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Analytical expressions for anisotropic tensor dimension

Formules analytiques pour la détermination de la dimension d'un tenseur anisotrope

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ABSTRACT

The vector space dimension of a linear behavior operator, such as the elasticity tensor, depends on the symmetry group of the material it is defined on. This Note aims at introducing an easy and analytical way to calculate this dimension knowing the material symmetry group. These general results will be illustrated in the case of classical and strain-gradient elasticity.

RÉSUMÉ

La dimension de l'espace vectoriel d'un opérateur linéaire de comportement, comme le tenseur d'élasticité, dépend des symétries du milieu matériel. L'objectif de cette Note est de présenter une méthode simple et analytique pour de déterminer ces dimensions connaissant le groupe de symétrie matérielle. Ces résultats généraux seront illustrés dans les cas de l'élasticité classique et du second gradient.

1. Introduction

The case of a general linear behavior will be considered here. This behavior is supposed to be modeled by a tensor, and attention will be focused on the link between the symmetry of the material domain and the number of coefficients needed to correctly define the tensor. In Section 2 some basic definitions about symmetries will be summed up. Some notes about classical tensor decompositions will then be recapped in Section 3. The fundamental notion of G-invariant harmonic tensor space will be introduced in Section 4. Our main result will be stated in Section 5. And finally some illustrations will be proposed in Section 6.

2. Physical and material symmetries

Hereafter $\mathcal{E}^3$ will be the 3-dimensional Euclidean physical space. Let G be a subgroup of the orthogonal group in 3-dimension $\mathrm{O}(3)$, $\mathrm{O}(3)$ is the group of isometries of $\mathcal{E}^3$. Let's define a material $\mathcal{M}$ as a 3-dimensional subset of $\mathcal{E}^3$. $\mathcal{M}$ is said G-invariant if the action of G's elements transforms $\mathcal{M}$ into itself. This set of operation, called the material symmetry group, will be denoted $\mathrm{G}_{\mathcal{M}}$:

$$\mathrm{G}_{\mathcal{M}} = \{\mathrm{Q} \in \mathrm{O}(3),\ \mathrm{Q} \star \mathcal{M} = \mathcal{M}\} \tag{1}$$

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where $\star$ stands for the Q action upon $\mathcal{M}$. Now consider a physical property $\mathcal{P}$ defined on $\mathcal{M}$, the set of operation letting the behavior be invariant is the physical symmetry group, denoted $\mathrm{G}_{\mathcal{P}}$:

$$\mathrm{G}_{\mathcal{P}} = \{\mathrm{Q} \in \mathrm{O}(3),\ \mathrm{Q} \star \mathcal{P} = \mathcal{P}\} \tag{2}$$

$\mathcal{P}$ will be described, in the following, by an nth-order tensor denoted T^n. $\mathbb{T}^n$ will stand for its related vector space. If the property is modeled by an even order tensor, the former definition could be reduced to the study of SO(3), the group of orthogonal transformations whose determinant equals $+1$. This hypothesis will be made in this Note.[1] The material symmetry group and the physical one are related by the mean of Neumann's principle [1]:

$$\mathrm{G}_{\mathcal{M}} \subseteq \mathrm{G}_{\mathcal{P}} \tag{3}$$

meaning that each operation leaving the material invariant leaves the physical property invariant. Nevertheless, as shown for tensorial properties using Hermann's theorems [2], physical properties can be more symmetrical than the material.

In $\mathcal{E}^3$ $\mathrm{G}_{\mathcal{P}}$ is conjugate with a *closed* subgroup of SO(3) [1]. The collection of those subgroups is [3]:

$$\Sigma := \{I, Z_p, D_p, \mathrm{SO}(2), \mathrm{O}(2), \mathcal{T}, \mathcal{O}, \mathcal{I}, \mathrm{SO}(3)\} \tag{4}$$

with I the identity group; Z_p the pth-order cyclic group, i.e. the symmetry group of a p-fold-invariant chiral figure; D_p the $(2p)$th-order dihedral group, i.e. the symmetry group of a regular p-gone[2]; SO(2) the continuous group of rotations and O(2) the continuous group of orthogonal transformations in 2-D. $\mathcal{T}$ will stand for the tetrahedron symmetry group, $\mathcal{O}$ for the octahedron one and $\mathcal{I}$ for the icosahedron one.

To study tensor symmetry classes an elementary parts decomposition is needed. Such a decomposition is, in literature, known as harmonic [4,3] or irreducible [5,6].

3. Tensor decomposition

3.1. Harmonic decomposition

The orthogonal irreducible decomposition of a tensor is known as harmonic decomposition. In $\mathcal{E}^3$, it is an O(3)-invariant decomposition well known in group representation theory. It allows us to decompose any finite order tensor as a sum of irreducible ones [6,5]. This decomposition can be written:

$$\mathrm{T}^n = \sum_{k,\tau} \mathrm{D}(n)^{k,\tau} \tag{5}$$

where tensors $\mathrm{D}(n)^{k,\tau}$ are components of the irreducible decomposition, k denotes the order of the harmonic tensor embedded in $\mathrm{D}(n)$ and τ separates same order terms. This decomposition establishes an isomorphism between $\mathbb{T}^n$ and a direct sum of harmonic tensor vector spaces $\mathbb{H}^k$ [4]. It is written

$$\mathbb{T}^n \cong \bigoplus_{k,\tau} \mathbb{H}^{k,\tau} \tag{6}$$

but as explained in [3] this decomposition is not unique. On the other side, the O(3)-isotypic decomposition [7] (Eq. (7)) grouping same order terms is unique:

$$\mathbb{T}^n \cong \bigoplus_{k=0}^{n} \alpha_k \mathbb{H}^k \tag{7}$$

with α_k being the $\mathbb{H}^k$ multiplicity in the decomposition. Harmonic tensors are totally symmetric and traceless. In $\mathbb{R}^3$ the associated vector space dimension is:

$$\dim \mathbb{H}^k = 2k + 1 \tag{8}$$

For simplicity sake, when there is no risk of misunderstanding, spaces $\alpha_k \mathbb{H}^k$ will be denoted K^{α_k}: i.e. the harmonic space order (K) to the power of its multiplicity (α_k). Family $\{\alpha_k\}$ is a function of tensor space order and index symmetries. Several methods exist to compute this family [5,6,8]. A classical example is the space of elasticity tensors. In 3-D, it is isomorphic to $0^2 \oplus 2^2 \oplus 4$ which is a 21-dimensional vector space [4]. This decomposition is O(3)-invariant, but can further be decomposed considering O(2) group action.

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[1] This hypothesis is made here just for sake of conciseness. The introduced method can be directly extended to odd order tensors.

[2] D_p contains Z_p and mirror symmetry.

3.2. Cartan decomposition

So O(2)-action will be considered here. Under this action the harmonic space $\mathbb{H}^k$ can be decomposed as [3]:

$$\mathbb{H}^k \cong \bigoplus_{j=0}^{k} \mathbb{K}_j^k \quad \text{with } \dim \mathbb{K}_j^k = \begin{cases} 1 & \text{if } j = 0 \\ 2 & \text{if } j \neq 0 \end{cases} \tag{9}$$

It is referred to as Cartan decomposition of a harmonic tensor space. The relation (9) implies a decomposition containing subspaces for each j within $[0;k]$. If $\mathbb{H}^{\star j}$ stands for the harmonic tensor space of order j in 2-D space, it could be shown that for each k $\mathbb{K}_j^k$ is isomorphic to $\mathbb{H}^{\star j}$. Therefore, $\mathbb{H}^k$ is isomorphic to the following space:

$$\mathbb{H}^k \cong \bigoplus_{j=0}^{k} \mathbb{H}^{*j} \tag{10}$$

The relation (7) may be thus rewritten:

$$\mathbb{T}^n \cong \bigoplus_{k=0}^{n} \alpha_k \left(\bigoplus_{j=0}^{k} \mathbb{H}^{*j} \right) \cong \bigoplus_{k=0}^{n} \sum_{j=k}^{n} \alpha_j \mathbb{H}^{*k} \cong \bigoplus_{k=0}^{n} \beta_k \mathbb{H}^{*k} \tag{11}$$

This O(2)-invariant decomposition grouping same order terms is the O(2)-isotypic decomposition of a 3-D tensor space.

Coming back to the elasticity example, one reminds that, in 3-D, its vector space is isomorphic to $0^2 \oplus 2^2 \oplus 4$. According to formula (11), its O(2)-isotypic decomposition will be $0^{*5} \oplus 1^{*3} \oplus 2^{*3} \oplus 3^* \oplus 4^*$.

4. Invariance condition

Knowing harmonic decomposition of the tensor vector space, dimension of any G-invariant subspace can easily be computed. In order to achieve such a goal, Z_p-invariance condition should be detailed for harmonic tensor.

4.1. Z_p-invariant harmonic tensor

Let $\mathrm{G}_{\mathrm{H}^k}$ be the group of transformations that let H^k be unchanged, i.e.:

$$\mathrm{Q} \in \mathrm{G}_{\mathrm{H}^k} \quad \Rightarrow \quad \mathrm{Q} \star \mathrm{H}^k = \mathrm{H}^k \tag{12}$$

$\mathbb{H}^k$ is isomorphic with $\bigoplus_{j=0}^{n} \mathbb{H}^{*j}$. So, any $\mathrm{H}^k \in \mathbb{H}^k$ is defined by a family of tensors: $\{\mathrm{H}^{*j}\}$. The family order is obviously $j+1$. Thus the invariance condition on H^k can be expressed by $j+1$ conditions on the elements of the O(2)-invariant decomposition. These conditions are of $j+1$ different types according to the 2-D harmonic tensor order, i.e.

$$\mathrm{Q} \star_j \mathrm{H}^{*j} = \mathrm{H}^{*j} \tag{13}$$

where $\star_j$ is SO(2)-action of on $\mathbb{H}^{*j}$. This action shall easily be expressed in the sequel.

Let $\mathrm{H}^{*j} = (s_j, t_j)$ be a $\mathbb{H}^{*j}$ vector. Consider a plane rotation, $\mathrm{Q}_{rot} \in \mathrm{SO}(2)$ belonging to Z_p. As shown in [3] Q_{rot} acts on $\mathbb{H}^{*j}$ as a $Z_{\frac{p}{j}}$ generator, i.e. the rotation order p divided by the Cartan subspace indices.

$$\mathrm{Q}_{rot} = \begin{pmatrix} \cos\frac{2\pi}{p} & -\sin\frac{2\pi}{p} \\ \sin\frac{2\pi}{p} & \cos\frac{2\pi}{p} \end{pmatrix}; \qquad \mathrm{Q}_{rot} \star \mathrm{H}^{*j} = \begin{pmatrix} \cos\frac{2j\pi}{p} & -\sin\frac{2j\pi}{p} \\ \sin\frac{2j\pi}{p} & \cos\frac{2j\pi}{p} \end{pmatrix} \begin{pmatrix} s_j \\ t_j \end{pmatrix} \tag{14}$$

So, if Q_{rot} belongs to $\mathrm{G}_{\mathrm{H}^k}$ then each H^{*j} must be Q_{rot}-invariant. The matrix of the Q_{rot} action on H^{*j} will be denoted $\mathrm{Q}_{rot}^{(j)}$. The invariance condition of H^{*j} is the solution of $(\mathrm{Q}_{rot}^{(j)} - \mathrm{Id})\mathrm{H}^{*j} = 0$. In other words, $\ker(\mathrm{Q}_{rot}^{(j)} - \mathrm{Id})$ has to be studied. A direct calculation shows that the H^{*j} invariance condition under Z_p-action is:

$$j = tp, \quad t \in \mathbb{N} \tag{15}$$

Thus, if $j \neq tp$ then H^{*j} is equal to 0. Therefore ${}^{Z_p}\mathbb{H}^k$, the Z_p-invariant $\mathbb{H}^k$ planar decomposition, will be:

$${}^{Z_p}\mathbb{H}^k = \bigoplus_{0 \leqslant m \leqslant \lfloor \frac{k}{p} \rfloor} \mathbb{H}^{*mp} \tag{16}$$

where $\lfloor . \rfloor$ is the floor function.

Let us consider $\mathbb{F}\mathrm{ix}_{\mathbb{H}^k}(Z_p)$ the linear subspace of $\mathbb{H}^k$ that contains elements fixed under Z_p-action; its dimension is:

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{H}^k}(Z_p) = 2\left\lfloor \frac{k}{p} \right\rfloor + 1 \tag{17}$$

This relation allows us to determine the dimension of any tensor space left fixed under any SO(3) subgroup action. A group decomposition will be introduced to highlight this fact.

4.2. Disjoint union decomposition

Any SO(3) subgroups could be decomposed into disjoint unions of cyclic groups. We got the following result [3]:

$$D_p = \dot{\cup}^p Z_2 \,\dot{\cup}\, Z_p; \qquad \mathcal{T} = \dot{\cup}^4 Z_3 \,\dot{\cup}^3\, Z_2; \qquad \mathcal{O} = \dot{\cup}^3 Z_4 \,\dot{\cup}^4\, Z_3 \,\dot{\cup}^6\, Z_2; \qquad \mathcal{I} = \dot{\cup}^6 Z_5 \,\dot{\cup}^{10}\, Z_3 \,\dot{\cup}^{15}\, Z_2 \tag{18}$$

where $\dot{\cup}^n$ stands for n repetitions of $\dot{\cup}$ (disjoint union) which definition is:

Definition 4.1. Let $H_1, \ldots, H_n$ be subgroups of a group Σ. We say that Σ is the disjoint union of $H_1, \ldots, H_n$ if:

$$\Sigma = H_1 \cup \cdots \cup H_n \tag{19}$$

$$H_i \cap H_j = e, \quad \text{for all } i \neq j \tag{20}$$

So the notation $\dot{\cup}$ is used to denote disjoint union and e stands for the identity of Σ.
We further have, in the same reference, the following theorem:

Theorem 4.2. *Let Γ be a compact Lie group acting on V and let $\Sigma \subset \Gamma$ be a Lie subgroup. If Σ admits a disjoint union decomposition, i.e. $\Sigma = H_1 \,\dot{\cup} \cdots \dot{\cup}\, H_k$ then*:

$$\dim\bigl(\mathbb{F}\mathrm{ix}(\Sigma)\bigr) = \frac{1}{|\Sigma|}\left[\sum_{i=1}^{k} |H_i| \dim\bigl(\mathbb{F}\mathrm{ix}(H_i)\bigr) - (k-1)\dim(V)\right] \tag{21}$$

where $|G|$ stands for the order of G. Application of Theorem 4.2 to subgroup of SO(3) acting on $\mathbb{H}^k$ allows us to obtain the dimension of G-invariant harmonic space.

Using such a result allows us to compute the dimension of any G-invariant harmonic subspace (G in the set of SO(3) closed sub-groups).

4.3. Dimension of G-invariant harmonic space

So a straightforward application of those results leads to the following set of relations:

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{H}^k}(D_p) = \begin{cases} \lfloor \frac{k}{p} \rfloor + 1 & k = 2n \\ \lfloor \frac{k}{p} \rfloor & k = 2n+1 \end{cases}; \qquad \dim \mathbb{F}\mathrm{ix}_{\mathbb{H}^k}(\mathcal{T}) = 2\left\lfloor \frac{k}{3} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor - k + 1 \tag{22}$$

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{H}^k}(\mathcal{O}) = \left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{k}{3} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor - k + 1; \qquad \dim \mathbb{F}\mathrm{ix}_{\mathbb{H}^k}(\mathcal{I}) = \left\lfloor \frac{k}{5} \right\rfloor + \left\lfloor \frac{k}{3} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor - k + 1 \tag{23}$$

The two former relations could be rewritten in order to make the tetrahedral symmetry to appear:

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{H}^k}(\mathcal{O}) = \left\lfloor \frac{k}{4} \right\rfloor - \left\lfloor \frac{k}{3} \right\rfloor + \dim \mathbb{F}\mathrm{ix}_{\mathbb{H}^k}(\mathcal{T}); \qquad \dim \mathbb{F}\mathrm{ix}_{\mathbb{H}^k}(\mathcal{I}) = \left\lfloor \frac{k}{5} \right\rfloor - \left\lfloor \frac{k}{3} \right\rfloor + \dim \mathbb{F}\mathrm{ix}_{\mathbb{H}^k}(\mathcal{T}) \tag{24}$$

Now, the knowledge of the harmonic decomposition of a tensor will allow us to construct, according to formula (7), the dimension of G-invariant tensor subspaces. Some new and analytical relations could be derived according to those formulas.

5. *G*-invariant tensor subspaces

Combining results previously obtained with the tensor space harmonic decomposition, the following formulas are obtained:

5.1. Z_p-invariance

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}(Z_p) = 2\sum_{k=0}^{n} \alpha_k \left\lfloor \frac{k}{p} \right\rfloor + \sum_{k=0}^{n} \alpha_k \tag{25}$$

When $p > k$ we obtain $\lfloor \frac{k}{p} \rfloor = 0$ and so $\beta_{ht} = \sum_{k=0}^{n} \alpha_k$ is the number of transverse hemitropic coefficients. β_{ht} is the dimension of a SO(2)-invariant tensor.

5.2. D_p-invariance

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}(D_p) = \sum_{k=0}^{n} \alpha_k \left\lfloor \frac{k}{p} \right\rfloor + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \alpha_{2k} \tag{26}$$

When $p > k$ we obtain $\lfloor \frac{k}{p} \rfloor = 0$ and so $\beta_{ht} = \sum_{k=0}^{n} \alpha_k$ is the number of transverse isotropic coefficients. β_{it} is the dimension of an O(2)-invariant tensor.

5.3. $\mathcal{T}$, $\mathcal{O}$ and $\mathcal{I}$-invariance

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}(\mathcal{T}) = \sum_{k=0}^{n} \alpha_k \left(2\left\lfloor \frac{k}{3} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor - k + 1 \right) \tag{27}$$

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}(\mathcal{O}) = \sum_{k=0}^{n} \alpha_k \left(\left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{k}{3} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor - k + 1 \right) \tag{28}$$

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}(\mathcal{I}) = \sum_{k=0}^{n} \alpha_k \left(\left\lfloor \frac{k}{5} \right\rfloor + \left\lfloor \frac{k}{3} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor - k + 1 \right) \tag{29}$$

The two former expressions could be recasted

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}(\mathcal{O}) = \dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}(D_4) - \dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}(D_3) + \dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}(\mathcal{T}) \tag{30}$$

and

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}(\mathcal{I}) = \dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}(D_5) - \dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}(D_3) + \dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}(\mathcal{T}) \tag{31}$$

or:

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}(\mathcal{I}) = \dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}(D_5) - \dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}(D_4) + \dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}(\mathcal{O}) \tag{32}$$

6. Examples

Two examples will be considered: the classical fourth-order tensor of elasticity and the sixth-order tensor of Mindlin strain gradient elasticity.

6.1. Classical elasticity

Let's consider $\mathbb{E}\mathrm{la}_{\mathrm{C}}$ the vector space of classical elasticity tensor. It is the vector space of fourth-order tensor possessing the classical minor and major index symmetries, i.e.:

$$\mathrm{C}_{\underline{(ij)}\,\underline{(kl)}} \tag{33}$$

where (.) stands for the minor symmetry and $\underline{.}$ for the major one. It has been shown [4] that this vector space can be decomposed as follows:

$$\mathbb{E}\mathrm{la}_{\mathrm{C}} \cong 2\mathbb{H}^0 \oplus 2\mathbb{H}^2 \oplus \mathbb{H}^4 \tag{34}$$

Table 1
The number of coefficients for each physical symmetry class.

$G_{\mathcal{M}}$	I	Z_2	D_2	Z_3	D_3	Z_4	D_4	$Z_{n>4}$, $D_{n>4}$	$\mathcal{T}$, $\mathcal{O}$	$\mathcal{I}$, SO(3)
$G_{\mathbb{E}\mathrm{la}_C}$	I	Z_2	D_2	Z_3	D_3	Z_4	D_4	O(2)	$\mathcal{O}$	SO(3)
dim	21	13	9	7	6	7	6	5	3	2

Table 2
17 different systems of symmetry exist for the sixth-order tensor of Mindlin strain gradient elasticity.

$G_{\mathcal{M}}$	I	Z_2	D_2	Z_3	D_3	Z_4	D_4	Z_5	D_5	Z_6	D_6	$Z_{n>6}$	$D_{n>6}$	$\mathcal{T}$	$\mathcal{O}$	$\mathcal{I}$	SO(3)
$G_{\mathbb{E}\mathrm{la}_A}$	I	Z_2	D_2	Z_3	D_3	Z_4	D_4	Z_5	D_5	Z_6	D_6	SO(2)	O(2)	$\mathcal{T}$	$\mathcal{O}$	$\mathcal{I}$	SO(3)
dim	171	91	51	57	34	45	28	35	23	33	22	31	21	17	11	6	5

and so $\mathbb{E}\mathrm{la}_C$ is defined by the following $\{\alpha_k\}$ family: $\{2, 0, 2, 0, 1\}$. The following relations are thus obtained:

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{E}\mathrm{la}_C}(Z_p) = 2\left(2\left\lfloor \frac{2}{p} \right\rfloor + \left\lfloor \frac{4}{p} \right\rfloor\right) + 5; \qquad \dim \mathbb{F}\mathrm{ix}_{\mathbb{E}\mathrm{la}_C}(D_p) = \left(2\left\lfloor \frac{2}{p} \right\rfloor + \left\lfloor \frac{4}{p} \right\rfloor\right) + 5 \tag{35}$$

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{E}\mathrm{la}_C}(\mathcal{O}) = \dim \mathbb{F}\mathrm{ix}_{\mathbb{E}\mathrm{la}_C}(\mathcal{T}) = 3; \qquad \dim \mathbb{F}\mathrm{ix}_{\mathbb{E}\mathrm{la}_C}\big(\mathrm{SO}(3)\big) = \dim \mathbb{F}\mathrm{ix}_{\mathbb{E}\mathrm{la}_C}(\mathcal{O}) = 2 \tag{36}$$

Using these relations the array shown in Table 1 could be constructed, which gives the number of coefficients for each physical symmetry class.

6.2. Mindlin strain gradient elasticity

Let's consider $\mathbb{E}\mathrm{la}_A$ the vector space of Mindlin strain gradient elasticity tensor. It is the vector space of sixth-order tensor possessing minor and major index symmetries [9], i.e.:

$$\mathrm{A}_{\underline{(ij)k}\,\underline{(lm)n}} \tag{37}$$

It has been shown [2] that this vector space can be decomposed as follows:

$$\mathbb{E}\mathrm{la}_A \cong 5\mathbb{H}^0 \oplus 4\mathbb{H}^1 \oplus 10\mathbb{H}^2 \oplus 5\mathbb{H}^3 \oplus 5\mathbb{H}^4 \oplus \mathbb{H}^5 \oplus \mathbb{H}^6 \tag{38}$$

and so $\mathbb{E}\mathrm{la}_A$ is defined by the following $\{\alpha_k\}$ family: $\{5, 4, 10, 5, 5, 1, 1\}$. The following relations are thus obtained:

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{E}\mathrm{la}_A}(Z_p) = 2\left(4\left\lfloor \frac{1}{p} \right\rfloor + 10\left\lfloor \frac{2}{p} \right\rfloor + 5\left\lfloor \frac{3}{p} \right\rfloor + 5\left\lfloor \frac{4}{p} \right\rfloor + \left\lfloor \frac{5}{p} \right\rfloor + \left\lfloor \frac{6}{p} \right\rfloor\right) + 31 \tag{39}$$

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{E}\mathrm{la}_A}(D_p) = 4\left\lfloor \frac{1}{p} \right\rfloor + 10\left\lfloor \frac{2}{p} \right\rfloor + 5\left\lfloor \frac{3}{p} \right\rfloor + 5\left\lfloor \frac{4}{p} \right\rfloor + \left\lfloor \frac{5}{p} \right\rfloor + \left\lfloor \frac{6}{p} \right\rfloor + 21 \tag{40}$$

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{E}\mathrm{la}_A}(\mathcal{T}) = 17; \qquad \dim \mathbb{F}\mathrm{ix}_{\mathbb{E}\mathrm{la}_A}(\mathcal{O}) = 11 \tag{41}$$

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{E}\mathrm{la}_C}(\mathcal{O}) = 6; \qquad \dim \mathbb{F}\mathrm{ix}_{\mathbb{E}\mathrm{la}_C}\big(\mathrm{SO}(3)\big) = 5 \tag{42}$$

Using these relations the array shown in Table 2 could be constructed. According to this procedure, it is shown that 17 different systems of symmetry exist for the sixth-order tensor of Mindlin strain gradient elasticity.

7. Conclusion

In this Note, analytical formulas giving the dimension of a subspace left fixed under SO(3)-subgroups action have been presented. To our knowledge most of those relations were currently unknown. To be applied to other linear behaviors the only thing to know is the harmonic decomposition of the tensor vector space. An easy method for computing this decomposition was proposed in [2]. Following the former approach extension to O(3)-subgroups is straightforward.

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