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A simple parameter-free entropy correction for approximate Riemann solvers

Une correction entropique non paramétrique simple pour les solveurs de Riemann approchés

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ABSTRACT

We present here a simple and general non-parametrized entropy-fix for the computation of fluid flows involving sonic points in rarefaction waves. It enables to improve the stability and the accuracy of approximate Riemann solvers. It is also applied to MHD flows.

RÉSUMÉ

On présente dans cette note une correction entropique non paramétrique simple et générale pour la simulation d'écoulements de fluides comportant des points soniques en zone de détente. Celle-ci permet d'accroître la stabilité et la précision de solveurs de Riemann approchés. Cette correction est aussi appliquée aux équations de la MHD idéale.

Version française abrégée

La simulation de systèmes hyperboliques de lois de conservation nécessite le développement de solveurs adaptés. Dans le cadre de la dynamique des gaz, de nombreux solveurs de Riemann ont été développés dans la littérature. Deux des plus connus sont vraisemblablement le schéma de Godunov [1] et le schéma de Roe [2]. Le schéma de Godunov, avec solveur de Riemann exact, a la propriété de contenir sa propre correction entropique aux points de détente sonique. De même, le schéma de Rusanov, [3], permet de simuler sans correction les détentes soniques. L'inconvénient de ce schéma réside bien entendu dans son assez faible précision (voir [4]).

Par contre, il est bien connu que le solveur de Riemann approché proposé par Roe nécessite une correction entropique. Si celle-ci n'est pas mise en oeuvre, le schéma fait apparaître un choc (non entropique) aux points soniques, pouvant conduire à des densités ou pressions physiquement inadmissibles, et même à un arrêt du calcul lorsque le rapport de pression initial est très important. De manière classique, on utilise pour le solveur de Roe une correction entropique telle que celle proposée dans [5] ou [6]. Les schémas VFRoe [7] et VFRoe-ncv [8] sont des simplifications du schéma de Godunov qui ne nécessitent

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pas le calcul et la diagonalisation d'une linéarisée de Roe. Ils peuvent être employés pour des systèmes pour lesquels on ne connaît pas de linéarisée de Roe facile à calculer. Il sont aussi intéressants du point de vue des performances algorithmiques. Comme le schéma de Roe, ils nécessitent une correction entropique dans les détentes soniques. L'analyse classique, proposée dans [5] ou [6] par exemple, utilise de façon essentielle les propriétés de la matrice de Roe et ne peut donc pas être étendue simplement aux schémas de type VFRoe.

L'approche proposée dans cette note consiste très simplement à localiser les interfaces entre deux volumes de contrôle séparant deux valeurs du nombre de Mach de part et d'autre de 1, dans les zones régulières, puis à remplacer localement sur cette interface le flux du solveur de Riemann approché par un flux de type Rusanov [3]. La consistance du flux et la conservativité du schéma sont préservées, l'ordre de convergence global du schéma reste inchangé, et la précision globale à maillage donné se trouve nettement améliorée par rapport à d'autres corrections entropiques paramétriques. Pour les équations de la dynamique des gaz, il est alors tout à fait possible d'envisager des calculs en présence de très faibles densités et pressions (Section 3) ; il en est de même pour la simulation de problèmes de magnétohydrodynamique ([9], et Section 4). Cette correction est également très utile si l'on cherche à réaliser certaines simulations nécessitant une formulation à section variable, [10]. L'approche permet de traiter des problèmes de Riemann mettant en jeu des rapports de pression (ou densité) de l'ordre de 10^5. Sur les maillages les plus grossiers, la perturbation résiduelle au niveau du point sonique est semblable à celle observée pour un schéma de Godunov exact. De nombreuses applications à d'autres systèmes deviennent alors envisageables [11]. L'extension au cadre multidimensionnel est immédiate.

1. Introduction

The computation of approximate solutions of hyperbolic systems of conservation laws requires to develop suitable Riemann solvers. The most well-known among them are probably the Godunov scheme [1] and the Roe scheme [2]. The first among the latter two does not require any entropy correction at sonic points in rarefaction waves, whereas the second one does. For Roe's approximate Riemann solver, modifications of the numerical flux have been proposed in [5,6]. The approach discussed herein is devoted to the VFRoe-ncv solver, for which no Roe linearization is required. Actually, the basic idea here is to take advantage of the fact that: (i) though it is not very accurate, the Rusanov scheme provides convergent approximations, and does not require any entropy correction, and: (ii) sonic points in smooth genuinely non-linear fields can be tracked in a very simple way. Hence, we propose to: (i) first locate interfaces between cells containing a subsonic and a supersonic states respectively, with continuous transition, and then: (ii) replace the numerical flux based on the approximate Riemann solver by a Rusanov type flux, [3], at this particular interface. We present the modified approximate Riemann solver in the next section, and then show some applications in the framework of gas dynamics and magneto-hydrodynamics.

2. A parameter-free entropy correction

We focus on a hyperbolic system of conservation laws, noting $W(x,t) \in \mathbb{R}^p$ the state variable, whose variations are governed by:

$$\frac{\partial W}{\partial t} + \frac{\partial f(W)}{\partial x} = 0 \tag{1}$$

$f(W) \in \mathbb{R}^p$ denotes the continuous flux. We also assume that there exists some Lax entropy–entropy flux pair $(\eta(W), f_\eta(W))$, such that the following entropy inequality holds: $0 \leqslant \frac{\partial \eta}{\partial t} + \frac{\partial f_\eta}{\partial x}$ for weak solutions of (1).

If W_i^n denotes the approximated mean value of $W(x,t)$ at time t^n over cell i, Finite Volume approximations are obtained by updating the following cell scheme, that is based on an Approximate Riemann Solver (ARS):

$$h_i\big(W_i^{n+1} - W_i^n\big) + \Delta t^n \big(F^{\mathrm{ARS}}\big(W_i^n, W_{i+1}^n\big) - F^{\mathrm{ARS}}\big(W_{i-1}^n, W_i^n\big)\big) = 0 \tag{2}$$

The two-point numerical flux $F^{\mathrm{ARS}}(W_L, W_R)$ is of course assumed to be consistent ($F^{\mathrm{ARS}}(W,W) = f(W)$), and the time step Δt^n is chosen in such a way that the CFL constraint associated to the scheme holds. At the interface Γ_{ij} separating two cells i and j, an entropy correction is needed for the Approximate Riemann Solver if some eigenvalue λ_k crosses 0 in a smooth wave, for some given index k associated with a genuinely non-linear (GNL) field.

The new scheme is thus obtained considering the Modified Approximate Riemann Solver (MARS):

– If $\lambda_k(W_L) \leqslant 0 \leqslant \lambda_k(W_R)$ in a GNL wave k, then:

$$F^{\mathrm{MARS}}(W_L, W_R) = F^{\mathrm{Rusanov}}(W_L, W_R) \tag{3}$$

– Otherwise:

$$F^{\mathrm{MARS}}(W_L, W_R) = F^{\mathrm{ARS}}(W_L, W_R) \tag{4}$$

We recall here that the Rusanov flux is defined by:

$$F^{\text{Rusanov}}(W_L, W_R) = \big(f(W_L) + f(W_R) - r_{LR}(W_R - W_L)\big)/2$$

We note $r_{LR} = \max(r(J(W_L)), r(J(W_R)))$, where $r(J(W))$ denotes the spectral radius of the Jacobian matrix $J(W) = \frac{\partial f(W)}{\partial W}$. The resulting scheme is obviously:

$$h_i\big(W_i^{n+1} - W_i^n\big) + \Delta t^n\big(F^{\text{MARS}}\big(W_i^n, W_{i+1}^n\big) - F^{\text{MARS}}\big(W_{i-1}^n, W_i^n\big)\big) = 0 \tag{5}$$

In order to be less diffusive it is also possible to replace formula (3) by

$$F^{\text{MARS}}(W_L, W_R) = F^{\text{ARS}}(W_L, W_R) - \frac{1}{2}\min\big(\big|\lambda_k(W_L)\big|, \big|\lambda_k(W_R)\big|\big)(W_R - W_L) \tag{6}$$

The justification of this formula is heuristic: if we consider a Riemann problem with a sonic point in the k-wave, the quantity $\min(|\lambda_k(W_L)|, |\lambda_k(W_R)|)$ is the minimal shift in the initial velocity that would remove the sonic point. Thus, we decide to add the corresponding numerical viscosity to the numerical flux in order to remove the possible non-physical solutions. An advantage of (6) upon (3) is that if F^{ARS} is continuous, then $F^{\text{MARS}}(W_L, W_R)$ is also continuous with respect to W_L and W_R.

3. Application to gas dynamics

We focus here on the gas dynamics equations, thus $W = (\rho, \rho U, \rho E)$ and $f(W) = (\rho U, \rho U^2 + P, U(\rho E + P))$, where ρ, U, P and $E = \epsilon(P, \rho) + U^2/2$ respectively denote the density, the velocity, the pressure and the total energy of the fluid. The pressure is given by a perfect gas law $P = (\gamma - 1)\rho\epsilon$ for a given constant $\gamma > 1$.

Our approximate VFRoe-ncv solver is described in [4]. We use the non-conservative variables $Z = (\rho, U, P)$ in order to predict approximate states in the linearized Riemann problem around each cell interface. We recall that this solver does not require the computation of a Roe matrix and thus it is simpler to implement than the classical Roe solver.

In this test case we assume that the perfect gas constant is $\gamma = 1.4$. The CFL number is fixed to 1/2. The initial condition of the Riemann problem is $W(x < 0, t = 0) = W_L$, $W(x > 0, t = 0) = W_R$, setting: $\rho_L = 0.01$, $U_L = 0$, $P_L = 5$, and $\rho_R = 1000$, $U_R = 0$, $P_R = 10^5$. Without any entropy fix, the VFRoe-ncv scheme [4] cannot even be used because of negative numerical densities. Fig. 1 shows the behaviour of the density at time $T = 0.0098842$ for the entropy-fixed scheme. We compare the corrections of formulas (3) and (6) with a standard Godunov scheme and the exact solution. The strong 3-rarefaction wave contains a sonic point located at the initial position of the discontinuity ($x = 0$). A zoom shows the behaviour of the density around the sonic point. It has been checked that only one sonic flag has been defined in the computational domain, at each time step.

We observe that the VFRoe-ncv scheme with the correction (6) has the same behavior and precision as the Godunov scheme: a small non-physical shock is visible but disappears under mesh refinements. Let us recall that the Godunov scheme does not require any entropy fix. The correction (3) is a little bit more diffusive, which explains a small constant density zone. The two schemes are more precise than the Rusanov scheme.

4. Application to magnetohydrodynamics

In this section, we apply the entropy correction to the VFRoe-ncv approximation of the ideal MagnetoHydroDynamics (MHD) equations. In the case of the MHD system supplemented by the divergence cleaning terms of [12], the conservative variables and the flux are given by

$$W = \left(\rho, \rho u^T, \frac{p}{\gamma - 1} + \frac{\rho u \cdot u + B \cdot B}{2}, B^T, \psi\right)^T$$

$$u = (u_1, u_2, u_3)^T, \qquad B = (B_1, B_2, B_3)^T, \qquad n = (1, 0, 0)^T$$

$$f(W) = \begin{pmatrix} \rho u \cdot n \\ \rho(u \cdot n)u + \big(p + \frac{B \cdot B}{2}\big)n - (B \cdot n)B \\ \big(\frac{\gamma p}{\gamma - 1} + \frac{\rho u \cdot u}{2} + B \cdot B\big)u \cdot n - (B \cdot u)(B \cdot n) \\ (u \cdot n)B - (B \cdot n)u + \psi n \\ c_h^2 B \cdot n \end{pmatrix} \tag{7}$$

The magnetic field is noted B and n is a unit vector. The unknown ψ is an auxiliary variable whose role is to damp the divergence of the magnetic field in order to recover numerically the constraint $\nabla \cdot B = 0$. The divergence errors are evacuated at the constant velocity c_h. In practice, c_h is chosen in such a way that it is higher than all the wave speeds of the MHD system. Let us recall that the numerical solution of the Riemann problem for the MHD is difficult to compute [14] (see also `http://hal.archives-ouvertes.fr/hal-00337063/en/`, pp. 8–14). Torrilhon has developed an interactive web page where it is possible to experiment this numerical resolution: `http://www.sam.math.ethz.ch/ matorril/mhdsolver/`.

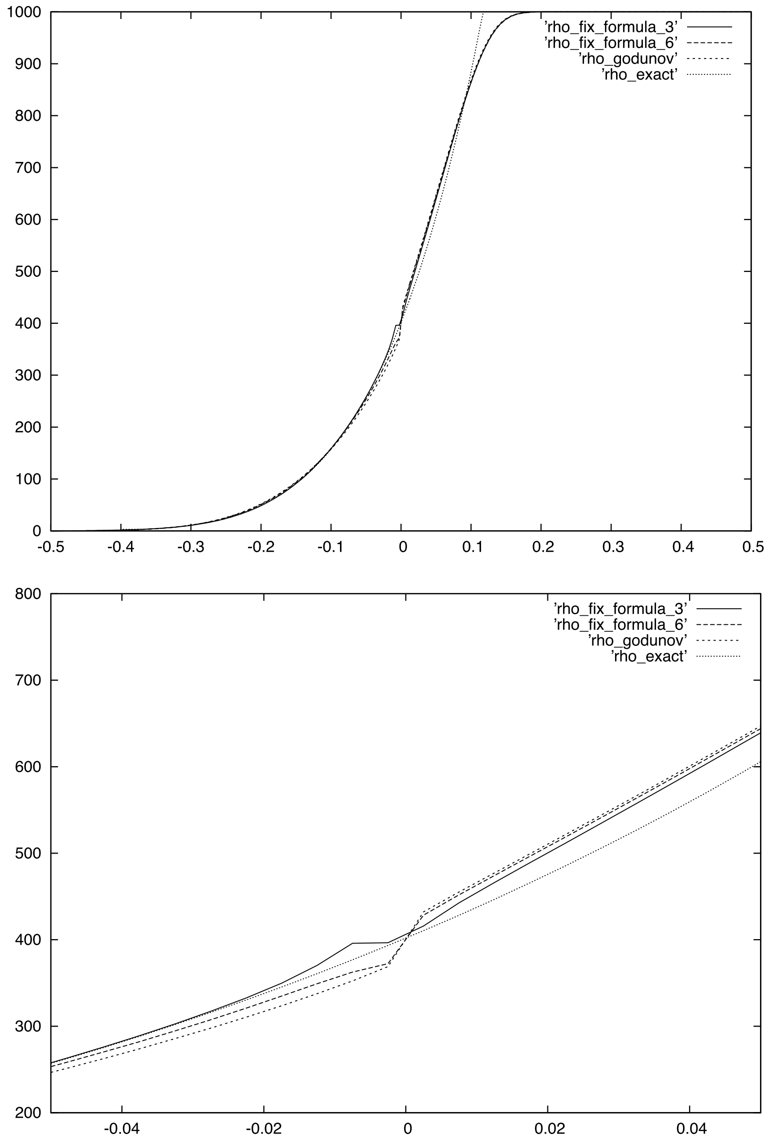

Fig. 1. Test case 1: density profiles obtained by using 200 cells. Comparison between Godunov, VFRoe-ncv with two entropy fixes and the exact solution. A zoom on the density profiles around the sonic point is displayed on the right.

Table 1
The data used in the calculations.

	ρ	u_1	u_2	u_3	P	B_1	B_2	B_3	ψ
State L	3	1.3	0	0	3	1.5	1	1	0
State R	1	1.3	0	0	1	1.5	cos(1.5)	sin(1.5)	0

Although Roe schemes exist for the MHD system (see [13] for instance), this is not the case for the modified MHD system with divergence cleaning. In addition, if it is easy to compute the eigenvalues of the MHD system with divergence cleaning, the expression of the eigenvectors is rather complicated. The VFRoe-ncv scheme is then interesting, because it is possible to compute the numerical flux without computing the eigenvector basis. The non-conservative variables that are used in the VFRoe-ncv scheme [8] are: $Z = (\rho, u^T, p, B^T, \psi)^T$. The left and right data of the Riemann problem are taken from [14]; see Table 1.

We perform a computation on the interval $[-1, 6]$. The jump of the initial condition is at $x = 0$, the polytropic constant is $\gamma = 5/3$ and the divergence cleaning velocity is $c_h = 3.8$. The CFL number is set to 0.8 and the number of cells to 2000. In Fig. 2, we compare the results obtained with the unfixed VFRoe-ncv scheme, the fixed VFRoe-ncv scheme (with formula (6), but formula (3) leads to very similar results), Rusanov scheme and the exact solution of the one-dimensional Riemann

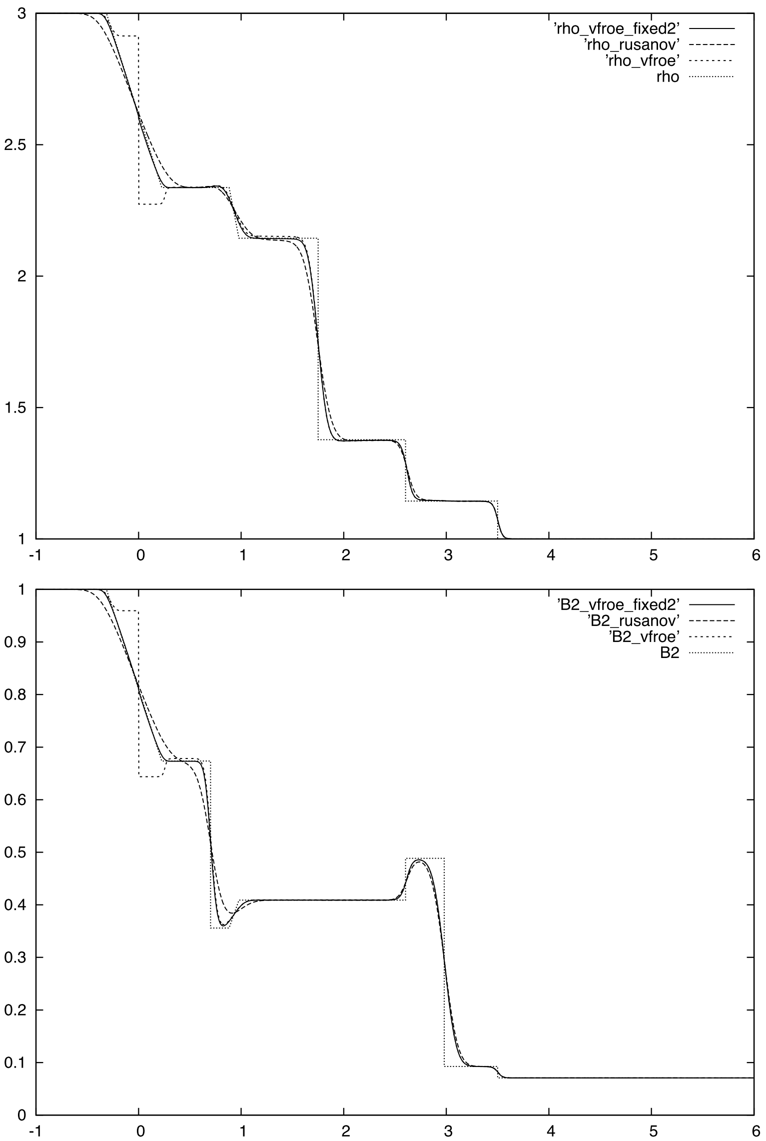

Fig. 2. Test case 2: density and second component of the magnetic field. Comparison between the VFRoe-ncv scheme, the VFRoe-ncv scheme with the parameter-free entropy fix and the Rusanov scheme.

problem. Without any entropy fix, a non-physical shock occurs in the sonic rarefaction wave. The non-physical shock is suppressed by the parameter-free entropy-fix scheme. In addition, the modified VFRoe-ncv scheme is more accurate than the Rusanov scheme.

Let us mention that many other approximate Riemann solvers have been developed for the MHD. For some of them it is even possible to prove some entropy estimates (see [15]). The advantage of our approach is that it is very simple and general and that the resulting scheme has the same precision as the Roe or the Godunov solver.

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