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On the tallest column

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ABSTRACT

A new approach is proposed to the study of the classical problem about the highest column. The existence and the uniqueness of the solution is proved for the first time. The method is based on the study of critical points of a suitable nonlinear functional.

RÉSUMÉ

On propose une nouvelle approche au problème classique de la forme d'une colonne la plus haute. On prouve pour la première fois qu'une telle colonne existe et est unique. La méthode est basée sur l'étude des points critiques d'une fonctionnelle nonlinéaire.

Version française abrégée

On considère une colonne verticale de l'hauteur H et de volume V de matériau de densité ρ et de module de Young E. Soit $A(z)$ l'aire et $I(z)$ le moment de l'inertie, par rapport à l'axe orthogonal au plan de l'inclinaison, de la section à l'hauteur z. Supposons que toute les sections soient semblables et orientées de la même manière. Alors $I(z)=\alpha A^2(z)$, où α est une constante determinée par la forme de la section. On suppose que le sommet de la colonne $z=H$ est libre et la base $z=0$ est encastrée. Soit $y(z)$ l'ecart lateral de la colonne par rapport à la position verticale dans le plan de l'inclinaison.

On considère le problème de determination de la forme de la colonne, caracterisée par $A(z)$, de telle manière que la colonne est stable et l'hauteur H est maximale possible. Par la théorie de Bernoulli–Euler l'ecart $y(z)$ satisfait l'équation

$$E\alpha A(z)^2 y_{zz} = \int_z^H \rho g A(t)\left[y(t)-y(z)\right]\mathrm{d}t, \quad 0 \leqslant z \leqslant H$$

La condition en $z=0$ implique que $y(0)=y'(0)=0$. On suppose que le volume $V=\int_0^H A(z)\,\mathrm{d}z$ est donné.

Notre résultat principal est le théorème suivant :

Théorème. *Il existe une solution du problème et elle est unique.*

Pour le démontrer on considère la foncionnelle nonlinéaire $G[v]=I_1+\sqrt{I_0 I_2}$, où

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$$I_j = \int_0^\infty v'(x)^3 v(x)^j \, dx, \quad j = 0, 1, 2$$

et on trouve son minimum sur la variété

$$M = \left\{ v \in C^1(0, \infty); \ v(0) = 0, \ \lim_{y \to 0} v'(y) y^{1/4} = 4 \right\}$$

dans la classe S_1 de fonctions v croissantes et telles que la fonction $v'(y)y^{1/4}$ est décroissante.

1. Let us consider a vertical column of height H and volume V of material of mass density ρ and Young's modulus E. Let $A(z)$ be the area and $I(z)$ the moment of inertia, about an axis perpendicular to the plane of bending, of the cross section at height z. Assume that all cross sections are similar and similarly oriented. Then $I(z) = \alpha A^2(z)$ where α is a constant determined by the shape of the cross section. We assume that the top of the column at $z = H$ is free and the bottom at $z = 0$ is clamped. Let $r(z)$ be the lateral deflection of the column from the vertical in a fixed vertical plane, called the plane of bending.

We consider the problem of determining the shape of the column, characterized by $A(z)$ in such a way that the column is stable against buckling and the height H is maximal possible. By the Bernoulli–Euler theory the deflection $r(z)$ satisfies the equation

$$E\alpha A(z)^2 r_{zz} = \int_z^H \rho g A(t) \left[r(t) - r(z) \right] dt, \quad 0 \leqslant z \leqslant H$$

The condition at $z = 0$ implies that $r(0) = r'(0) = 0$ and the volume $V = \int_0^H A(z) \, dz$ is given. The condition at $z = H$ is

$$\lim_{z \to H} A(z)^2 r'(z) = 0$$

Setting

$$x = z/H, \qquad a(x) = HA(xH)/V, \qquad \eta(x) = r(xH)/H, \qquad \lambda = \rho g H^4 / \alpha E V$$

we obtain

$$a^2(x)\eta_{xx} = \lambda \int_x^1 a(t) \left[\eta(t) - \eta(x) \right] dt, \quad 0 \leqslant x \leqslant 1$$

$$\eta(0) = \eta'(0) = 0, \qquad \lim_{x \to 1} a(x)^2 \eta'(x) = 0, \qquad \int_0^1 a(x) \, dx = 1$$

Let us name *critical* the minimal value of λ for which this boundary problem has a non-trivial solution η. The problem now is to find the function $a(x) \geqslant 0$ such that the critical value of λ is maximal possible.

Setting $\varphi(x) = \eta'(x)$ and differentiating the equation with respect to x we obtain

$$\left(a^2(x)\varphi'(x) \right)' + \lambda \int_x^1 a(t) \, dt \, \varphi(x) = 0, \quad 0 \leqslant x \leqslant 1$$

$$\varphi(0) = 0, \qquad \lim_{x \to 1} a^2(x)\varphi'(x) = 0$$

The problem is to find a column shape $a(x) \geqslant 0$ such that the λ is as large as possible and $\varphi(x) > 0$ in $]0, 1[$.

2. L. Euler [1] posed and solved the problem of buckling of prismatic columns under self-weight. Keller and Niordsen [2] formulated and solved numerically the problem for columns of variable sections when the height and the volume of the column are fixed. Their solution was criticized by Cox and McCarthy [3], who fairly have remarked that the existence of the optimal column cannot be taken for granted, and that the problem can have a continuous spectrum, so that the eigenvalue does not vary smoothly over the class of admissible designs, calling in doubt the method and the result of [2]. However, they themselves also have not proved the existence of the tallest column. Unfortunately their constructions are based on a wrong formula for the Green function in Section 3 of [3].

Here we prove for the first time the existence and the uniqueness of the optimal column, using our method proposed in [4], where a similar problem was studied for the weightless columns. In particular we justify the results of [2] and the numerical method of Keller and Niordsen. Our principal idea is to consider the suitable nonlinear functional whose critical points are the min–max points of the variational problem.

3. State now the problem as follows:

$\mathcal{P}$: *to find a positive function a in* $C(0,1)$ *such that*

$$\int_0^1 a(t)\,\mathrm{d}t = 1 \tag{1}$$

and the critical load

$$\lambda = \inf_\varphi L[a,\varphi], \quad \textit{where } L[a,\varphi] \equiv \inf_\varphi \frac{\int_0^1 a(x)^2\varphi'(x)^2\,\mathrm{d}x}{\int_0^1(\int_x^1 a(t)\,\mathrm{d}t)\varphi(x)^2\,\mathrm{d}x} \tag{2}$$

is maximal.

Here the infimum is taken over the class of all positive functions φ from $C^1(0,1)$ such that

$$\varphi(0) = 0, \qquad a^2(1)\varphi'(1) = 0 \tag{3}$$

The Euler equation has the form

$$\big(a^2(x)\varphi'(x)\big)' + \lambda\int_x^1 a(t)\,\mathrm{d}t\,\varphi(x) = 0 \tag{4}$$

4. Let S_0 be the class of monotone increasing functions φ such that $\varphi(0) = 0$; let A_0 be the class of monotone decreasing functions a such that $a(1) = 0$ and $\int_0^1 a(t)\,\mathrm{d}t = 1$.

Lemma 4.1. *Let φ be a function from the class S_0 and $\varphi'(0) = 1$. Let*

$$a(x) = \frac{c}{\varphi'(x)^2} + \frac{\lambda}{2\varphi'(x)^2}\int_0^x \varphi(t)^2\,\mathrm{d}t, \quad \textit{where } c = a(0) \tag{5}$$

and

$$\lambda = L[a,\varphi], \qquad \int_0^1 a(t)\,\mathrm{d}t = 1 \tag{6}$$

Then

$$F[\varphi] = \frac{2}{\lambda}$$

where

$$F[\varphi] = I_1 + \sqrt{I_0 I_2}, \qquad I_j = \int_0^1 \frac{\mathrm{d}x}{\varphi'(x)^2}\left(\int_0^x \varphi(t)^2\,\mathrm{d}t\right)^j, \quad j = 0,1,2 \tag{7}$$

The proof is simple but rather technical. It will be published in an other article.

5. Now we consider the functional $F[\varphi] = I_1 + \sqrt{I_0 I_2}$.

In order to describe the class of admissible functions φ suppose that both equations (4) and (5) hold. From Eq. (4) we see that the function $a(x)^2\varphi'(x)$ is decreasing and Eq. (5) implies that the function $a(x)\varphi'(x)^2$ is increasing. Using the second equality in (3) we conclude that $\varphi'(x) \geqslant 0$ in $(0,1)$.

Therefore, the functions $a(x)^2\varphi'(x)^4$ and $\varphi'(x)^3$ are increasing, and the functions $a(x)^4\varphi'(x)^2$ and $a(x)^3$ are decreasing. Thus the function $a(x)$ is monotone decreasing and the function $\varphi'(x)$ is monotone increasing. Since $\varphi(0) = 0$ we see that $\varphi(x)$ is an increasing positive convex function.

Since the functions $\varphi(x)$ and $\varphi'(x)$ are monotone increasing there exists the inverse function $\psi \in C^1[0,\infty)$ such that $\psi(\varphi(x)) = x$ for $x \in [0,1)$. It is easy to check that

$$\psi(0) = 0, \qquad \psi'(0) = 1, \qquad \psi(\infty) = 1, \qquad \psi'(\infty) = 0, \qquad \psi(x) \geqslant 0, \qquad \psi'(x) \geqslant 0$$

and

$$I_j = \int_0^\infty \psi'(x)^3 \left(\int_0^x t^2 \psi'(t)\, \mathrm{d}t \right)^j \mathrm{d}x, \quad j = 0, 1, 2$$

Setting

$$u(x) = \int_0^x t^2 \psi'(t)\, \mathrm{d}t, \qquad y = x^4/4, \qquad v(y) = u(x)$$

we obtain that $\mathrm{d}y = x^3\, \mathrm{d}x$, $u'(x) = x^2 \psi'(x) = v'(y) x^3$ and

$$I_j = \int_0^\infty \frac{u'(x)^3}{x^6} u(x)^j\, \mathrm{d}x = \int_0^\infty v'(y)^3 v(y)^j\, \mathrm{d}y, \quad j = 0, 1, 2$$

and

$$\psi(\infty) = \int_0^\infty \psi'(t)\, \mathrm{d}t = \int_0^\infty \frac{u'(x)}{x^2}\, \mathrm{d}x = \int_0^\infty \frac{v'(y)}{2\sqrt{y}}\, \mathrm{d}y = \int_0^\infty \frac{v(y)}{4y\sqrt{y}}\, \mathrm{d}y = 1$$

Let us remark that $u(0) = 0$ and for small x we have $u'(x) = x^2(1 + o(1)) = \sqrt{4y}(1 + o(1))$. Therefore, as $x \to 0$,

$$v'(y) = u'(x) x^{-3} = 1/x\big(1 + o(1)\big) = (4y)^{-1/4}\big(1 + o(1)\big)$$

6. Consider the following problem:

$\mathcal{P}_1$: *to find the minimum of*

$$G[v] = I_1 + \sqrt{I_0 I_2} = \int_0^\infty v'(y)^3 v(y)\, \mathrm{d}y + \left(\int_0^\infty v'(y)^3\, \mathrm{d}y \int_0^\infty v'(y)^3 v(y)^2\, \mathrm{d}y \right)^{1/2}$$

on the manifold

$$M = \left\{ v \in C^1(0, \infty);\ v(0) = 0,\ \lim_{y \to 0} v'(y) y^{1/4} = \frac{1}{\sqrt{2}},\ \int_0^\infty \frac{v(y)}{y\sqrt{y}}\, \mathrm{d}y = 4 \right\}$$

in the class S_1 of increasing functions v such that the function $v'(y)y^{1/4}$ is decreasing.

Theorem 6.1. *There exists a unique solution of problem $\mathcal{P}_1$.*

Proof. The existence follows easily from the compactness of a minimizing sequence by the Helly theorem. The proof of the uniqueness is more complicated, it is based on the asymptotic analysis of solutions to the Lagrange–Euler equation. □

7.

Lemma 7.1. *For any function $a \in A_0$ there exists a function $\varphi \in S_0$ such that* (5) *holds.*

The proof uses an a priori estimate of solutions to Eq. (5), which guarantees the existence of the solution on the whole interval $[0, 1[$.

8.

Theorem 8.1. *There exists a unique solution of problem $\mathcal{P}$.*

Proof. *Existence.* Let $w_0(y)$ be the point of minimum of the functional G, found in Theorem 6.1, so that $G[w_0] = 2/\lambda$. Using the above construction we can find the corresponding functions $\varphi_0(x)$ and $a_0(x)$. By Lemma 4.1, $L[a_0, \varphi_0] = \lambda$. We can verify that $\inf_{\varphi \in S_0} L[a_0, \varphi] = \lambda$ and for any admissible a we have

$$\inf_{\varphi \in S_0} L[a, \varphi] \leqslant \lambda$$

so that a_0 is the solution to problem $\mathcal{P}$.

Uniqueness. Let a_0, φ_0 be the functions defined above and $L[a_0, \varphi_0] = \lambda$.

As we have seen above, $F[\varphi_0] = 2/\lambda$. We can verify that this is the minimal possible value of the functional F on M. Therefore, φ_0 is the minimum point of F on M and by Theorem 6.1 is unique. Thus, a_0 is also unique. $\square$

Remark. Theorem 8.1 holds with some small modifications in the proof in the case of a loaded column, when the Bernoulli–Euler equation has the form

$$E\alpha A(z)^2 y_{zz} = \int_z^H \rho g A(t)\big[y(t) - y(z)\big]\,\mathrm{d}t + P\big[y(H) - y(z)\big], \quad 0 \leqslant z \leqslant H$$

where P is a vertical load at the top of the column.

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