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On inflating closed mylar shells

Sur le gonflage des ballons de mylar fermés

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ABSTRACT

We discuss the inflating of a closed thin shell made of inextensible flexible material like mylar. The problem is to determine the extremal form of the shell, when it is inflated to the maximal possible volume. We introduce a variational problem which describes the inflating of rotationally symmetric shells. The main result presents a criteria for a rotationally symmetric shell to admit volume increasing deformations without surface stretching. Moreover explicit solutions are found for cylindrical and biconical shells.

RÉSUMÉ

On étudie le gonflage d'un ballon mince fermé produit d'un matériel non-extensible et souple comme mylar. La question est de déterminer la forme extrémal du ballon, quand il est gonflé jusqu'au volume maximal possible. On présente un problème variationnel qui décrit le gonflage des ballons de rotation. Le résultat essentiel est un critère pour un ballon de rotation d'admettre déformations qui accroissent le volume sans étendre la surface. En plus, des solutions explicites sont trouvées au cas des ballons cylindrique et biconique.

1. Introduction

We consider a closed thin shell F made of some inextensible flexible material like the mylar (a polyester made of the extremely thin sheets of great tensile strength). If one starts to inflate the shell with air, its surface suffers some deformations aimed at increasing the volume inside the shell. In the deformation process, the shell surface does not stretch because of the material properties. On the other hand it easily admits various small wrinkles, crumples and folds due to a high flexibility of the mylar, cf. [1]. *The main problem is to determine the extremal form of the shell, F_{max}, when it is inflated to the maximal possible volume*. Our intuition and everyday experience as well as numerical experiments [2], suggest that for any initial shell F this problem has a unique solution F_{max} which represents a rather regular surface inheriting symmetry properties of F. However actually this quite difficult problem is very far from to be solved.

Evidently, if F is spherical, then it cannot be inflated without surface stretching, i.e. $F_{max} = F$, due to the well-known isoperimetric property of the sphere. Shells that have the same property, $F_{max} = F$, are called *the sandbags* [3,4].

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The only one non-trivial case, when the explicit solution was obtained with some mathematical rigorousity, has been discussed by W. Paulsen [5], see also [6]. It deals with a shell formed by two circular disks sewed along their boundaries. In this case it turns out that the inflated shell F_{max} is not spherical, as one might expect, but represents a well-defined closed convex surface of rotation, a *Paulsen balloon*, whose form is similar to a squeezed ellipsoid. To our knowledge, for other kinds of mylar shells, different from spheres and two-covered discs, the stated problem is not solved. Notice that the case of polyhedral shells was treated in [3,7,8]. The main result states that no one polyhedral shell is a sandbag. However, the extremal shell F_{max} is not determined even for simplest case, when F is a cube or a tea-bag.

In this Note we discuss the inflating of closed *convex rotationally symmetric* shells. The simplest examples of such shells are right cylinders, cones and bicones, ellipsoids of rotation, etc. We consider a geometrical model, which leads us to some 1-D variational problem with constraints. Writing down the corresponding Euler–Lagrange equation, we analyze its solutions and describe corresponding extremal shells. As a consequence, a criterion for the initial shell F to be a sandbag is obtained. Roughly speaking, F is a sandbag if it is sufficiently elongated along the axis of rotation, like a long cylinder. For instance, an ellipsoid $x^2/a^2 + y^2/a^2 + z^2/c^2 = 1$ is a sandbag, iff $a \leqslant c\sqrt{2}$. Moreover, in the case when F is either a right cylinder or a right bicone, the extremal shell F_{max} is described explicitly with the help of Paulsen balloon's strips and caps.

Shells similar to mylar ones may arise in aeronautic engineering as well as in microbiology, so we hope our results will be useful for practical applications.

2. Inflating: geometric model

Let F be a closed convex rotationally symmetric surface. Consider a short transformation $\Phi : F \to \tilde{F}$. By definition, the shortness means that for an arbitrary curve Γ in F its length is greater or equal to the length of the corresponding curve $\Phi(\Gamma)$ in $\tilde{F}$ [9]. Usually $\tilde{F}$ is referred to as short w.r.t. F and denoted by $\tilde{F} \prec F$. The set of surfaces obtained by short transformations of F is denoted by $W_F = \{\tilde{F} \in R^3 | \tilde{F} \prec F\}$.

Our goal is *to find a surface F_{max} in W_F enclosing the maximal volume*. This general problem is rather difficult, because short transformations may be highly irregular and even shrink F to curves or points, so transformed surfaces may have very complicated structure. However, when we ask for the volume maximization of surfaces of rotation, it is natural to reduce the class of transformations by making some additional assumptions, cf. [5].

Definition. A short transformation $\Phi : F \to \tilde{F}$ is said to be *special short* if it satisfies the following conditions:

(i) Φ is piecewise smooth of class C^1;
(ii) $\tilde{F}$ is rotationally symmetric;
(iii) Φ maps parallels and meridians of F to parallels and meridians of $\tilde{F}$ respectively, moreover the lengths of meridians are preserved, whereas the lengths of parallels are non-increasing.

Assumption (ii) is based on the observation that the extremal surface F_{max} should inherit the symmetry properties of the initial surface F, so it should remain rotationally symmetric. Assumption (iii) seems to be slightly doubtful, but this is just what was seen for the Paulsen mylar balloons [5,6].

The class of surfaces obtained from F by special short transformations will be denoted by W_F^0. Clearly $W_F^0 \subset W_F$. We believe that in order to find F_{max} one can replace W_F by W_F^0.

Conjecture. $\sup_{\tilde{F} \in W_F} Vol(\tilde{F}) = \sup_{\tilde{F} \in W_F^0} Vol(\tilde{F})$.

This conjecture is a really challenging mathematical problem, however from the physical point of view it seems to be rather reasonable. Benefits of the proposed replacement is that the original problem may be reduced to a much more simple analytic problem.

3. Initial shell representation

Let F be obtained by rotating of a plane curve γ around the z-axis. The meridian γ is represented as

$$x = u(s), \qquad z = v(s) \tag{1}$$

where $s \in [0, L]$ is the arc length of γ counted from the top $P(s = 0)$ to the bottom $Q(s = L)$, L denotes the total length of γ. We assume that γ is piecewise smooth of class C^2, i.e. γ is C^2-smooth everywhere in $[0, L]$ except a finite number of singular points $0 < s_1 < \cdots < s_N < L$. It means that F may contain a finite number of singular parallels. Moreover conical singularities at the poles P and Q are also admitted.

Due to the choice of parametrization, we have $(u')^2 + (v')^2 \equiv 1$. Since F is closed and convex, the function $u(s)$ satisfies the following conditions:

$$u(0)=0, \qquad u(L)=0, \qquad u'(0)>0, \qquad u'(L)<0 \tag{2}$$

$$u(s)>0, \quad s\in(0,L) \tag{3}$$

$$u' \text{ is non-increasing} \tag{4}$$

whereas $v(s)$ is determined up to an additive constant by the formula $v(s)=-\int_0^s \sqrt{1-(u')^2}\,\mathrm{d}s$.

We emphasize that all the geometric properties of F can be described in terms of the function $u(s)$, which will be referred to as the *profile* of F. In particular, the volume enclosed by F is calculated as follows:

$$V=\pi\int_0^L u^2\sqrt{1-\left(u'\right)^2}\,\mathrm{d}s \tag{5}$$

4. Volume functional and variational problem

Given the profile $u(s)$, consider its variation $\tilde{u}(s,\varepsilon)=u(s)+\varepsilon w(s)$, where $w(s)$ is assumed to be piecewise C^1-smooth, satisfying the initial conditions $w(0)=w(L)=0$ and ensuring $\tilde{u}(s)>0$, $s\in(0,L)$ for all ε in an interval I. Setting $\tilde{v}=-\int_0^s\sqrt{1-(\tilde{u}')^2}\,\mathrm{d}s$ and writing a parametric representation like (1), we obtain a variation $\tilde{\gamma}_\varepsilon$ of the initial meridian γ. Consequently, rotating $\tilde{\gamma}_\varepsilon$, we obtain a variation $\tilde{F}_\varepsilon$ of the initial surface F.

For any ε, the surface $\tilde{F}_\varepsilon$ is closed, piecewise smooth, rotationally symmetric. The corresponding transformation $F\to\tilde{F}_\varepsilon$ maps meridians and parallels of F to meridians and parallels of $\tilde{F}_\varepsilon$. The meridians in F and in $\tilde{F}_\varepsilon$ have the same lengths equal to L. As for parallels, a parallel with radius $u(s)$ in F is mapped to a parallel with radius $\tilde{u}(s;\varepsilon)$ in $\tilde{F}_\varepsilon$.

The volume enclosed by $\tilde{F}_\varepsilon$ is equal to $V=\pi\int_0^L\tilde{u}^2\sqrt{1-(\tilde{u}')^2}\,\mathrm{d}s$, it depends obviously on ε. The following statement may be proved by a direct calculation:

Proposition 4.1. *The following first variation formula holds*:

$$\begin{aligned}\frac{1}{\pi}\frac{\mathrm{d}V}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}&=\int_0^L\frac{u}{(1-(u')^2)^{3/2}}\left(uu''+2\left(1-\left(u'\right)^2\right)\right)w\,\mathrm{d}s-\frac{u^2u'}{\sqrt{1-(u')^2}}w\bigg|_0^L\\&\quad+\sum_i u^2(s_i)\left(\lim_{s\to s_i+0}\frac{u'}{\sqrt{1-(u')^2}}-\lim_{s\to s_i-0}\frac{u'}{\sqrt{1-(u')^2}}\right)w(s_i)\end{aligned} \tag{6}$$

The last term in (6) means the sum over the singular points s_i of $u(s)$. The integrand in (6) is geometrically invariant and may be expressed in terms of the principal curvatures of F, whereas other summands may be described via the angle between the vector tangent to meridian and the positive direction of the x-axis. Notice that the expression in parentheses in the last sum is negative, since u' is non-increasing.

Formula (6) holds for an arbitrary variation of $u(s)$. On the other hand, we search to maximize the volume of F with respect to short transformations in W_F^0, so not any variation is admissible. Namely, it is easy to see that the transformation $F\to\tilde{F}_\varepsilon$ is short, so $\tilde{F}_\varepsilon$ belongs to W_F^0, if and only if $u(s)\geqslant\tilde{u}(s;\varepsilon)$, i.e. if $\varepsilon w(s)\leqslant 0$. Without loss of generality one can suppose that $\varepsilon\geqslant 0$ and $w(s)\leqslant 0$, such variations will be referred to as admissible.

Let us analyze (6). Due to (2), the second term in (6) vanishes if $w(s)\equiv 0$ in neighborhoods of end points in $[0,L]$, so it does not affect the volume. The third term either vanishes, if $w(s)\equiv 0$ in neighborhoods of singular points s_j, or it is positive, if $w(s)<0$ in a neighborhood of a singular point s_j. Hence, if $w(s)$ is chosen to be negative in a sufficiently small neighborhood of s_j and vanishes elsewhere, then $\frac{\mathrm{d}V}{\mathrm{d}\varepsilon}(0)>0$. Thus, the presence of singular points allows us to increase the volume. Finally the sign of the first term in (6) depends on the sign of $u''+2(1-(u')^2)$. The further analysis leads to the following result:

Proposition 4.2. *If $u(s)$ has at least one singular point in $(0,L)$ or if $uu''+2(1-(u')^2)<0$ somewhere in $[0,L]$, then there is an admissible variation $\tilde{u}(s,\varepsilon)$ with $\frac{\mathrm{d}V}{\mathrm{d}\varepsilon}|_{\varepsilon=0}>0$.*

Corollary. *If a closed convex surface of rotation F contains at least one singular parallel or its profile function $u(s)$ satisfies $uu''+2(1-(u')^2)<0$ somewhere in $[0,L]$, then F admits volume increasing special short transformations.*

Let us discuss the extremal profile $u_{max}(s)$, which maximizes the volume functional. Clearly, we have:

$$u_{max}(s)=0, \qquad u_{max}(L)=0 \tag{7}$$

$$0\leqslant u_{max}(s)\leqslant u(s), \quad s\in[0,L] \tag{8}$$

Next, if there exists a subinterval $(a,b) \subset (0,L)$ such that $u_{max}(s) < u(s)$ everywhere in (a,b) and $u_{max}(a) = u(a)$, $u_{max}(b) = u(b)$, then due to (6) the function u_{max} has to satisfy in (a,b) the equation

$$uu'' + 2\bigl(1 - (u')^2\bigr) = 0 \tag{9}$$

Finally, $u_{max}(s)$ has to be C^1-smooth, i.e. without singular points in $(0,L)$, as discussed above.

Eq. (9) may be solved in terms of Jacobi's elliptic functions [10], its solution is given by

$$\xi(s) = \frac{1}{C_1}\,\mathrm{sn}(C_1 s + C_2, i)$$

Analyzing the properties of $\xi(s)$, one can prove the following technical result:

Lemma. *If there exists a solution $v(s)$ of* (9)*, which satisfies $v(s) < u(s)$ everywhere in (a,b) and $v(a) = u(a)$, $v(b) = u(b)$, then $u'' + 2(1-(u')^2) < 0$ somewhere in $[a,b]$ or $u(s)$ has a singular point in $[a,b]$.*

Consequently, if $u(s)$ is smooth and satisfies $uu'' + 2(1-(u')^2) \geqslant 0$ everywhere in $[0,L]$, then u_{max} has to coincide with $u(s)$ everywhere in $[0,L]$. Thus we obtain the following criterion for F to be a sandbag.

Proposition 4.3. *If $u(s)$ satisfies $uu'' + 2(1-(u')^2) \geqslant 0$ and does not contain any singular point in $(0,L)$, then any admissible variation of $u(s)$ decreases the volume, so $u_{max} \equiv u$.*

Corollary. *If a closed convex surface of rotation F does not contain singular parallels and if its profile function $u(s)$ satisfies $uu'' + 2(1-(u')^2) \geqslant 0$ everywhere in $[0,L]$, then F is a sandbag in W_F^0, i.e. it does not admit volume increasing special short transformations.*

Below we discuss some examples in order to illustrate Propositions 4.2 and 4.3.

5. Explicit solutions

5.1. Sphere

Let F be a sphere of radius R. Then $u = R\sin\dfrac{s}{R}$, $s \in [0,\pi R]$. It is easy to calculate, that $uu'' + 2(1-(u')^2) = \sin^2\frac{s}{R} \geqslant 0$. Hence F is a sandbag by Proposition 4.3.

5.2. Cylinders

Let F be a circular cylinder with base radius R and height $2H$. Then

$$u = \begin{cases} s, & s \in [0,R] \\ R, & s \in [R, R+2H] \\ -s + 2H + 2R, & s \in [R+2H, 2R+2H] \end{cases} \tag{10}$$

Since $u(s)$ has two singular points $s_1 = R$, $s_2 = R + 2H$, the volume enclosed by F may be increased by small short variations due to Proposition 4.2.

The form of the maximal profile u_{max} depends on the difference between the ratio H/R and the first lemniscate constant $L_1 = \int_0^1 \frac{1}{\sqrt{1-t^4}}\,dt \approx 1.311$. If the cylinder is "low" so that $H/R \leqslant L_1 - 1$, then u_{max} is given by the formula

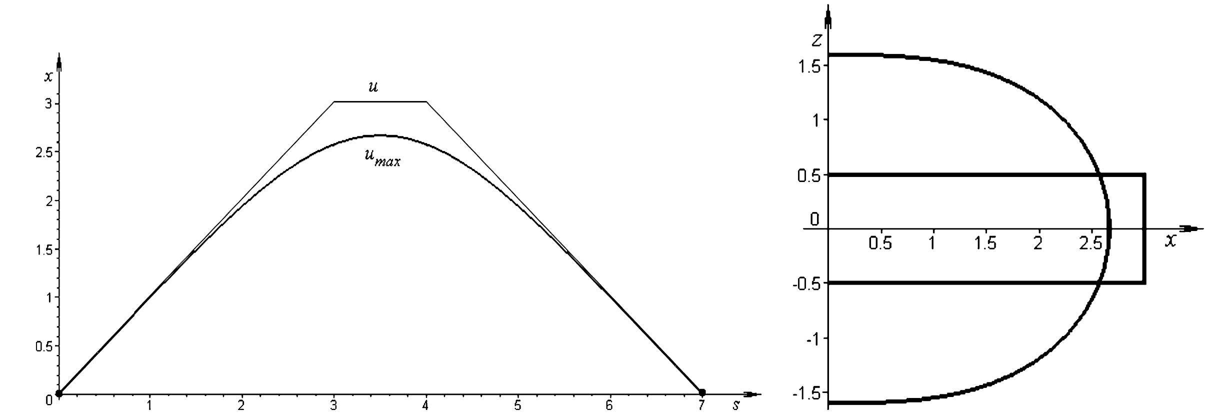
$$u_{max} = \frac{R+H}{L_1}\,\mathrm{sn}\!\left(\frac{L_1}{R+H}s, i\right), \quad s \in [0, 2R+2H] \tag{11}$$

The profile u_{max} corresponds to the Paulsen mylar balloon, whose meridian has the same length as the meridian of the initial cylinder. Fig. 1 shows a) the profiles u and u_{max}, b) the corresponding meridians. Fig. 2 demonstrates the cylinder F and the extremal surface F_{max}, which is a Paulsen mylar balloon.

If the cylinder is high so that $H/R > L_1 - 1$, then u_{max} is given by the following formula (see Fig. 3a):

$$u_{max} = \begin{cases} R\,\mathrm{sn}\bigl(\frac{s}{R}, i\bigr), & s \in [0, RL_1] \\ R, & s \in [RL_1, R(2-L_1)+2H] \\ R\,\mathrm{sn}\bigl(\frac{s-2H}{R} + 2(L_1-1), i\bigr), & s \in [R(2-L_1)+2H, 2R+2H] \end{cases} \tag{12}$$

The profile u_{max} corresponds to a C^1-smooth surface F_{max} composed of three parts: two parts are halves of a Paulsen mylar balloon, and the third one is cylindrical (see Figs. 3b and 4).



(a) (b)

Fig. 1. (a) “Low” cylinder: the profiles u and u_{max}. (b) “Low” cylinder: the meridians γ and γ_{max}.

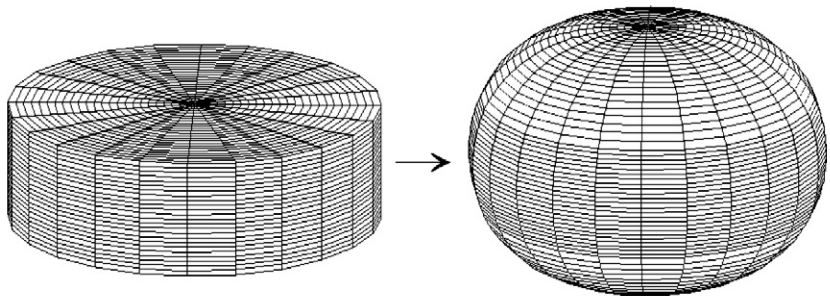
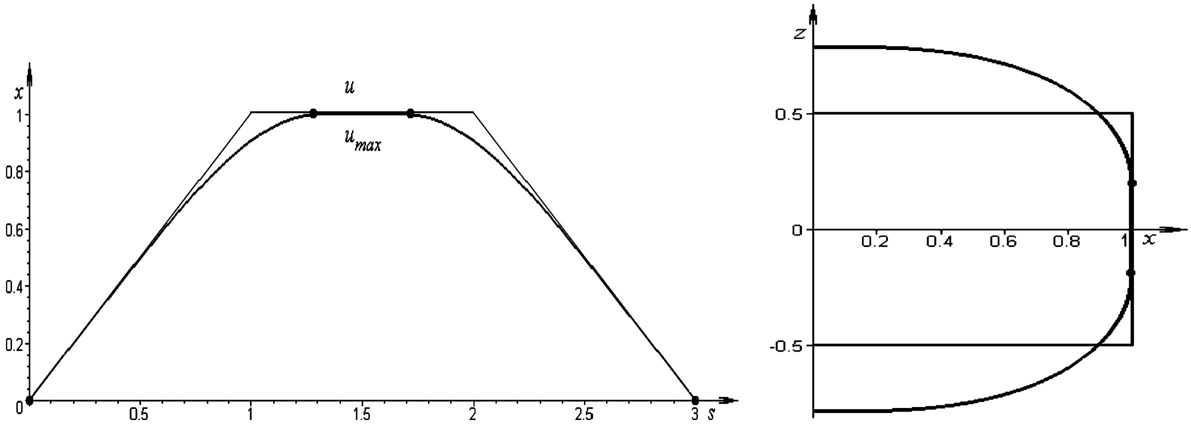


Fig. 2. “Low” cylinder: non-inflated (left) and fully inflated (right).



(a) (b)

Fig. 3. (a) “High” cylinder: the profiles u and u_{max}. (b) “High” cylinder: the meridians γ and γ_{max}.

5.3. Bicones

Let F be a circular bicone with base radius R and height $2H$. Then

$$u = \begin{cases} \frac{R}{\sqrt{R^2+H^2}} s, & s \in [0, \sqrt{R^2+H^2}] \\ -\frac{R}{\sqrt{R^2+H^2}} s + 2R, & s \in [\sqrt{R^2+H^2}, 2\sqrt{R^2+H^2}] \end{cases} \tag{13}$$

Since $u(s)$ has a singular point $s_1 = \sqrt{R^2+H^2}$, the volume enclosed by F may be increased by small short variations due to Proposition 4.2. The maximal profile u_{max} is given by the following formula (see Fig. 5a):

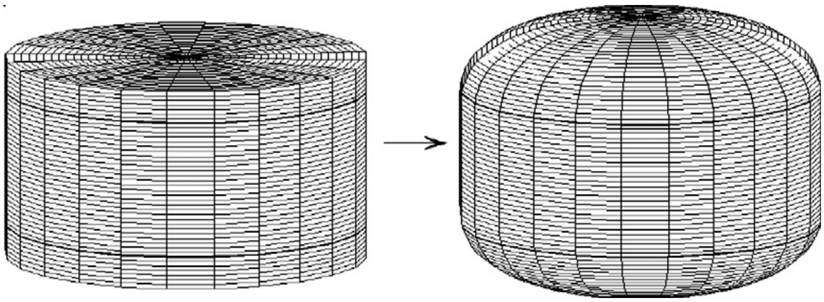


Fig. 4. “High” cylinder: non-inflated (left) and fully inflated (right).

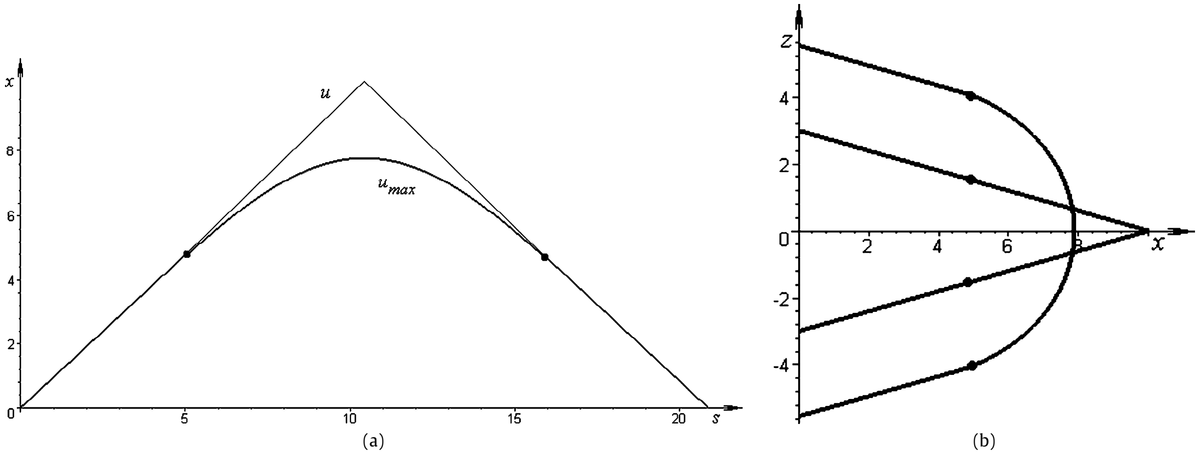


Fig. 5. (a) Bicone: the profiles u and u_{max}. (b) Bicone: the meridians γ and γ_{max}.

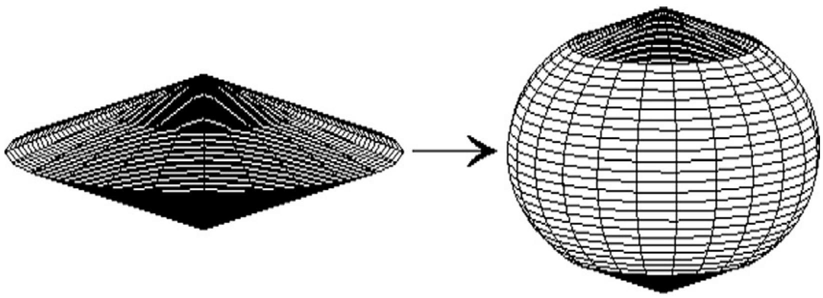


Fig. 6. Bicone: non-inflated (left) and fully inflated (right).

$$u_{max} = \begin{cases} \frac{R}{\sqrt{R^2+H^2}} s, & s \in [0, a] \\ \frac{1}{C} \operatorname{sn}(C(s - \sqrt{R^2+H^2}) + L_1, i), & s \in [a, 2\sqrt{R^2+H^2} - a] \\ -\frac{R}{\sqrt{R^2+H^2}} s + 2R, & s \in [2\sqrt{R^2+H^2} - a, 2\sqrt{R^2+H^2}] \end{cases} \tag{14}$$

where $a = a(R, H)$, $C = C(R, H)$. The profile u_{max} corresponds to a C^1-smooth surface composed of three parts: two ones are parts of the initial bicone, the third one is a strip of a Paulsen mylar balloon (see Figs. 5b and 6).

5.4. Ellipsoid of rotation

Let F be an ellipsoid given by $x^2/a^2 + y^2/a^2 + z^2/c^2 = 1$. Then $u(s) = a\cos(t(s))$, where $t(s) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is determined by $t' = 1/\sqrt{a^2 \sin^2 t + c^2 \cos^2 t}$. In this case

$$u'' + 2\big(1 - (u')^2\big) = \frac{c^2 \cos^2 t}{(a^2 \sin^2 t + c^2 \cos^2 t)^2} \big(2a^2 \sin^2 t + 2c^2 \cos^2 t - a^2\big)$$

An elementary analysis demonstrates, that $2a^2\sin^2 t + 2c^2\cos^2 t - a^2 \geqslant 0$ if $2c^2 \geqslant a^2$. Thus F is a sandbag, if $2c^2 \geqslant a^2$, due to Proposition 4.3.

On the other hand, if $2c^2 < a^2$ then $2a^2\sin^2 t + 2c^2\cos^2 t - a^2 < 0$ holds in a subinterval of $[0, L]$, so $u'' + 2(1 - (u')^2) < 0$ holds too. Hence, due to Proposition 4.2, one can increase the volume of F by special short transformations. In order to obtain $u_{max}(s)$, one have to replace $u(s)$ by a solution of (9) in some subinterval of $[0, L]$. From the geometrical point of view, in order to obtain the corresponding extremal surface F_{max}, we have to replace some strip of F by a strip of a Paulsen mylar balloon.

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