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Comparing hinged and supported rectangular plates

Comparaison d'une plaque rectangulaire simplement appuyée et d'une plaque charnière

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ABSTRACT

We consider the Kirchhoff–Love model for the supported plate, that is, the fourth order differential equation $\Delta^2 u = f \leqslant 0$ in a two-dimensional bounded domain Ω with the condition $u_{|\partial\Omega} \geqslant 0$ and supplemented with natural boundary conditions. We show that the solution differs from the solution of the hinged plate problem, that is, the bi-Laplace equation with $u = \Delta u = 0$ on the boundary, in the case of a rectangular domain.

RÉSUMÉ

On considère le modèle de Kirchhoff–Love pour des plaques minces simplement appuyées, c'est à dire l'équation aux dérivées partielles du quatrième ordre $\Delta^2 u = f \leqslant 0$ sur une domaine borné Ω de dimension deux avec la condition $u_{|\partial\Omega} \geqslant 0$ et supplementée avec les conditions naturelles. Nous démontrons que la solution de ce problème n'est pas identique à la solution d'une plaque charnière dans le cas où cette plaque est rectangulaire. Dans cet dernier cas, les conditions de bord sont $u = \Delta u = 0$.

1. Introduction

1.1. Description and motivation

In the mathematical and engineering literature the (simply) supported and hinged boundary conditions of a transversally loaded plate are often confused. A reason for that may be the expectation that a plate will behave similarly to a beam: when negatively loaded, it will go down. Such a maximum principle type result is not however clear for a two-dimensional plate. In other words, one should differentiate between the following boundary conditions:

- *hinged:* the deflection of the plate is zero on the boundary;
- *supported:* the deflection of the plate cannot become negative on the boundary.

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The main question is:

Does a plate which is supported at its boundary by walls of constant height and is pushed downwards, touch this supporting structure everywhere?

The purpose of this Note is to illustrate, within the framework of the Kirchhoff–Love model, that a negatively loaded, simply supported rectangular plate will exhibit bending moments, concentrated at the corners, which will force the plate to lift there. We consider the rectangular plate, since this not only represents the most relevant shape, but also since the 90° angles mark a special case in the sense that the solution has a regularity that is not shared by domains with other angles. In that case, one has to consider the Williams–Kondrat'ev–Grisvard [1–3] approach for the behavior of the solution near a corner. This will be done in a forthcoming paper [4], where one will also be able to find the proofs of the following results.

The model of a hinged plate shaped as a regular n-polygon has been studied by Babuška. He noticed in [5] that the solution of the hinged problem on a disk is not approximated by the solutions of the hinged problem on a regular n-polygon when $n \to \infty$. See also [6,7]. Both this 'paradox' named after Babuška, and the lifting at corners proven here is due to a δ-type moment at corners points. The mathematical formulations and proofs however are rather distinct.

1.2. The mathematical setting

The Kirchhoff–Love model for thin elastic plates can be considered as the Euler–Lagrange equation that arises in the following minimization problem:

Find $u_0 : \Omega \to \mathbb{R}$ in an appropriate family of functions $\mathcal{V}$ such that:

$$J_\sigma(u_0) = \min_{u \in \mathcal{V}} J_\sigma(u), \quad \textit{with } J_\sigma(u) = \int\limits_\Omega \left(\frac{1}{2}(\Delta u)^2 + (1-\sigma)\big(u_{xy}^2 - u_{xx}u_{yy}\big) - fu \right) \mathrm{d}x\,\mathrm{d}y \tag{1}$$

Here Ω represents the shape of the plate and $u(x)$ represents the deflection at $x \in \Omega$ under the vertical load density $f(x)$. The parameter σ denotes the Poisson ratio of the plate, constant for the homogeneous situation considered here and, depending on the material, varying from -1 up to $\frac{1}{2}$. The minimal value of the functional corresponds to the elastic energy of the deformed plate. Introducing boundary conditions through an appropriate set of functions $\mathcal{V}$, one models the different cases:

- hinged case: $\mathcal{V} = H_0(\Omega) := W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$,
- supported case: $\mathcal{V} = H_+(\Omega) := \{u \in W^{2,2}(\Omega);\ u^- := -\min(u,0) \in W_0^{1,2}(\Omega)\}$.

2. Existence results

We establish the existence of a minimizer for J_σ in the aforementioned cases. When the plate is hinged, existence is straightforward.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be open, bounded with Lipschitz boundary. Let J_σ be as in* (1) *with $-1 < \sigma < 1$ and $f \in L^2(\Omega)$. Then J_σ possesses a unique minimizer in $H_0(\Omega)$.*

The proof of the above theorem is a direct application of the Riesz representation theorem, since the square root of the sum of the L^2 norms of the second derivatives of a function is a norm in $H_0(\Omega)$ when $\partial\Omega$ is Lipschitz.

Proving the existence of minimizers in $H_+(\Omega)$ is not so straightforward; the functional is not coercive in $H_+(\Omega)$, which is also a closed subset of $W^{2,2}(\Omega)$. Moreover, we will need to impose some condition on f since, otherwise, if $f > 0$ then for a fixed $u_0 \in H_+(\Omega)$ and $t > 0$ we have that $u_0 + t \in H_+(\Omega)$ and $\lim_{t\to\infty} J_\sigma(u_0 + t) = -\infty$.

Theorem 2.2. *Let $\Omega \in \mathbb{R}^2$ with a Lipschitz boundary and $-1 < \sigma < 1$. Moreover, assume that $0 \not\equiv f \in L^2(\Omega)$ such that*

$$\int\limits_\Omega f\zeta\,\mathrm{d}\lambda < 0, \quad \textit{for all nontrivial } \zeta \in H_+(\Omega) \textit{ with } \|\zeta_{xx}\|_2 = \|\zeta_{xy}\|_2 = \|\zeta_{yy}\|_2 = 0 \tag{2}$$

Then, there exists a unique minimizer $u_\sigma \in H_+(\Omega)$ of J_σ satisfying $J'_\sigma(u_\sigma; v - u_\sigma) \geqslant 0$ for all $v \in H_+(\Omega)$.

The proof of existence is based on a standard regularization argument. Uniqueness comes from the fact that J'_σ is a monotone operator. The proof is based on techniques found in [8].

3. The regularity of hinged rectangular plates

From now on we restrict ourselves to the rectangle $\mathcal{R} = (0,a) \times (0,b)$ with $a, b > 0$.

3.1. An extension and a density lemma

Using the symmetries of our rectangle and elliptic regularity one can deploy a reflection argument to show the following:

Lemma 3.1. *For $u : \overline{\mathcal{R}} \to \mathbb{R}$ define*

$$Eu(x,y) = \begin{cases} u(x,y), & \text{for } (x,y) \in \mathcal{R} \\ -u(-x,y), & \text{for } (-x,y) \in \mathcal{R} \\ 0, & \text{elsewhere} \end{cases} \tag{3}$$

and set $\mathcal{R}_0 = (-a,a) \times (0,b)$. Then the operator $E : H_0(\mathcal{R}) \to H_0(\mathcal{R}_0)$ is continuous.

This lemma allows us to prove an approximation theorem for Sobolev functions with boundary consitions. We recall the following definitions:

$$C^k(\overline{\mathcal{R}}) = \left\{u \in C^k(\mathcal{R});\ \partial_\alpha u \text{ bounded, uniformly continuous in } \mathcal{R},\ \forall \alpha \in \mathbb{N} \times \mathbb{N} \text{ with } |\alpha| \leqslant k\right\}$$

$$C^\infty(\overline{\mathcal{R}}) = \bigcap_{k=0}^{\infty} C^k(\overline{\mathcal{R}})$$

$$C_0(\overline{\mathcal{R}}) = \left\{u \in C(\overline{\mathcal{R}});\ u = 0 \text{ on } \partial\mathcal{R}\right\}$$

Since $\mathcal{R}$ is bounded and has a Lipschitz boundary there exists a total extension operator for $\mathcal{R}$ (see [9, Theorem 5.24, p. 154]) and thus $C^\infty(\overline{\mathcal{R}})$ coincides with the space of functions in $C^\infty(\mathbb{R}^2)$, restricted to $\overline{\mathcal{R}}$. Applying Lemma 3.1 we can prove the following:

Corollary 3.2. $\overline{C^\infty(\overline{\mathcal{R}}) \cap C_0(\overline{\mathcal{R}})}^{\|\cdot\|_{2,2}} = H_0(\mathcal{R})$.

3.2. Regularity of hinged plates

The argument is proven indirectly by showing that a hinged rectangular plate solves the Navier bi-Laplace problem which solves in turn the iterated Dirichlet Laplace. Consider the following two problems:

$$\begin{cases} -\Delta u = w & \text{in } \mathcal{R} \\ u = 0 & \text{on } \partial\mathcal{R} \end{cases} \quad \text{and} \quad \begin{cases} -\Delta w = f & \text{in } \mathcal{R} \\ w = 0 & \text{on } \partial\mathcal{R} \end{cases} \quad (4), \qquad \begin{cases} \Delta^2 u = f & \text{in } \mathcal{R} \\ \Delta u = u = 0 & \text{on } \partial\mathcal{R} \end{cases} \tag{5}$$

Lemma 3.3. *If $f \in L^2(\mathcal{R})$ then the weak solution $u \in H_0(\mathcal{R})$ of* (5) *satisfies $u \in W^{4,2}(\mathcal{R})$ and thus $(u, -\Delta u) \in H_0(\mathcal{R}) \times H_0(\mathcal{R})$ solves* (4)*.*

A straightforward integration by parts yields the following:

Lemma 3.4. *Let $u \in C^\infty(\overline{\mathcal{R}}) \cap C_0(\overline{\mathcal{R}})$. Then for all $v \in C^\infty(\overline{\mathcal{R}})$*

$$\int_{\mathcal{R}} (u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy})\,\mathrm{d}x\,\mathrm{d}y = -\int_{\partial\mathcal{R}} u_{\tau n}v_\tau\,\mathrm{d}\tau = 2[u_{xy}v]_{(0,0)\,\&\,(a,b)}^{(a,0)\,\&\,(0,b)} + \int_{\partial\mathcal{R}} u_{\tau\tau n}v\,\mathrm{d}\tau \tag{6}$$

where $[\Psi]_{(0,0)\,\&\,(a,b)}^{(a,0)\,\&\,(0,b)} = \Psi(a,0) + \Psi(0,b) - \Psi(0,0) - \Psi(a,b)$.

Now using Corollary 3.2 one can finally show the equivalence between a hinged plate and the Navier bi-Laplace.

Corollary 3.5. *For all $u \in H_0(\mathcal{R})$*

$$\int_{\mathcal{R}} \left(u_{xx}u_{yy} - u_{xy}^2\right)\mathrm{d}x\,\mathrm{d}y = 0$$

that is, the minimizer $u \in H_0(\mathcal{R})$ of J_σ coincides with the weak solution of (5)*.*

For polygonal domains this result is found in Lemma 2.2.2 of [3]. See also [10]. For u a solution of $-\Delta u = f$ in $W_0^{1,2}(\Omega)$ one finds $u \in W^{2,2}(\Omega)$ for generic $f \in L^2(\Omega)$ if and only if the polygonal domain has no concave corners. See [11].

4. The comparison argument

Finally we can state our main result. The right angles of our rectangle will allow us to deploy an argument based on Serrin's corner point lemma [12]. This is due to the fact that a hinged plate will exhibit concentrated bending moments, in the form of the mixed second order derivative, at the corners of $\mathcal{R}$, as implied by Lemma 3.4.

Theorem 4.1. *Let $f \in L^2(\mathcal{R})$ with $0 \not\equiv f \leqslant 0$. Then, the weak solution of (5) does not minimize J_σ in $H_+(\mathcal{R})$.*

5. Conclusion

According to the Kirchhoff–Love model, the last result illustrates that a supported rectangular plate which is pushed downwards will not stick on its supporting boundary but will have a positive deflection there. Consequently, in contrast to a widely spread belief, the Navier boundary conditions that are indeed satisfied by a hinged rectangular plate, will not be satisfied by a supported one; the elastic energy of a plate having a unilateral support is strictly less than the elastic energy of a purely hinged one.

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