with a concentric void with radius a. This type of RVE is widely used in the literature to study either the mechanical properties of porous materials or the void nucleation and growth [15,21]. The RVE is subjected to a dynamic hydrostatic tension $P(t)$ at the outer surface. Supposing the solid is isotropic, the displacement $\boldsymbol{u}$ in spherical coordinates $\{r,\theta,\phi\}$ only has a nonzero component u_r due to symmetry, i.e. $\boldsymbol{u} = u_r\boldsymbol{e}_r = u\boldsymbol{e}_r$, where $\boldsymbol{e}_r$ is the unit vector along the radial direction. The equilibrium equation for stress tensor $\boldsymbol{\sigma}$ under spherical coordinates is given by:

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi}) = \rho\ddot{u} \tag{1}$$

where ρ is the mass density. For isotropic linear elastic materials, the constitutive equation writes: $\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I}$, where λ and μ are Lame constants, $\boldsymbol{\varepsilon}$ is the strain tensor, and $\boldsymbol{I}$ is the second-order identity tensor. The symmetric condition gives $\varepsilon_{\theta\theta} = \varepsilon_{\phi\phi}$ and $\sigma_{\theta\theta} = \sigma_{\varphi\varphi}$. For small deformations, the strain components follow $\varepsilon_{rr} = \partial u/\partial r$ and $\varepsilon_{\theta\theta} = u/r$. Substituting these relations into Eq. (1) leads to:

$$\frac{\partial^2 u}{\partial r^2} + 2\frac{\partial}{\partial r}\left(\frac{u}{r}\right) = \frac{\ddot{u}}{c_l^2} \tag{2}$$

where $c_l = \sqrt{(\lambda + 2\mu)/\rho}$ is the longitudinal wave speed of elastic solid. For arbitrary $P(t)$ on the outer boundary of RVE, we can obtain the load amplitude $p(\omega)$ in the frequency domain by using Fourier transformation:

$$p(\omega) = \int_{-\infty}^{\infty} P(t)e^{-i\omega t}\,\mathrm{d}t \tag{3}$$

For linear elastic materials, the displacement under a dynamic load $P(t)$ can be calculated according to superposition principle:

$$u(r,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_p(r,\omega)e^{i\omega t}\,\mathrm{d}\omega \tag{4}$$

where $u_p(r,\omega)$ is the displacement amplitude for RVE subjected to a load $p(\omega)\exp(i\omega t)$. Substituting Eq. (4) into Eq. (2) gives:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\partial^2 U}{\partial\tilde{r}^2} + 2\frac{\partial}{\partial\tilde{r}}\left(\frac{U}{\tilde{r}}\right) + U\right)e^{i\omega t}\,\mathrm{d}\omega = 0 \tag{5}$$

where the dimensionless radius $\tilde{r} = r\omega/c_l = 2\pi r/\lambda_l$ with $\lambda_l = 2\pi c_l/\omega$ is the longitudinal wavelength and $U(\tilde{r},\omega) = u_p(r,\omega)$. Eq. (5) indicates that the term in the brackets must be zero, thus the solution is:

$$U(\tilde{r}) = c_1\left(\frac{\sin\tilde{r}}{\tilde{r}^2} - \frac{\cos\tilde{r}}{\tilde{r}}\right) + c_2\left(\frac{\cos\tilde{r}}{\tilde{r}^2} + \frac{\sin\tilde{r}}{\tilde{r}}\right) \tag{6}$$

c_1 and c_2 can be determined by the boundary conditions $\sigma_{rr}(\tilde{a}) = 0$ and $\sigma_{rr}(\tilde{r}_0) = p\exp(i\omega t)$, where $\tilde{r}_0 = r_0(\omega/c_l)$ and $\tilde{a} = a(\omega/c_l)$.

The stress amplitudes are obtained as:

$$\frac{\sigma_{rr}}{p} = \frac{\tilde{r}_0^3}{\tilde{r}^3}\frac{S_1\cos\tilde{d} + S_2\sin\tilde{d}}{S_1^0\cos\tilde{d}_0 + S_2^0\sin\tilde{d}_0}, \qquad \frac{\sigma_{\theta\theta}}{p} = \frac{\tilde{r}_0^3}{\tilde{r}^3}\frac{S_3\cos\tilde{d} + S_4\sin\tilde{d}}{S_1^0\cos\tilde{d}_0 + S_2^0\sin\tilde{d}_0} \tag{7}$$

where $\tilde{d} = \tilde{r} - \tilde{a}$, $\tilde{d}_0 = \tilde{r}_0 - \tilde{a}$. The terms $S_i = S_i(\tilde{r},\tilde{a})$ $(i = 1,2,3,4)$ are defined as:

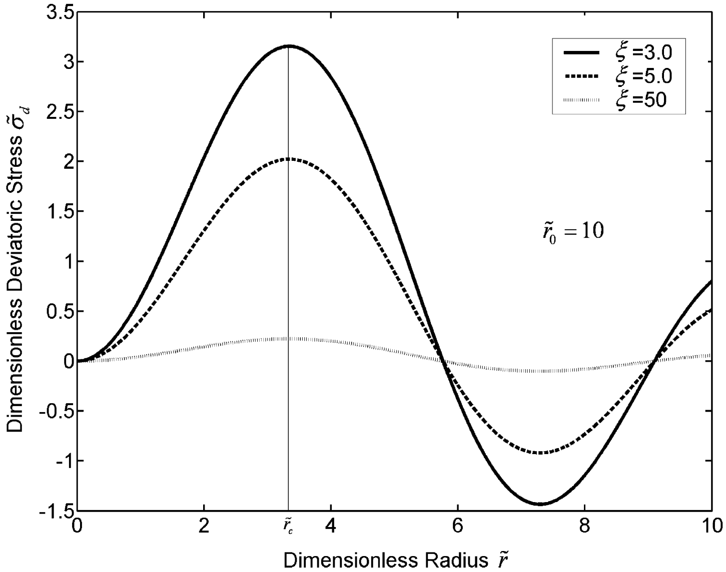

Fig. 1. Dimensionless deviatoric stress $\tilde{\sigma}_d = \sigma_d/p$ variation with radius for dynamic loading case with $\tilde{a} = 0$.

$$
\begin{aligned}
S_1(\tilde{r},\tilde{a}) &= 4(\tilde{r}-\tilde{a})(4+\tilde{a}\tilde{r}\xi) \\
S_2(\tilde{r},\tilde{a}) &= 4\xi(\tilde{r}+\tilde{a})^2 - (\tilde{a}\tilde{r}\xi+4)^2 - 16\tilde{a}\tilde{r} \\
S_3(\tilde{r},\tilde{a}) &= 2\big(\tilde{a}\tilde{r}\big(\tilde{a}\xi + 2\tilde{r}(\xi-2)\big) - 4(\tilde{r}-\tilde{a})\big) \\
S_4(\tilde{r},\tilde{a}) &= \big(8\tilde{a}\tilde{r} + \big(4-\xi\tilde{a}^2\big)\big(\tilde{r}^2(\xi-2)+2\big)\big)
\end{aligned} \tag{8}
$$

and $S_i^0 = S_i(\tilde{r}_0,\tilde{a})$. $\xi = \lambda/\mu + 2 = 2(1-\nu)/(1-2\nu)$ is uniquely determined by Poisson's ratio ν of solid. Generally, $0 < \nu < 0.5$, thus $\xi > 2$. The hydrostatic stress $\boldsymbol{\sigma}_m = \sigma_m \boldsymbol{I}$ and the deviatoric stress $\boldsymbol{\sigma}_d = (\sigma_d/3)(2\boldsymbol{e}_r \otimes \boldsymbol{e}_r - \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta - \boldsymbol{e}_\phi \otimes \boldsymbol{e}_\phi)$ are calculated by $\sigma_m = (\sigma_{rr} + 2\sigma_{\theta\theta})/3$ and $\sigma_d = \sigma_{rr} - \sigma_{\theta\theta}$, where $\{\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi\}$ are unite vectors in spherical coordinates, and the symbol "$\otimes$" denotes tensor product.

The von Mises stress $\sigma_e = \sqrt{(3/2)\boldsymbol{\sigma}_d : \boldsymbol{\sigma}_d}$ is equivalent to $|\sigma_d|$, the absolute value of σ_d. For a static loading case with a loading p on the surface of RVE, the solution writes:

$$
\sigma_e^s = p\frac{3r_0^3}{2r^3}\frac{a^3}{r_0^3 - a^3}, \qquad \sigma_m^s = p\frac{r_0^3}{r_0^3 - a^3} \tag{9}
$$

Eq. (9) suggests that for static loading case, von Mises stress σ_e^s monotonically decreasing with the increase of r, thus the initiation of yielding is always on the surface of the void. For $a = 0$, Eq. (9) gives $\sigma_e^s = 0$, meaning that the material never yields. However, under dynamic loading case, for $a = 0$, Eq. (7) gives:

$$
\tilde{\sigma}_d = \frac{2\tilde{r}_0^3}{\tilde{r}^3}\frac{3\tilde{r}\cos\tilde{r} - (3-\tilde{r}^2)\sin\tilde{r}}{4\tilde{r}_0\cos\tilde{r}_0 - (4-\tilde{r}_0^2\xi)\sin\tilde{r}_0} \tag{10}
$$

where $\tilde{\sigma}_d = \sigma_d/p$ is the dimensionless deviatoric stress. It is clear that $\tilde{\sigma}_d$ is not uniform along the radial direction, as shown in Fig. 1. This indicates that for dynamic loading cases, the material may yield even in the absence of the void. The radius $\tilde{r}_c$ denoting the site where the maximum value of σ_d ($\sigma_d^{\max}$) is achieved can be calculated by solving the equation $\partial\sigma_d/\partial\tilde{r} = 0$, i.e. $\tan\tilde{r}_c = \tilde{r}_c(9-\tilde{r}_c^2)/(9-4\tilde{r}_c^2)$. The first positive root gives $\tilde{r}_c \approx 3.3421$. If $|\sigma_d^{\max}|$ achieves the yielding limit σ_y, yielding will initiate at $\tilde{r} = \tilde{r}_c$. Since r_c only depends on the wavelength λ_l, for a RVE with size $r_0 < r_c$, the initiation of yielding would always occur at the outer surface of RVE. This is quite different with the conclusion for the static loading cases.

Once the yielding initiates at $r = r_c$, plastic region will form in the vicinity of $\tilde{r} = \tilde{r}_c$, and the solid will divide into three regions, i.e. one plastic region between two elastic regions. When the strain in the plastic region reaches the fracture strain, the plastic region will be fractured, and the inner elastic region will separate from the mother solid as debris and a void with the radius slightly larger than r_c will form.

Fig. 2 shows the variation of dimensionless deviatoric stress $\tilde{\sigma}_d$ with the depth $\tilde{d} = \tilde{r} - \tilde{a}$ from the void surface for different void sizes. It is found that for materials with initial void size $\tilde{a}$ smaller than a specific value $\tilde{a}_1$, $\tilde{\sigma}_d$ decreases rapidly with the increase of $\tilde{d}$. When $\tilde{a}$ is larger than $\tilde{a}_1$ and smaller than a critical value $\tilde{a}_c$, the decrease of $\tilde{\sigma}_d$ may be slow, and when $\tilde{a}$ is larger than $\tilde{a}_c$, $\tilde{\sigma}_d$ increases with the increase of $\tilde{d}$. Thus, $\tilde{a} < \tilde{a}_c$ is definitely a necessary condition for yielding to initiate on the void surface.

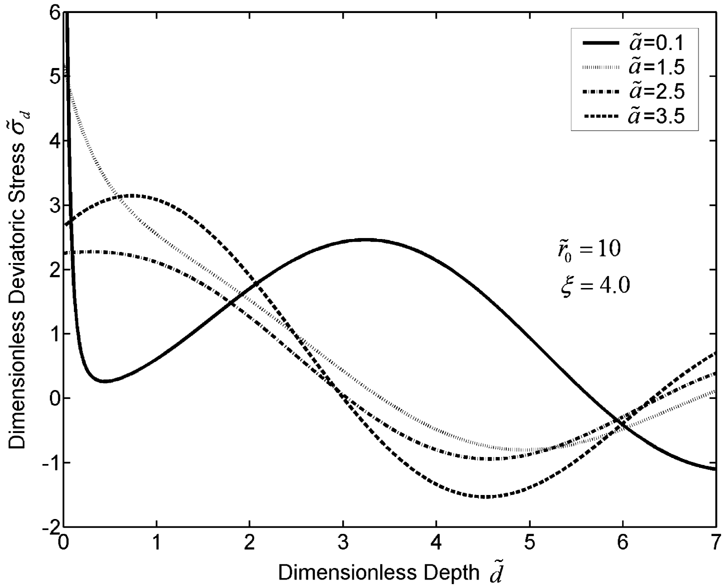

Fig. 2. Dimensionless deviatoric stress $\tilde{\sigma}_d = \sigma_d/p$ variation with the depth from the void surface for different void sizes.

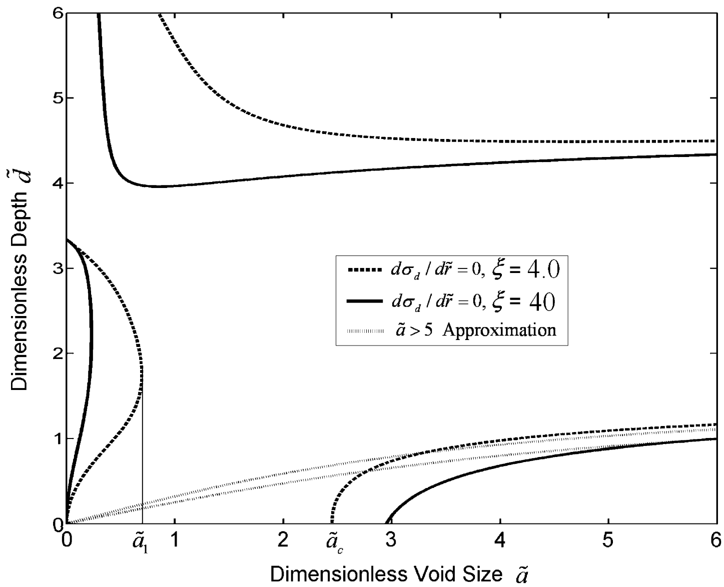

Fig. 3. The depth correspondence to maximum value of σ_d ($\partial\sigma_d/\partial\tilde{r} = 0$) in the inner solid near void surface for different void sizes.

Now we consider the specific value of $\tilde{a}_c$. The derivation of Eq. (7) gives:

$$\sigma_d \frac{\partial \sigma_d}{\partial \tilde{r}}\bigg|_{\tilde{r}=\tilde{a}} = \frac{p^2}{\tilde{a}} \frac{4\tilde{r}_0^6(3\xi - 4)(12 + (\tilde{a}^2 - 9)\xi)}{(S_1^0 \cos\tilde{d}_0 + S_2^0 \sin\tilde{d}_0)^2} \tag{11}$$

When $\tilde{a} < \tilde{a}_c$, $\tilde{\sigma}_d$ will at first decrease as $\tilde{r}$ increases from the void surface according to Fig. 2, which corresponds to $\sigma_d(\partial\sigma_d/\partial\tilde{r})|_{\tilde{r}=\tilde{a}} < 0$. Since $3\xi - 4 > 0$, Eq. (11) indicates $\tilde{a} < \tilde{a}_c = \sqrt{9 - 12/\xi} < 3$. That is to say, for any ξ values ($\xi > 2$), the necessary condition for initial yielding locating on the void surface should be $\tilde{a} < 3$. For the void nucleation occurring under dynamic loading as discussed above, the formed void size $\tilde{r}_c \approx 3.3421 > 3$; this implies that the subsequent yielding will always occur in the inner solid instead of on the void surface. The location of the maximum value of σ_d in the inner solid corresponds to where the yielding initiates and can be found by solving $\partial\sigma_d/\partial\tilde{r} = 0$. Fig. 3 shows the depth of the potential yielding nucleation sites from the surface of the void $\tilde{d}$ for different void sizes $\tilde{a}$. If the void size $\tilde{a} > \tilde{a}_c$, the depth $0 < \tilde{d} < \pi/2$, which indicates $\tilde{d}$ will not exceed $\pi/2$, i.e. $d_{\max} = \lambda_l/4$. For $\tilde{a} > 5$, $\tilde{d} = \arctan((\xi - 1)^{-1}\tilde{a}\xi/4)$ is a good approximation.

3. Conclusion and discussion

In summary, the initiation of yielding for compressible materials with a central void under dynamic load is investigated. For void size larger than a special value $a_c \approx 0.4775\lambda_l$ (i.e. $\tilde{a}_c = 3$), the yielding would always initiate in the solid with some distance from the void surface. This distance never exceeds 1/4 of the longitudinal wavelength. For dynamic void nucleation, initiation location is uniquely determined by the wavelength, independent on the RVE size or other material parameters. The size of the forming void is always larger than a_c, and debris is left in the void.

The research may promote the understanding of the void formation of materials under impact loading. For ductile materials, the elastic–plastic boundary forming during the loading process may induce additional reflection and transmission waves to make the problem more complex. For brittle materials, once a spherical void forms, with the applying of subsequent loading, a new void will form in the solid, and eventually multi-concentric hollow spheres may form. The site of the subsequent-forming hollow spheres with large radius is at a depth near $\lambda_l/4$ from the void surface, which is nearly uniform. This allows the possible controlling of the thickness of the forming hollow spheres in radial direction by changing the loading frequencies. The forming structure may be anticipated to be used in some special regions, such as thermal protection, and micro-wave shielding.

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References

[1] A.L. Gurson, Continuum theory of ductile rupture by void nucleation and growth: Part I. Yield criteria and flow rules for porous ductile media, J. Eng. Technol. 99 (1977) 2–15.
[2] Y. Huang, J.W. Hutchinson, V. Tvergaard, Cavitation instabilities in elastic–plastic solids, J. Mech. Phys. Solids 39 (1991) 223–241.
[3] C. Bordreuil, J.C. Boyer, E. Salle, On modelling the growth and the orientation changes of ellipsoidal voids in a rigid plastic matrix, Modell. Simul. Mater. Sci. Eng. 11 (2003) 365–380.
[4] B. Chen, X. Peng, J. Fan, S. Chen, An elastoplastic constitutive description based on an ellipsoidal void model, Mater. Sci. Eng. A 423 (2006) 230–236.
[5] N.A. Fleck, J.W. Hutchinson, Void growth in shear, Proc. R. Soc. London, Ser. A 407 (1986) 435–458.
[6] H.L. Li, Z.P. Huang, Effect of strain rate-sensitivity on statistical evolution of microvoids, Sci. China, Ser. A 39 (1996) 1332–1339.
[7] J.B. Leblond, M. Gologanu, External estimate of the yield surface of an arbitrary ellipsoid containing a confocal void, C. R. Mecanique 336 (2008) 813–819.
[8] Y.S. Lee, Development of constitutive equations for plastic deformation of a porous material using numerical experiments, J. Mater. Process. Technol. 130–131 (2002) 161–167.
[9] A. Molinari, T.W. Wright, A physical model for nucleation and early growth of voids in ductile materials under dynamic loading, J. Mech. Phys. Solids 53 (2005) 1476–1504.
[10] V. Monchiet, E. Charkaluk, D. Kondo, An improvement of Gurson-type models of porous materials by using Eshelby-like trial velocity fields, C. R. Mecanique 335 (2007) 32–41.
[11] K.L. Pan, Z.P. Huang, X. Ji, Elliptical void growth in shear, Proc. R. Soc. London, Ser. A 451 (1995) 553–570.
[12] G.V. Zhdanova, Scattering of plane longitudinal elastic waves by a slender cavity of revolution. The case of axial incidence, Math. USSR Sb. 49 (1984) 305–323.
[13] C.F. Ying, R. Truell, Scattering of a plane longitudinal wave by a spherical obstacle in an isotropically elastic solid, J. Appl. Phys. 27 (1956) 1086–1097.
[14] A.S. Ovsyannikov, V.A. Starikov, Scattering of a plane longitudinal steady wave at a cavity in the form of a body of revolution, Int. Appl. Mech. 21 (1985) 542–546.
[15] X.Y. Wu, K.T. Ramesh, T.W. Wright, The dynamic growth of a single void in a viscoplastic material under transient hydrostatic loading, J. Mech. Phys. Solids 51 (2003) 1–26.
[16] T.W. Wright, K.T. Ramesh, Dynamic void nucleation and growth in solids: A self-consistent statistical theory, J. Mech. Phys. Solids 56 (2008) 336–359.
[17] T.C. Tszeng, Quasistatic and dynamic growth of sub-microscale spherical voids, Mech. Mater. 41 (2009) 584–598.
[18] T.W. Wright, K.T. Ramesh, Statistically informed dynamics of void growth in rate dependent materials, Int. J. Impact Eng. 36 (2009) 1242–1249.
[19] A. Molinari, S. Mercier, Micromechanical modelling of porous materials under dynamic loading, J. Mech. Phys. Solids 49 (2001) 1497–1516.
[20] M. Ortiz, A. Molinari, Effect of strain hardening and rate sensitivity on the dynamic growth of a void in a plastic material, J. Appl. Mech. 59 (1992) 48–53.
[21] M. Danielsson, D.M. Parks, M.C. Boyce, Constitutive modeling of porous hyperelastic materials, Mech. Mater. 36 (2004) 347–358.