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Inverse problems

Unique determination of unknown boundaries in an elastic plate by one measurement

Détermination unique des frontières inconnues dans une plaque élastique par une mesure

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A B S T R A C T

In this article we study two inverse problems for a thin elastic plate subjected to a given couple field applied at its boundary. One problem consists in determining an unknown portion of the exterior boundary of the plate subjected to homogeneous Neumann conditions, while the other problem concerns with the determination of a rigid inclusion inside the plate. In both cases, under the assumption that the plate is made by isotropic material, we prove uniqueness with one measurement.

R É S U M É

Dans cet article, on considère deux problèmes inverses pour une plaque mince élastique soumise à une distribution donnée de moments sur une partie de son bord. Le premier problème consiste à déterminer une portion inconnue du bord, supposée libre d'efforts. Le second problème correspond à l'identification d'une inclusion rigide dans la plaque. Pour les deux problèmes, l'identifiabilité au moyen d'une seule mesure est prouvée, sous l'hypothèse d'un comportement isotrope du matériau constitutif de la plaque.

1. Introduction

In several applications of nondestructive methods in structural mechanics one deals with inverse problems for determining an unknown or inaccessible portion of the boundary of a body by measurements taken on an accessible part of its boundary. As an example of practical application, this kind of problems arises in damage assessment of mechanical specimens which are possibly defective due to the presence of interior rigid inclusions induced during the manufacturing process, see, for example, [1].

In this article we shall deal with two inverse problems with unknown boundaries for thin elastic plates. Let us first consider the case in which the unknown boundary is some portion of the exterior component of the boundary of the plate. Suppose that the middle surface of the plate is a bounded domain Ω in $\mathbb{R}^2$ with a sufficiently smooth boundary $\partial\Omega$, and assume that a part of $\partial\Omega$, say I, is not known. The inverse problem consists in determining I by a nondestructive method collecting Cauchy data measurements on the accessible part of the boundary $\partial\Omega$ represented by a sub-arc Γ of $\partial\Omega$, with $\Gamma \cup I = \partial\Omega$. More precisely, we assume that the inaccessible part I of $\partial\Omega$ is free and we accept to work within

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the Kirchhoff–Love theory of thin, elastic plates under infinitesimal deformations. Therefore, given a nontrivial couple field $\hat{M}$ on Γ, the statical equilibrium of the plate is described by the following Neumann problem:

$$\begin{cases} \operatorname{div}\big(\operatorname{div}\big(\mathbb{P}\nabla^2 w\big)\big) = 0, & \text{in } \Omega \quad (1)\\ \big(\mathbb{P}\nabla^2 w\big)n \cdot n = -\hat{M}_n, & \text{on } \Gamma \quad (2)\\ \operatorname{div}\big(\mathbb{P}\nabla^2 w\big) \cdot n + \big(\big(\mathbb{P}\nabla^2 w\big)n \cdot \tau\big),_s = (\hat{M}_\tau),_s, & \text{on } \Gamma \quad (3)\\ \big(\mathbb{P}\nabla^2 w\big)n \cdot n = 0, & \text{on } I \quad (4)\\ \operatorname{div}\big(\mathbb{P}\nabla^2 w\big) \cdot n + \big(\big(\mathbb{P}\nabla^2 w\big)n \cdot \tau\big),_s = 0, & \text{on } I \quad (5) \end{cases}$$

where $w = w(x_1, x_2)$, $w \in H^2(\Omega)$, is the transversal displacement of the point (x_1, x_2) of the plate and $\hat{M}_\tau$, $\hat{M}_n$ denote, respectively, the twisting and bending moments applied on Γ; see, for example, [2]. We assume that the plate is made by isotropic material with fourth order tensor $\mathbb{P}$ given by

$$\mathbb{P}A = B\big((1-\nu)A^{sym} + \nu(\operatorname{tr} A)I_2\big) \tag{6}$$

for every 2×2 matrix A, where B is the bending stiffness of the plate and ν is the Poisson's coefficient of the material, see Section 3. In the above equations, n denotes the unit outer normal to $\partial\Omega$, τ is the unit tangent vector and s is the arclength (we refer to Section 2 for the precise definitions).

Under suitable regularity assumptions on the coefficients of the plate tensor $\mathbb{P}$ and on the boundary of $\partial\Omega$, we prove that the unknown boundary I is uniquely determined by applying a single, nontrivial couple field $\hat{M}$ on Γ and measuring on Γ the corresponding transversal displacement w and its normal derivative $\frac{\partial w}{\partial n}$, see Theorem 3.1 for a precise statement.

A second problem considered in this article is that associated to the unique determination of a rigid inclusion inside the plate. Let D, $D \Subset \Omega$, be an open simply connected subset of Ω representing the inclusion. Our aim is to identify D by applying a couple field $\hat{M}$ at the boundary $\partial\Omega$ and by measuring the induced transversal displacement and its normal derivative on an open portion Γ of $\partial\Omega$. In this case, working as before within the Kirchhoff–Love theory of thin, elastic and isotropic plates under infinitesimal deformations, the transversal displacement $w \in H^2(\Omega)$ satisfies the following mixed boundary value problem

$$\begin{cases} \operatorname{div}\big(\operatorname{div}\big(\mathbb{P}\nabla^2 w\big)\big) = 0, & \text{in } \Omega \setminus \overline{D} \quad (7)\\ \big(\mathbb{P}\nabla^2 w\big)n \cdot n = -\hat{M}_n, & \text{on } \partial\Omega \quad (8)\\ \operatorname{div}\big(\mathbb{P}\nabla^2 w\big) \cdot n + \big(\big(\mathbb{P}\nabla^2 w\big)n \cdot \tau\big),_s = (\hat{M}_\tau),_s, & \text{on } \partial\Omega \quad (9)\\ w|_{\overline{D}} \equiv \text{ affine function}, & \text{in } \overline{D} \quad (10)\\ \dfrac{\partial w^e}{\partial n} = \dfrac{\partial w^i}{\partial n}, & \text{on } \partial D \quad (11) \end{cases}$$

coupled with the *equilibrium conditions* for the rigid inclusion D

$$\int_{\partial D} \big(\operatorname{div}\big(\mathbb{P}\nabla^2 w^e\big) \cdot n + \big(\big(\mathbb{P}\nabla^2 w^e\big)n \cdot \tau\big),_s\big)g - \big(\big(\mathbb{P}\nabla^2 w^e\big)n \cdot n\big)g,_n = 0, \quad \text{for every affine function } g \tag{12}$$

In above equations, n denotes the unit outer normal to $\Omega \setminus \overline{D}$ and we have defined $w^e \equiv w|_{\Omega\setminus\overline{D}}$ and $w^i \equiv w|_{\overline{D}}$.

Under suitable regularity assumptions on the coefficients of the plate tensor and on the boundary of Ω and D, in Theorem 3.2 we prove uniqueness with one boundary measurement.

We now briefly comment the findings of this article. Our results show that the two inverse problems posed above can be uniquely solved by a single boundary measurement. A similar conclusion is true for analogous inverse problems of uniqueness (and, in some cases, of stability) for electrical conductors (see, for example, [3–6]), in the thermic setting [7–11] and in two and three dimensional linear elasticity ([12,13] and [14,15]), where the unique determination of unknown boundaries has been proved using a single boundary measurement. All these papers deal with equations or systems of equations of the second order. The corresponding boundary value inverse problems for the fourth order operator describing the statical equilibrium of a plate are less studied. In [16], for example, it was shown that, in general, a cavity in a plate is uniquely determined by applying *two linearly independent* couple fields and by measuring at the boundary the corresponding transversal displacement and its normal derivative. It is worth noticing that the inverse problem of determining an unknown portion of the boundary $\partial\Omega$ of the plate and that of determining an unknown cavity inside the plate Ω are two variants of the same problem. In fact, the equilibrium problem in presence of a cavity D is described by (1)–(5), with $\Gamma = \partial\Omega$ and $I = \partial D$. Therefore, the difference between the two cases is merely of topological character and in fact in proving Theorem 3.1 we take advantage of the fact that the *regular* boundaries of the plates share a common portion Γ. It remains an open problem whether a cavity in a plate can be uniquely recovered by a single boundary measurement.

The article is organized as follows. In Section 2 we collect some notation. In Section 3 we provide the formulation of the direct problems and we state our main results, namely Theorem 3.1 and Theorem 3.2. Section 4 contains the proofs of the two theorems.

2. Notation

We shall denote by $B_r(P)$ the disk in $\mathbb{R}^2$ of radius r and center P. Let $\{e_1, e_2, e_3\}$ be the canonical basis of $\mathbb{R}^3$.

Definition 2.1 *($C^{k,\alpha}$ regularity).* Let $\mathcal{U}$ be a bounded domain in $\mathbb{R}^2$. Given k, α, with $k \in \mathbb{N}$, $0 < \alpha \leqslant 1$, we say that a portion S of $\partial\mathcal{U}$ is of *class* $C^{k,\alpha}$ if there exists $\rho_0 > 0$ such that for any $P \in S$, there exists a rigid transformation of coordinates under which we have $P = 0$ and

$$\mathcal{U} \cap B_{\rho_0}(0) = \{x = (x_1, x_2) \in B_{\rho_0}(0) \mid x_2 > \psi(x_1)\}$$

where ψ is a $C^{k,\alpha}$ function on $(-\rho_0, \rho_0)$.

Given a bounded domain $\mathcal{U}$ in $\mathbb{R}^2 = \text{span}\{e_1, e_2\}$ such that $\partial\mathcal{U}$ is of class $C^{k,\alpha}$, with $k \geqslant 1$, we consider as positive the orientation of the boundary induced by the outer unit normal n in the following sense. Given a point $P \in \partial\mathcal{U}$, let us denote by $\tau = \tau(P)$ the unit tangent at the boundary in P obtained by applying to n a counterclockwise rotation of angle $\frac{\pi}{2}$, that is

$$\tau = e_3 \times n \tag{13}$$

where $\times$ denotes the vector product in $\mathbb{R}^3$.

Given any connected component $\mathcal{C}$ of $\partial\mathcal{U}$ and fixed any point $P_0 \in \mathcal{C}$, let us consider an arclength parametrization $\varphi(s) = (x_1(s), x_2(s))$, $s \in [0, l(\mathcal{C})]$, such that $\varphi(0) = P_0$ and $\varphi'(s) = \tau(\varphi(s))$. Here $l(\mathcal{C})$ denotes the length of $\mathcal{C}$.

Throughout the article, we denote by $w_{,\alpha}$, $w_{,s}$, and $w_{,n}$ the derivatives of a function w with respect to the x_α variable, $\alpha = 1, 2$, to the arclength s and to the normal direction n, respectively, and similarly for higher order derivatives.

We denote by $\mathbb{M}^2$ the space of 2×2 real valued matrices and by $\mathcal{L}(X, Y)$ the space of bounded linear operators between Banach spaces X and Y.

For all 2×2 matrices A, B and for every $\mathbb{L} \in \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2)$, we use the following notation:

$$(\mathbb{L}A)_{\alpha\beta} = L_{\alpha\beta\gamma\delta} A_{\gamma\delta} \tag{14}$$

$$A \cdot B = A_{\alpha\beta} B_{\alpha\beta} \tag{15}$$

$$|A| = (A \cdot A)^{\frac{1}{2}} \tag{16}$$

$$A^{sym} = \frac{1}{2}\left(A + A^T\right) \tag{17}$$

Notice that here and in the sequel, summation over repeated indexes is implied.

Finally, let us introduce the space of the affine functions on $\mathbb{R}^2$

$$\mathcal{A} = \{g(x_1, x_2) = ax_1 + bx_2 + c,\ a, b, c \in \mathbb{R}\} \tag{18}$$

3. Formulation of the inverse problems and statement of the main results

Let us consider a thin plate $\Omega \times [-\frac{h}{2}, \frac{h}{2}]$ with middle surface represented by a simply connected bounded domain Ω in $\mathbb{R}^2$ and having uniform thickness h, $h \ll \text{diam}(\Omega)$. We assume that $\partial\Omega$ is of class $C^{1,1}$.

Let the plate be made of nonhomogeneous linear elastic material with elasticity tensor $\mathbb{C}(x) \in \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2)$ and let us assume that body forces inside $\Omega \times [-\frac{h}{2}, \frac{h}{2}]$ are absent. We denote by $\hat{M}$ a couple field acting on the boundary $\partial\Omega$.

We shall assume throughout that the material of the plate is isotropic, that is the *plate tensor* $\mathbb{P} = \frac{h^3}{12}\mathbb{C}$ is defined by

$$\mathbb{P}A = B\left[(1 - \nu)A^{sym} + \nu(\text{tr}\, A)I_2\right] \tag{19}$$

for every 2×2 matrix A, where I_2 is the 2×2 identity matrix and $\text{tr}(A)$ denotes the trace of the matrix A. The *bending stiffness* (per unit length) of the plate is given by the function

$$B(x) = \frac{h^3}{12}\left(\frac{E(x)}{1 - \nu^2(x)}\right) \tag{20}$$

where the Young's modulus E and the Poisson's coefficient ν can be written in terms of the Lamé moduli λ, μ of the material as follows

$$E(x) = \frac{\mu(x)(2\mu(x) + 3\lambda(x))}{\mu(x) + \lambda(x)}, \qquad \nu(x) = \frac{\lambda(x)}{2(\mu(x) + \lambda(x))} \tag{21}$$

Working in the framework of the linear elasticity for infinitesimal deformations and under the assumptions of the Kirchhoff–Love theory, the statical equilibrium problem of the plate is described by the following Neumann problem

$$\begin{cases} M_{\alpha\beta,\alpha\beta} = 0, & \text{in } \Omega \quad (22) \\ M_{\alpha\beta} n_\alpha n_\beta = \hat{M}_n, & \text{on } \partial\Omega \quad (23) \\ M_{\alpha\beta,\beta} n_\alpha + (M_{\alpha\beta} n_\beta \tau_\alpha)_{,s} = -(\hat{M}_\tau)_{,s}\,, & \text{on } \partial\Omega \quad (24) \end{cases}$$

Here

$$M_{\alpha\beta} = M_{\alpha\beta}(w) = -P_{\alpha\beta\gamma\delta} w_{,\gamma\delta}\,, \quad \alpha,\beta,\delta,\gamma = 1,2 \tag{25}$$

where $w(x_1, x_2)$ is the *transversal displacement* of the point $x = (x_1, x_2)$ belonging to the middle surface Ω. Moreover, following a standard convention in plate theory, the boundary couple field $\hat{M}$ is represented in local coordinates as

$$\hat{M} = \hat{M}_\tau n + \hat{M}_n \tau, \quad \text{on } \partial\Omega \tag{26}$$

where $\hat{M}_\tau = \hat{M} \cdot n$, $\hat{M}_n = \hat{M} \cdot \tau$ denote respectively the *twisting moment* and the *bending moment* applied at the boundary. Here, $\cdot$ denotes the scalar product in $\mathbb{R}^2$.

Conditions (23), (24) express the local equilibrium conditions on the bending moment and on the transversal forces acting on the boundary $\partial\Omega$, respectively. For details about the mechanical meaning of the functions $M_{\alpha\beta}$, we refer to [17].

The plate tensor $\mathbb{P}$ is assumed to satisfy the following assumptions:

(I) *Regularity*

$$\mathbb{P} \in C^{1,1}\big(\mathbb{R}^2, \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2)\big) \tag{27}$$

(II) *Ellipticity* (*strong convexity*) There exist positive constants α_0, γ_0 such that

$$\mu(x) \geqslant \alpha_0 > 0, \quad 2\mu(x) + 3\lambda(x) \geqslant \gamma_0 > 0, \quad \text{for every } x \in \mathbb{R}^2 \tag{28}$$

It follows easily that

$$\mathbb{P}A \cdot A \geqslant \frac{h^3}{12}\xi_0 |A|^2, \quad \text{in } \mathbb{R}^2 \tag{29}$$

for every 2×2 symmetric matrix A, where $\xi_0 = \min\{2\alpha_0, \gamma_0\}$.

Our first result, concerning the determination of unknown boundaries, will be based on the following assumptions on the domain Ω. Let Γ, I be two closed, nonempty sub-arcs of the boundary $\partial\Omega$ such that

$$\Gamma \cup I = \partial\Omega, \qquad \Gamma \cap I = \{Q, R\} \tag{30}$$

where Q, R are two distinct points of $\partial\Omega$. Here Γ represents the accessible portion of the boundary, whereas I represents the inaccessible, unknown portion of the boundary to be determined.

On the assigned couple field $\hat{M}$ let us require the following assumptions:

$$\hat{M} \in L^2\big(\Gamma, \mathbb{R}^2\big), \qquad (\hat{M}_n, \hat{M}_{\tau,s}) \not\equiv 0 \tag{31}$$

$$\int_\Gamma \hat{M}_\alpha = 0, \quad \alpha = 1,2 \tag{32}$$

$$\operatorname{supp}(\hat{M}) \Subset \Gamma \tag{33}$$

Conditions (32) express the global equilibrium of the plate in terms of the cartesian representation $\hat{M} = M_2 e_1 + M_1 e_2$ of the boundary couple field.

In this case the statical equilibrium problem of the plate takes the following form

$$\begin{cases} M_{\alpha\beta,\alpha\beta} = 0, & \text{in } \Omega \quad (34) \\ M_{\alpha\beta} n_\alpha n_\beta = \hat{M}_n, & \text{on } \Gamma \quad (35) \\ M_{\alpha\beta,\beta} n_\alpha + (M_{\alpha\beta} n_\beta \tau_\alpha)_{,s} = -(\hat{M}_\tau)_{,s}\,, & \text{on } \Gamma \quad (36) \\ M_{\alpha\beta} n_\alpha n_\beta = 0, & \text{on } I \quad (37) \\ M_{\alpha\beta,\beta} n_\alpha + (M_{\alpha\beta} n_\beta \tau_\alpha)_{,s} = 0, & \text{on } I \quad (38) \end{cases}$$

It is well known that under the above assumptions the Neumann problem (34)–(38) is solvable and that its weak solution $w \in H^2(\Omega)$ is determined up to the addition of an affine function, see [17, Proposition 3.4] for details.

Let us notice that the role of the second condition in (31) is to avoid that problem (34)–(38) admits affine solutions.

Theorem 3.1 *(Unique determination of unknown boundaries with one measurement). Let Ω_1, Ω_2 be two simply connected bounded domains in $\mathbb{R}^2$ such that $\partial\Omega_i$, $i = 1, 2$, are of class $C^{4,1}$. Let $\partial\Omega_i = I_i \cup \Gamma$, $i = 1, 2$, where I_i and Γ are the inaccessible and the accessible parts of the boundaries $\partial\Omega_i$, respectively. Let us assume that Ω_1 and Ω_2 lie on the same side of Γ and that conditions* (30) *are satisfied by both pairs $\{I_1, \Gamma\}$ and $\{I_2, \Gamma\}$. Let the plate tensor $\mathbb{P}$ be given by* (19)*, with Lamé moduli λ and μ of class $C^{2,1}(\mathbb{R}^2)$, satisfying the strong convexity condition* (28)*. Let $\hat{M} \in L^2(\Gamma, \mathbb{R}^2)$ be a boundary couple field satisfying conditions* (31), (33)*. Let $w_i \in H^2(\Omega_i)$ be a solution to the Neumann problem* (34)–(38) *in $\Omega = \Omega_i$, $i = 1, 2$. If*

$$w_1 = w_2, \qquad \frac{\partial w_1}{\partial n} = \frac{\partial w_2}{\partial n}, \quad \text{on } \Gamma \tag{39}$$

then

$$\Omega_1 = \Omega_2 \tag{40}$$

Let us consider now the inverse problem of determining an unknown rigid inclusion inside the plate by measuring the transversal displacement and its normal derivative at the boundary $\partial\Omega$ of the plate.

Let D, $D \Subset \Omega$, be an open simply connected subset of Ω of class $C^{1,1}$, representing a rigid inclusion inside Ω. On the assigned couple field $\hat{M}$ let us require the following assumptions:

$$\hat{M} \in H^{-\frac{1}{2}}\big(\partial\Omega, \mathbb{R}^2\big), \quad (\hat{M}_n, \hat{M}_{\tau,s}) \not\equiv 0 \tag{41}$$

$$\int_{\partial\Omega} \hat{M}_\alpha = 0, \quad \alpha = 1, 2 \tag{42}$$

$$\operatorname{supp}(\hat{M}) \Subset \Gamma \tag{43}$$

where Γ is a nonempty open portion of $\partial\Omega$.

In this case, the statical equilibrium problem of the plate takes the following form

$$\begin{cases} M_{\alpha\beta,\alpha\beta} = 0, & \text{in } \Omega \setminus \overline{D} \quad (44)\\ M_{\alpha\beta}n_\alpha n_\beta = \hat{M}_n, & \text{on } \partial\Omega \quad (45)\\ M_{\alpha\beta,\beta}n_\alpha + (M_{\alpha\beta}n_\beta\tau_\alpha)_{,s} = -(\hat{M}_\tau)_{,s}, & \text{on } \partial\Omega \quad (46)\\ w|_{\overline{D}} \in \mathcal{A}, & \quad (47)\\ \dfrac{\partial w^e}{\partial n} = \dfrac{\partial w^i}{\partial n}, & \text{on } \partial D \quad (48) \end{cases}$$

coupled with the *equilibrium conditions* for the rigid inclusion D

$$\int_{\partial D} \big(\big(M_{\alpha\beta,\beta}n_\alpha + (M_{\alpha\beta}n_\beta\tau_\alpha)_{,s}\big)g - M_{\alpha\beta}n_\alpha n_\beta g_{,n}\big) = 0, \quad \text{for every } g \in \mathcal{A} \tag{49}$$

In (48), n denotes the unit outer normal to $\Omega \setminus \overline{D}$ and we have defined $w^e \equiv w|_{\Omega\setminus\overline{D}}$ and $w^i \equiv w|_{\overline{D}}$. Notice that in (49) $M_{\alpha\beta} = M_{\alpha\beta}(w^e)$. More precisely, denoting

$$H^2_D(\Omega) = \big\{w \in H^2(\Omega) \,\big|\, \exists h \in \mathcal{A} \text{ s.t. } w|_{\overline{D}} = h\big\} \tag{50}$$

a weak solution to problem (44)–(49) is a function $w \in H^2_D(\Omega)$ satisfying

$$\int_\Omega M_{\alpha\beta}(w)v_{,\alpha\beta} = \int_{\partial\Omega} \hat{M}_{\tau,s}v + \hat{M}_n v_{,n}, \quad \text{for every } v \in H^2_D(\Omega) \tag{51}$$

By standard variational arguments, it can be proven that problem (44)–(49) admits a weak solution which is determined up to the addition of an affine function.

Theorem 3.2 *(Unique determination of a rigid inclusion with one measurement). Let Ω be a simply connected domain in $\mathbb{R}^2$ such that $\partial\Omega$ is of class $C^{1,1}$ and let D_i, $i = 1, 2$, be two simply connected domains compactly contained in Ω, such that ∂D_i is of class $C^{3,1}$, $i = 1, 2$. Moreover, let Γ be a nonempty open portion of $\partial\Omega$, of class $C^{3,1}$. Let the plate tensor $\mathbb{P}$ be given by* (19)*, with Lamé moduli λ and μ of class $C^{1,1}(\overline{\Omega})$, and satisfying the strong convexity condition* (28)*. Let $\hat{M} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)$ be a boundary couple field satisfying* (41)–(43)*. Let w_i, $i = 1, 2$, be solutions to the mixed problem* (44)–(49)*, with $D = D_i$.*

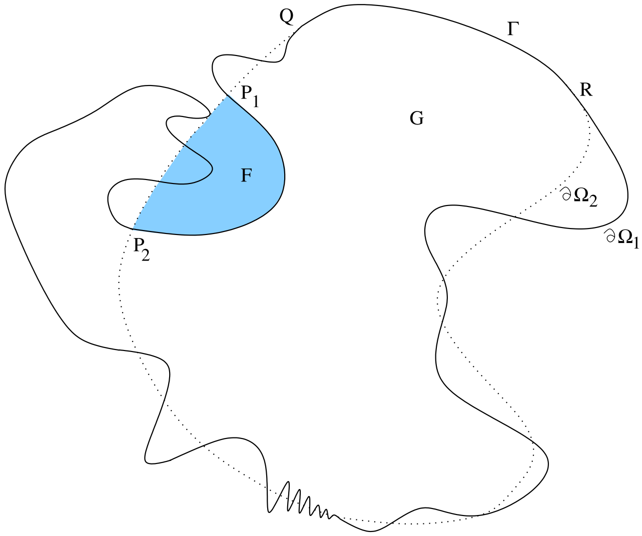

Fig. 1. The connected component F of $\Omega_2 \setminus \overline{G}$.

If

$$w_1 = w_2, \qquad \frac{\partial w_1}{\partial n} = \frac{\partial w_2}{\partial n}, \quad \text{on } \Gamma \tag{52}$$

then

$$D_1 = D_2 \tag{53}$$

Remark 1. Let us notice that the weak unique continuation of the solutions to the plate equation holds under $C^{1,1}-$ regularity of the Lamé moduli λ, μ (see [16, Proposition 2 and Remark 1]). The stronger regularity assumptions made in Theorem 3.1 have been introduced to give a classical meaning to the Neumann boundary conditions, as required by the arguments of the proof.

4. Proof of Theorems 3.1 and 3.2

4.1. Proof of Theorem 3.1

Let us choose $P_0 \in \Gamma \setminus \{Q, R\}$ and let $\varphi_i(s)$ be the arclength parametrization of $\partial\Omega_i$ such that $\varphi_i(0) = P_0$, $\varphi_i'(s) = \tau(\varphi_i(s))$, for $i = 1, 2$. We may assume that $Q = \varphi_i(s_i')$, $R = \varphi_i(s_i'')$, with $s_i' < s_i''$.

By the regularity assumptions on the boundaries $\partial\Omega_i$ and on the Lamé moduli λ, μ, and since $\operatorname{supp}(\hat{M}) \Subset \Gamma$, there exists a neighborhood $\mathcal{U}_i$ of I_i in $\overline{\Omega}_i$ such that $w_i \in H^5(\mathcal{U}_i)$, for $i = 1, 2$, see for details [16, Proposition 1]. By Sobolev embedding theorems (see for instance [18]), it follows that

$$w_i \in C^3(\mathcal{U}_i) \quad \text{for } i = 1, 2 \tag{54}$$

and therefore the homogeneous boundary conditions (34)–(38) are satisfied in the classical sense on I_i, $i = 1, 2$.

Let G be the connected component of $\Omega_1 \cap \Omega_2$ such that $\Gamma \subset \partial G$.

Let us prove, for instance, that $\Omega_2 \subset \Omega_1$. We have that

$$\Omega_2 \setminus \overline{\Omega}_1 \subset \Omega_2 \setminus \overline{G} \tag{55}$$

so that, if we prove that $\Omega_2 \setminus \overline{G} = \emptyset$ then $\Omega_2 \subset \overline{\Omega}_1$ and, by the regularity of Ω_1, it follows that $\Omega_2 \subset \Omega_1$.

Let us assume, by contradiction, that there exists a nonempty connected component F of $\Omega_2 \setminus \overline{G}$, see Fig. 1.

By the definition of G and F, it follows that

$$(\partial\Omega_1 \cap \partial F) \setminus \partial G \subset \partial\Omega_2 \cap \partial F \tag{56}$$

so that

$$\partial\Omega_1 \cap \partial F = (\partial\Omega_1 \cap \partial F \cap \partial G) \cup \big((\partial\Omega_1 \cap \partial F) \setminus \partial G\big) \subset (\partial\Omega_1 \cap \partial F \cap \partial G) \cup (\partial\Omega_2 \cap \partial F) \tag{57}$$

$$\partial F = (\partial\Omega_2 \cap \partial F) \cup (\partial\Omega_1 \cap \partial F) \subset (\partial\Omega_2 \cap \partial F) \cup (\partial\Omega_1 \cap \partial F \cap \partial G) \tag{58}$$

Let us define

$$\Sigma_2 = \partial F \cap \partial \Omega_2 \tag{59}$$

$$\Sigma_1 = \partial F \setminus \Sigma_2 \subset \partial \Omega_1 \cap \partial G \tag{60}$$

We have

$$\partial F = \Sigma_1 \cup \Sigma_2 \tag{61}$$

$$\Sigma_1 \cap \Sigma_2 = \emptyset \tag{62}$$

$$\partial F \cap \big(\Gamma \setminus \{Q, R\}\big) = \emptyset \tag{63}$$

By (63) and since Σ_2 is closed in $\partial\Omega_2$, any (nonempty) connected component of Σ_2 is either a single point or a closed subarc of $\partial\Omega_2$ having distinct endpoints.

By (63) and since Σ_1 is open in $\partial\Omega_1$, any (nonempty) connected component of Σ_1 is an open subarc of $\partial\Omega_1$ having distinct endpoints P_1, P_2 belonging to Σ_2.

Claim. *Σ_1 consists of a single open arc γ with distinct endpoints P_1, $P_2 \in \Sigma_2$ and Σ_2 consists of a single closed arc τ with the same endpoints P_1 and P_2.*

Proof of the Claim. First, let us notice that $\Sigma_1 \neq \emptyset$. In fact, otherwise, $\partial F = \Sigma_2 \subset \partial\Omega_2$, so that $\partial F = \partial\Omega_2$ contradicting (63).

Let γ be a connected component of Σ_1, with distinct endpoints P_1, $P_2 \in \Sigma_2$, and let τ be the closed sub-arc of $\partial\Omega_2$, having endpoints P_1 and P_2, which does not intersect $\Gamma \setminus \{Q, R\}$. By (62), $\gamma \cap \tau \subset \Sigma_1 \cap \partial\Omega_2 = \Sigma_1 \cap \partial F \cap \partial\Omega_2 = \Sigma_1 \cap \Sigma_2 = \emptyset$, so that $\gamma \cup \tau$ is the boundary of a bounded domain which we denote by H. If we prove that $F = H$ then the Claim follows.

The domain G does not intersect $\partial H = \gamma \cup \tau \subset \partial\Omega_1 \cup \partial\Omega_2$ and therefore either $G \subset H$ or $G \subset \Omega_2 \setminus \overline{H}$. Let us see that the latter case occurs.

By contradiction, let us assume that $G \subset H$. In this case we have that $\Gamma \subset \overline{G} \subset \overline{H}$. On the other hand, $(\Gamma \setminus \{Q, R\}) \cap \gamma \subset (\Gamma \setminus \{Q, R\}) \cap \partial F = \emptyset$. Similarly, by the choice of τ, $(\Gamma \setminus \{Q, R\}) \cap \tau = \emptyset$. It follows that $\Gamma \setminus \{Q, R\} \subset H$.

The open sub-arc of $\partial\Omega_2$ $\sigma = \partial\Omega_2 \setminus \tau$ is a connected set which does not intersect $\partial H = \gamma \cup \tau$, since $\gamma \cap \partial\Omega_2 = \emptyset$. Therefore, since $\sigma \supset \Gamma \setminus \{Q, R\}$, we have that $\sigma \subset H$.

It follows that $\overline{\Omega}_2 \subset H \cup \tau$. Hence, given any point $P \in \gamma$, we have that $\mathrm{d}(P, \overline{\Omega}_2) > 0$, contradicting $\gamma \subset \partial G \subset \overline{\Omega}_2$. We have thus proved that $G \subset \Omega_2 \setminus \overline{H}$.

Therefore, given $S \in \gamma$, since $\gamma \subset \partial G \cap \partial\Omega_1$, by the definition of G and by the regularity of $\partial\Omega_1$, it follows that there exists $\rho > 0$ such that $B_\rho(S) \setminus H \subset \overline{G}$. On the other hand, since $\gamma \subset \partial F$, $F \cap B_\rho(S) \neq \emptyset$ so that, being $F \cap \overline{G} = \emptyset$ by definition of F, $B_\rho(S) \cap F \cap H \neq \emptyset$. Now, $F \cap \partial H = F \cap (\tau \cup \gamma) \subset F \cap \partial F = \emptyset$ and, by the connectedness of F, we have that $F \subset H$.

If Σ_1 had another connected component, say γ', then γ' would be an open arc contained in H. Given $T \in \gamma' \subset H$, there would exist a neighborhood V of T contained in H and therefore, since $T \in \partial G$, $H \cap G \supset V \cap G \neq \emptyset$, contradicting $G \cap H = \emptyset$.

Hence $\Sigma_1 = \gamma$ and therefore $\Sigma_2 = \tau$ and $F = H$. The proof of the Claim is complete. □

Let ν be the outer unit normal to F, defined on $\gamma \cup \tau \setminus \{P_1, P_2\}$, and let us denote by n^i and τ^i the normal and tangent vectors to $\partial\Omega_i$, $i = 1, 2$. Since $F \subset \Omega_2$, we have that $\nu = n^2$ on $\tau \setminus \{P_1, P_2\}$. Since $\gamma \subset \partial G \cap \partial\Omega_1$, and recalling that $\overline{F} \cap G = \emptyset$, for every $P \in \gamma$, there exists a neighborhood U of P such that $(U \setminus \overline{F}) \cap G \neq \emptyset$ and, since $G \subset \Omega_1$ and by the regularity of Ω_1, $(U \setminus \overline{F}) \subset \Omega_1$. It follows that $\nu = -n^1$ on γ.

To fix ideas, let us assume that P_2 follows P_1 along γ according to the positive orientation of $\partial\Omega_1$; since $\nu = -n^1$ on γ and $\nu = n^2$ on $\tau \setminus \{P_1, P_2\}$, also P_2 follows P_1 along τ according to the positive orientation of $\partial\Omega_2$. Let $s_1, s_2 \in \mathbb{R}$, $s_1 < s_2$ such that $P_1 = \varphi_1(s_1)$, $P_2 = \varphi_1(s_2)$.

The function $w = w_1 - w_2$ satisfies the following Cauchy problem:

$$\begin{cases} M_{\alpha\beta,\alpha\beta}(w) = 0, & \text{in } G \quad (64)\\ w = 0, & \text{on } \Gamma \quad (65)\\ \dfrac{\partial w}{\partial n} = 0, & \text{on } \Gamma \quad (66)\\ M_{\alpha\beta}(w) n_\alpha n_\beta = 0, & \text{on } \Gamma \quad (67)\\ M_{\alpha\beta,\beta}(w) n_\alpha + \big(M_{\alpha\beta}(w) n_\alpha \tau_\beta\big)_{,s} = 0, & \text{on } \Gamma \quad (68) \end{cases}$$

From the uniqueness of the solution to the Cauchy problem (64)–(68) (see, for instance, [16, Proposition 3]) and from the weak unique continuation property (see, for instance, [16, Proposition 2 and Remark 1]), we have that $w \equiv 0$ in G, that is

$$w_1 \equiv w_2, \quad \text{in } G \tag{69}$$

By (54), w_i coincide with all their derivatives up to the third order in $\overline{G}$. Let us first apply integration by parts to the equation $M_{\alpha\beta,\alpha\beta}(w_2)=0$ in F. By using (69) and (38) we obtain

$$\begin{aligned}0&=\int_F M_{\alpha\beta,\alpha\beta}(w_2)=\int_{\partial F} M_{\alpha\beta,\beta}(w_2)\nu_\alpha\\&=-\int_\gamma M_{\alpha\beta,\beta}(w_1)n^1_\alpha+\int_\tau M_{\alpha\beta,\beta}(w_2)n^2_\alpha=\int_\gamma \big(M_{\alpha\beta}(w_1)n^1_\beta\tau^1_\alpha\big)_{,s}-\int_\tau \big(M_{\alpha\beta}(w_2)n^2_\beta\tau^2_\alpha\big)_{,s}\\&=\big(M_{\alpha\beta}(w_1)n^1_\beta\tau^1_\alpha\big)(P_2)-\big(M_{\alpha\beta}(w_1)n^1_\beta\tau^1_\alpha\big)(P_1)+\big(M_{\alpha\beta}(w_2)n^2_\beta\tau^2_\alpha\big)(P_1)-\big(M_{\alpha\beta}(w_2)n^2_\beta\tau^2_\alpha\big)(P_2)\end{aligned}\tag{70}$$

so that

$$\big(M(w_2)n^1\cdot\tau^1\big)(P_2)-\big(M(w_2)n^2\cdot\tau^2\big)(P_2)=\big(M(w_2)n^1\cdot\tau^1\big)(P_1)-\big(M(w_2)n^2\cdot\tau^2\big)(P_1):=K\tag{71}$$

where M is the 2×2 matrix of entries $M_{\alpha\beta}$ given by (25).

Now, let us apply integration by parts to the equation $M_{\alpha\beta,\alpha\beta}(w_2)w_2=0$ in F, obtaining

$$\begin{aligned}0&=\int_F M_{\alpha\beta,\alpha\beta}(w_2)w_2=\int_F M_{\alpha\beta}(w_2)w_{2,\alpha\beta}+\int_{\partial F} M_{\alpha\beta,\beta}(w_2)\nu_\alpha w_2-\int_{\partial F} M_{\alpha\beta}(w_2)\nu_\beta w_{2,\alpha}\\&:=\int_F M_{\alpha\beta}(w_2)w_{2,\alpha\beta}+I_1-I_2\end{aligned}\tag{72}$$

Recalling (34)–(38) and by using the following relations, which hold for any function $u\in C^1(\Omega_i)$,

$$u_{,\alpha}=n^i_\alpha u_{,n}+\tau^i_\alpha u_{,s},\quad \text{on } \partial\Omega_i,\ \alpha=1,2\tag{73}$$

we can compute

$$I_1=-\int_\gamma M_{\alpha\beta,\beta}(w_1)n^1_\alpha w_1+\int_\tau M_{\alpha\beta,\beta}(w_2)n^2_\alpha w_2=\int_\gamma \big(M_{\alpha\beta}(w_1)n^1_\beta\tau^1_\alpha\big)_{,s}w_1-\int_\tau \big(M_{\alpha\beta}(w_2)n^2_\beta\tau^2_\alpha\big)_{,s}w_2\tag{74}$$

$$\begin{aligned}I_2&=-\int_\gamma M_{\alpha\beta}(w_1)n^1_\beta w_{1,\alpha}+\int_\tau M_{\alpha\beta}(w_2)n^2_\beta w_{2,\alpha}\\&=-\int_\gamma \big(M_{\alpha\beta}(w_1)n^1_\beta n^1_\alpha\big)w_{1,n}-\int_\gamma \big(M_{\alpha\beta}(w_1)n^1_\beta\tau^1_\alpha\big)w_{1,s}+\int_\tau \big(M_{\alpha\beta}(w_2)n^2_\beta n^2_\alpha\big)w_{2,n}+\int_\tau \big(M_{\alpha\beta}(w_2)n^2_\beta\tau^2_\alpha\big)w_{2,s}\\&=-\int_\gamma \big(M_{\alpha\beta}(w_1)n^1_\beta\tau^1_\alpha\big)w_{1,s}+\int_\tau \big(M_{\alpha\beta}(w_2)n^2_\beta\tau^2_\alpha\big)w_{2,s}\end{aligned}\tag{75}$$

$$\begin{aligned}I_1-I_2&=\int_\gamma \big(M_{\alpha\beta}(w_1)n^1_\beta\tau^1_\alpha w_1\big)_{,s}-\int_\tau \big(M_{\alpha\beta}(w_2)n^2_\beta\tau^2_\alpha w_2\big)_{,s}\\&=\big(M(w_1)n^1\cdot\tau^1\big)(P_2)w_1(P_2)-\big(M(w_1)n^1\cdot\tau^1\big)(P_1)w_1(P_1)\\&\quad+\big(M(w_2)n^2\cdot\tau^2\big)(P_1)w_2(P_1)-\big(M(w_2)n^2\cdot\tau^2\big)(P_2)w_2(P_2)\end{aligned}\tag{76}$$

By (71), (72) and (76) and recalling that the solutions w_i coincide with all their derivatives up to the third order at P_1 and P_2, we have

$$\int_F M_{\alpha\beta}(w_2)w_{2,\alpha\beta}=I_2-I_1=K\big(w_1(P_1)-w_1(P_2)\big)\tag{77}$$

If $K=0$, then

$$0=I_1-I_2=-\int_F M_{\alpha\beta}(w_2)w_{2,\alpha\beta}\geqslant\frac{h^3}{12}\xi_0\int_F\big|\nabla^2 w_2\big|^2\tag{78}$$

and, F being a nonempty open set, w_2 coincides with an affine function h in F. By the weak unique continuation property, $w_2\equiv h$ in Ω_2, contradicting the choice of the nontrivial Neumann data $\hat{M}$ on Γ, see (31). Therefore, if $K=0$, we have

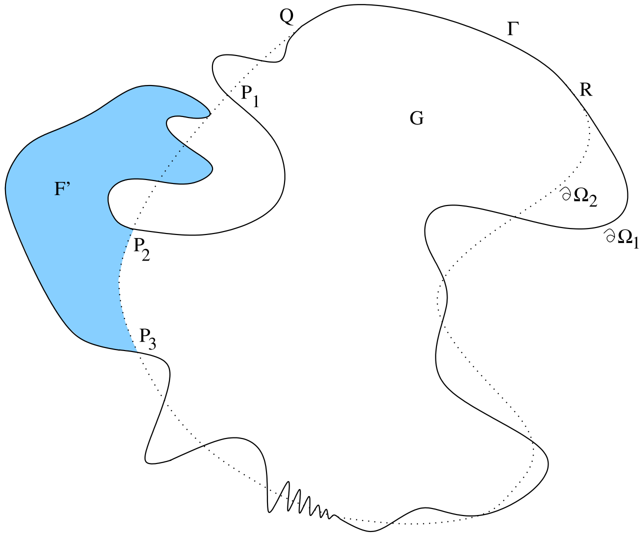

Fig. 2. The connected component F' of $\Omega_1 \setminus \overline{G}$.

a contradiction, that is $F = \emptyset$, and the thesis is proved. It is to notice that, by (71), $K = 0$ certainly holds when $\partial\Omega_1$ and $\partial\Omega_2$ are tangent either at P_1 or at P_2. Hence it remains to consider the case when $K \neq 0$, which implies that $\partial\Omega_1$ and $\partial\Omega_2$ are tangent neither at P_1 nor at P_2. In this case, P_2 is an isolated point of $\partial\Omega_1 \cap \partial\Omega_2$ and therefore, by the regularity of $\partial\Omega_i$, $i = 1, 2$, there exists $\rho > 0$ such that $B_\rho(P_2) \setminus (\partial\Omega_1 \cup \partial\Omega_2) = D_1 \cup D_2 \cup D_3 \cup D_4$, where $D_1 = F \cap B_\rho(P_2)$, $D_2 = (\Omega_1 \setminus \overline{\Omega}_2) \cap B_\rho(P_2)$, $D_3 = \Omega_1 \cap \Omega_2 \cap B_\rho(P_2) = G \cap B_\rho(P_2)$, $D_4 = B_\rho(P_2) \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$. Starting from $\Omega_1 \setminus \overline{\Omega}_2 \subset \Omega_1 \setminus \overline{G}$, and by performing the symmetric construction reversing the roles of Ω_1 and Ω_2, it is clear that there exists a connected component F' of $\Omega_1 \setminus \overline{G}$ such that $D_2 \subset F'$, see Fig. 2. Similarly, $\partial F' = \gamma' \cup \tau'$, where τ' is an open sub-arc of $\partial\Omega_2 \cap \partial G$ and γ' is a closed sub-arc of $\partial\Omega_1$ having the same endpoints P_2, $P_3 \in \gamma'$, with $P_3 = \varphi_1(s_3)$, for some $s_3 > s_2$.

By performing similar integration by parts for the equation $M_{\alpha\beta,\alpha\beta}(w_1) = 0$ in F', we have

$$\big(M(w_2)n^1 \cdot \tau^1\big)(P_3) - \big(M(w_2)n^2 \cdot \tau^2\big)(P_3) = \big(M(w_2)n^1 \cdot \tau^1\big)(P_2) - \big(M(w_2)n^2 \cdot \tau^2\big)(P_2) = K \tag{79}$$

Since we are assuming that $K \neq 0$, we have again that $\partial\Omega_1$ and $\partial\Omega_2$ are not tangent at P_3 and we can repeat the above arguments, constructing an increasing sequence s_n, such that $P_n = \varphi_1(s_n) \in \partial\Omega_1 \cap \partial\Omega_2$ and

$$\big(M(w_2)n^1 \cdot \tau^1\big)(P_n) - \big(M(w_2)n^2 \cdot \tau^2\big)(P_n) = K \tag{80}$$

The sequence s_n is bounded above by s_1'', where we recall that $R = \varphi_1(s_1'')$, since $(\Omega_2 \setminus \overline{G}) \cap \Gamma = \emptyset$ and $(\Omega_1 \setminus \overline{G}) \cap \Gamma = \emptyset$. Therefore s_n converges to some $\bar{s}$ and P_n converges to $\overline{P} = \varphi_1(\bar{s}) \in \partial\Omega_1 \cap \partial\Omega_2$. Thus $\overline{P}$ is a cluster point for $\partial\Omega_1 \cap \partial\Omega_2$, so that $\partial\Omega_1$ and $\partial\Omega_2$ are tangent at $\overline{P}$, implying that

$$\big(M(w_2)n^1 \cdot \tau^1\big)(\overline{P}) - \big(M(w_2)n^2 \cdot \tau^2\big)(\overline{P}) = 0 \tag{81}$$

On the other hand, by the regularity of the solutions and of the boundaries, we have that

$$\begin{aligned} K &= \lim_{n\to\infty} K = \lim_{n\to\infty} \big(M(w_2)n^1 \cdot \tau^1\big)(P_n) - \big(M(w_2)n^2 \cdot \tau^2\big)(P_n) \\ &= \big(M(w_2)n^1 \cdot \tau^1\big)(\overline{P}) - \big(M(w_2)n^2 \cdot \tau^2\big)(\overline{P}) = 0 \end{aligned} \tag{82}$$

obtaining a contradiction and concluding the proof of the theorem.

4.2. Proof of Theorem 3.2

Let G be the connected component of $\Omega \setminus (\overline{D_1 \cup D_2})$ such that $\Gamma \subset \partial G$. For $i = 1, 2$, let $h_i \in \mathcal{A}$ be such that $w_i|_{\overline{D}_i} = h_i$, and let us set $v_i = w_i - h_i$. Hence $v_i \equiv 0$ in $\overline{D}_i$ and, by (48), $\frac{\partial v_i^e}{\partial \nu} = 0$ on ∂D_i. Let us notice that, since v_i satisfies homogeneous Dirichlet conditions on the $C^{3,1}$ boundary ∂D_i, by regularity results we have that $v_i \in H^4(\widetilde{\Omega} \setminus D_i)$, for every $\widetilde{\Omega}$, $D_i \Subset \widetilde{\Omega} \Subset \Omega$, $i = 1, 2$ (see, for example, [19]). By Sobolev embedding theorems (see, for instance, [18]), we have that v_i and ∇v_i are continuous up to ∂D_i, $i = 1, 2$. Therefore

$$v_i \equiv 0, \qquad \nabla v_i^e \equiv 0, \quad \text{on } \partial D_i \tag{83}$$

Let us set $h = h_2 - h_1$, $h(x) = ax_1 + bx_2 + c$. Then we have that the function $w = v_1 - v_2 - h$ satisfies the following Cauchy problem

$$\begin{cases} M_{\alpha\beta,\alpha\beta} = 0, & \text{in } G & (84)\\ w = 0, & \text{on } \Gamma & (85)\\ \dfrac{\partial w}{\partial n} = 0, & \text{on } \Gamma & (86)\\ M_{\alpha\beta}(w) n_\alpha n_\beta = 0, & \text{on } \Gamma & (87)\\ M_{\alpha\beta,\beta}(w) n_\alpha + \big(M_{\alpha\beta}(w) n_\alpha \tau_\beta\big)_{,s} = 0, & \text{on } \Gamma & (88)\end{cases}$$

From the uniqueness of the solution to the Cauchy problem (84)–(88) (see, for instance, Proposition 3 in [16]) and from the weak unique continuation property (see, for example, Proposition 2 and Remark 1 in [16]), we have that

$$w \equiv 0, \quad \text{in } G \tag{89}$$

Let us prove for instance that $D_2 \subset D_1$. We have that

$$D_2 \setminus \overline{D}_1 \subset \Omega \setminus (\overline{D_1 \cup G}) \tag{90}$$

$$\partial\big(\Omega \setminus (\overline{D_1 \cup G})\big) = \Sigma_1 \cup \Sigma_2 \tag{91}$$

where $\Sigma_2 = \partial D_2 \cap \partial G$ and $\Sigma_1 = \partial(\Omega \setminus (\overline{D_1 \cup G})) \setminus \Sigma_2 \subset \partial D_1$.

We can distinguish the following two cases:

(i) $\partial D_1 \cap \Sigma_2 \neq \emptyset$;
(ii) $\partial D_1 \cap \Sigma_2 = \emptyset$.

If (i) holds, then there exists $P_0 \in \partial D_1 \cap \Sigma_2$. Then, by (83) and (89), $h(P_0) = 0$. Moreover, given a sequence of points $P_n \in G$ converging to P_0, again by (83) and (89), we have that

$$0 = \nabla w(P_n) = \nabla v_1(P_n) - \nabla v_2(P_n) - (a, b) \tag{92}$$

$$0 = \lim_{n\to\infty} \nabla w(P_n) = \nabla v_1^e(P_0) - \nabla v_2^e(P_0) - (a, b) = -(a, b) \tag{93}$$

that is $h \equiv c$, but $h(P_0) = 0$, so that $h \equiv 0$, that is $v_1 \equiv v_2$ in G.

Integrating by parts the equation $M_{\alpha\beta,\alpha\beta}(v_1)v_1 = 0$ in $\Omega \setminus (\overline{D_1 \cup G})$ we obtain

$$-\int\limits_{\Omega\setminus(\overline{D_1\cup G})} M_{\alpha\beta}(v_1) v_{1,\alpha\beta} = \int\limits_{\partial(\Omega\setminus(\overline{D_1\cup G}))} M_{\alpha\beta},_\beta(v_1)\nu_\alpha v_1 - \int\limits_{\partial(\Omega\setminus(\overline{D_1\cup G}))} M_{\alpha\beta}(v_1)\nu_\beta v_{1,\alpha} \tag{94}$$

where ν denotes the outer unit normal to $\Omega \setminus (\overline{D_1 \cup G})$. Let us notice that $\nu = n^1$ on Σ_1 and $\nu = -n^2$ on Σ_2. By (91) and (83), and using the fact that $v_1 = v_2$, $\nabla v_1 = \nabla v_2$ in Σ_2, we have

$$0 = -\int\limits_{\Omega\setminus(\overline{D_1\cup G})} M_{\alpha\beta}(v_1) v_{1,\alpha\beta} \geqslant \frac{h^3}{12}\xi_0 \int\limits_{D_2\setminus\overline{D}_1} \big|\nabla^2 v_1\big|^2 \tag{95}$$

where $\xi_0 > 0$ is the ellipticity constant appearing in (29). If the open set $D_2 \setminus \overline{D}_1$ were nonempty then, by the weak unique continuation principle, w_1 coincides with an affine function in $\Omega \setminus \overline{D}_1$, contradicting the choice of the nontrivial Neumann data $\hat{M}$ on $\partial\Omega$. Therefore, $D_2 \subset \overline{D}_1$ and, since D_2 is open and by the regularity of ∂D_1, it follows that $D_2 \subset D_1$.

In case ii), it is easy to see that either $\overline{D}_1 \cap \overline{D}_2 = \emptyset$ or $\overline{D}_1 \subset D_2$. Let us consider in detail the first situation, the second being similar. If $\overline{D}_1 \cap \overline{D}_2 = \emptyset$ then, integrating by parts the equation $M_{\alpha\beta},_{\alpha\beta}(v_1)v_1 = 0$ in D_2, we obtain

$$-\int\limits_{D_2} M_{\alpha\beta}(v_1) v_{1,\alpha\beta} = -\int\limits_{\partial D_2} M_{\alpha\beta,\beta}(v_1) n_\alpha v_1 + \int\limits_{\partial D_2} M_{\alpha\beta}(v_1) n_\beta v_{1,\alpha} \tag{96}$$

By using the relations (73) for $\Omega_i = \Omega \setminus \overline{D}_i$ and by the regularity of ∂D_2, Eq. (96) can be rewritten as follows

$$\begin{aligned} -\int\limits_{D_2} M_{\alpha\beta}(v_1) v_{1,\alpha\beta} &= -\int\limits_{\partial D_2} \big(M_{\alpha\beta,\beta}(v_1) n_\alpha v_1 - M_{\alpha\beta}(v_1) n_\beta \tau_\alpha v_{1,s}\big) + \int\limits_{\partial D_2} M_{\alpha\beta}(v_1) n_\alpha n_\beta v_{1,n} \\ &= -\int\limits_{\partial D_2} \big(M_{\alpha\beta,\beta}(v_1) n_\alpha + \big(M_{\alpha\beta}(v_1) n_\beta \tau_\alpha\big)_{,s}\big) v_1 + \int\limits_{\partial D_2} M_{\alpha\beta}(v_1) n_\alpha n_\beta v_{1,n} \end{aligned} \tag{97}$$

By using Eq. (49) with $g = h \in \mathcal{A}$, by (97) and recalling that $v_2 = v_1 - h = 0$, $v_{2,n} = v_{1,n} - h_{,n} = 0$ on ∂D_2, we have

$$-\int_{D_2} M_{\alpha\beta}(v_1)v_{1,\alpha\beta} = -\int_{\partial D_2} \big(M_{\alpha\beta,\beta}(v_1)n_\alpha + \big(M_{\alpha\beta}(v_1)n_\beta\tau_\alpha\big)_{,s}\big)(v_1 - h) + \int_{\partial D_2} M_{\alpha\beta}(v_1)n_\alpha n_\beta(v_{1,n} - h_{,n}) = 0 \tag{98}$$

As seen for case (i), we have that v_1 coincides with an affine function h in D_2. If D_2 is nonempty then, by the weak unique continuation property, $v_1 \equiv h$ in $\Omega \setminus \overline{D}_1$, contradicting the choice of the nontrivial Neumann data $\hat{M}$ on $\partial\Omega$. Therefore $D_2 = \emptyset$. Similarly, one can prove that $D_1 = \emptyset$ and therefore $D_1 = D_2$.

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