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Extension of the non-uniform warping theory to an orthotropic composite beam

Extension aux sections composites orthotropes de la théorie du gauchissement non uniforme des poutres

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ABSTRACT

This Note proposes an extension to composite section of the non-uniform (out-of-plane) warping beam theory recently established for homogeneous and isotropic beam by R. El Fatmi (C. R. Mecanique 335 (2007) 467–474). For the present work, which constitutes a first step of this extension, the cross-section is assumed to be symmetric and made by orthotropic materials; however, Poisson's effects (called here in-plane warping) are also taken into account. Closed form results are given for the structural behavior of the composite beam and for the expressions of the 3D stresses; these ones, easy to compare with 3D Saint Venant stresses, make clear the additional contribution of the new internal forces induced by the non-uniformity of the (in and out) warpings. As first numerical applications, results on torsion and shear-bending of a cantilever sandwich beam are presented.

RÉSUMÉ

Cette Note propose une extension aux sections composites de la théorie gérérale du gauchissement non uniforme récemment établie pour une poutre homogène isotrope et de section quelconque par El Fatmi (C. R. Mecanique 335 (2007) 467–474). Cette extension se restreint, dans la présente Note, aux sections symétriques à phases orthotropes, mais introduit aussi la prise en compte des effets Poisson dans la déformation des sections. Les résultats théoriques concernent le comportement généralisé de la poutre et l'expression 3D des contraintes; celles-ci, comparées à celles de Saint Venant, montrent explicitement la contribution de chacun des nouveaux efforts intérieurs induits par la non uniformité des gauchissements et des effets Poisson. Comme première application, quelques résultats numériques obtenus pour la torsion et la flexion simple d'une poutre console de section sandwhich sont présentés.

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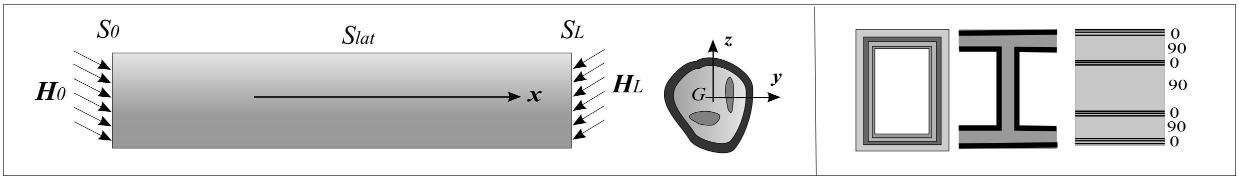

Fig. 1. The 3D (extended) Saint Venant problem and examples of *so*-CS.

1. Introduction

For composite section, unlike homogeneous and isotropic one, (out-of-plane) warpings are not only due to torsional moment and shear forces, but may also be induced by axial force and bending moments [1,2]. Furthermore, several elastic couplings between extensional, flexural or torsional deformations may occur, even if the composite cross-section is symmetric [1,3,4,2].

Recently, a general non-uniform warping beam theory (denoted by NUW-BT) for homogeneous elastic and isotropic beam, including torsional and shear forces effects, has been established by El Fatmi [5,6]. The theoretical development of this NUW-BT is completely based on the knowledge of the 3D solution of the *original* Saint Venant (SV) problem which provides, in particular, the SV-warping functions of the cross-section which are those commonly considered as the reference to describe the natural warpings of a cross-section. This makes this theory free from the classical assumptions [3] and valid for any shape of cross-section [6]. Using now the 3D *extended* SV-solution established for composite beam [7], NUW-BT may be rewritten for any elastic composite cross-section. However, the problem is much more complex and this extension should be handled with a great care. Besides, in NUW-BT, Poisson's effects are absent in the displacement model; this assumption, acceptable for a homogeneous section, should be reconsidered for a composite one, specially when the gaps between the rigidities of the materials are important.

The extension of NUW-BT will be then restricted, as a first step, to the case of symmetric cross-section (denoted by *so*-CS) made by orthotropic materials for which the principal material coordinates coincide with those of the beam. Indeed, it is shown in [4] that, for *so*-CS, warpings are (at least in SV-theory) only due to the torsion and the shear forces, and without any elastic coupling, which may simplify the problem. This situation seems to be similar to the homogeneous and isotropic case, but the section is now composite and each material is orthotropic. Further, to better take into account the composite nature of the cross-section, Poisson's effects (called "in-plane warping" in this Note) will also be considered; this to detect eventual coupling between in- and out-of-plane warpings. The present theory is based on a kinematics including the out-of-plane warpings due torsion and shear forces and the in-plane warpings due to axial force and bending moments. Starting from a displacement model including six independent warping parameters and using as (out or in) warping modes *the* SV-warping functions of the composite section, the corresponding non-uniform warping beam theory is derived. It should be noted that the theoretical development of this theory is completely based on the properties of 3D SV-solution. Thus, it is necessary in the present Note, to first recall this solution and to specify its properties for the particular case of *so*-CS.

2. The extended Saint Venant problem and its solution for *so*-CS

The (extended) SV-problem is a 3D equilibrium elastic problem (Fig. 1). The composite beam is along the $\boldsymbol{x}$-axis and is occupying a prismatic domain Ω of a constant cross-section S and length L. S_{lat} is the lateral surface and S_0 and S_L are the extremity cross-sections. The beam is in equilibrium only under surface force densities $\boldsymbol{H}_0$ and $\boldsymbol{H}_L$ acting on S_0 and S_L, respectively. The materials of the cross-section are anisotropic and perfectly bonded together. SV-solution satisfies all the equations of the linearized equilibrium problem, except the boundary conditions on (S_0, S_L) which are satisfied only in terms of resultant. Let $M = x\boldsymbol{x} + \boldsymbol{X}$ denotes a point of Ω, where $\boldsymbol{X}$ belongs to S, $\boldsymbol{\xi}$ the displacement vector, and $\boldsymbol{\sigma}$ the stress tensor; SV-solution, as given in [7], is (upperbar will be used for SV-quantities):

$$\bar{\boldsymbol{\xi}} = \boldsymbol{u}(x) + \boldsymbol{\omega}(x) \wedge \boldsymbol{X} + \mathcal{A}(\boldsymbol{X}) \cdot \boldsymbol{R}(x) + \mathcal{B}(\boldsymbol{X}) \cdot \boldsymbol{M}(x) \tag{1}$$

$$\bar{\boldsymbol{\sigma}} \cdot \boldsymbol{x} = \mathcal{A}^0(\boldsymbol{X}) \cdot \boldsymbol{R}(x) + \mathcal{B}^0(\boldsymbol{X}) \cdot \boldsymbol{M}(x) \tag{2}$$

$$\begin{bmatrix} \boldsymbol{\gamma} \\ \boldsymbol{\chi} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}' + \boldsymbol{x} \wedge \boldsymbol{\omega} \\ \boldsymbol{\omega}' \end{bmatrix} = \bar{\boldsymbol{\Lambda}} \begin{bmatrix} \boldsymbol{R} \\ \boldsymbol{M} \end{bmatrix}, \qquad \begin{pmatrix} \boldsymbol{R} = (N, T^y, T^z) \\ \boldsymbol{M} = (M^x, M^y, M^z) \end{pmatrix} \tag{3}$$

where the cross-sectional stresses ($\boldsymbol{R} = \langle \boldsymbol{\sigma} \cdot \boldsymbol{x} \rangle$, $\boldsymbol{M} = \langle \boldsymbol{X} \wedge (\boldsymbol{\sigma} \cdot \boldsymbol{x}) \rangle$) verify the 1D equilibrium equations: $\boldsymbol{R}' = \boldsymbol{0}$; $\boldsymbol{M}' + \boldsymbol{x} \wedge \boldsymbol{R} = \boldsymbol{0}$ ($\langle(\cdot)\rangle$ denotes $\int_S (\cdot)\,\mathrm{d}S$ and $(\cdot)'$ denotes the derivative with respect to x). $(N, T^y, T^z, M^x, M^y, M^z)$ are the axial force, the shear forces, the torsional moment, and the bending moments, respectively. The linear operators ($\bar{\boldsymbol{\Lambda}}, \mathcal{A}, \mathcal{B}\mathcal{A}^0\mathcal{B}^0$), which are characteristics of the cross-section (shape and materials), verify the following properties:

$$\langle \mathcal{A}^0 \rangle = \mathcal{I}, \qquad \langle \mathcal{B}^0 \rangle = \mathcal{O}, \qquad \langle \boldsymbol{X} \wedge \mathcal{A}^0 \rangle = \mathcal{O}, \qquad \langle \boldsymbol{X} \wedge \mathcal{B}^0 \rangle = \mathcal{I} \tag{4}$$

$$\langle \mathcal{A}^{0^t} \mathcal{A} \rangle = \langle \mathcal{A}^{0^t} \mathcal{B} \rangle = \langle \mathcal{B}^{0^t} \mathcal{A} \rangle = \langle \mathcal{B}^{0^t} \mathcal{B} \rangle = \mathcal{O} \tag{5}$$

Let $\mathbf{K}$ denote the elasticity tensor. $\bar{\boldsymbol{\Lambda}}$, the structural compliance operator of the beam, is the result of the identification [7]:

$$\langle \bar{\boldsymbol{\sigma}}^{\alpha} \boldsymbol{K}^{-1} \bar{\boldsymbol{\sigma}}^{\beta} \rangle = \begin{bmatrix} \boldsymbol{R}^{\alpha} \\ \boldsymbol{M}^{\alpha} \end{bmatrix}^{t} \bar{\boldsymbol{\Lambda}} \begin{bmatrix} \boldsymbol{R}^{\beta} \\ \boldsymbol{M}^{\beta} \end{bmatrix} \quad \forall(\alpha, \beta) \tag{6}$$

where $(\bar{\boldsymbol{\sigma}}^{\alpha}, \boldsymbol{R}^{\alpha}, \boldsymbol{M}^{\alpha})$ and $(\bar{\boldsymbol{\sigma}}^{\beta}, \boldsymbol{R}^{\beta}, \boldsymbol{M}^{\beta})$ are the stresses and cross-sectional resultants corresponding to the SV-solutions associated to the data $(\boldsymbol{H}_0^{\alpha}, \boldsymbol{H}_L^{\alpha})$ and $(\boldsymbol{H}_0^{\beta}, \boldsymbol{H}_L^{\beta})$, respectively.

Let *so*-CS designates a y–z-symmetric composite cross-section where each material is x–y–z-orthotropic. For *so*-CS $\mathcal{A}$, $\mathcal{B}$ and the 1D-constitutive behavior of the beam (derived from $\bar{\boldsymbol{\Lambda}}$) have the following forms [4]:

$$\mathcal{A} = \begin{bmatrix} 0 & \phi^y & \phi^z \\ U_y^x & 0 & 0 \\ U_z^x & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} N \\ M^y \\ M^z \end{bmatrix} = \bar{\boldsymbol{\Gamma}}_1 \begin{bmatrix} \gamma_x \\ \chi_y \\ \chi_z \end{bmatrix} \quad \text{with } \bar{\boldsymbol{\Gamma}}_1 = \begin{bmatrix} \widetilde{EA} & 0 & 0 \\ 0 & \widetilde{EI_y} & 0 \\ 0 & 0 & \widetilde{EI_z} \end{bmatrix}$$
$$\mathcal{B} = \begin{bmatrix} \phi^x & 0 & 0 \\ 0 & U_y^y & U_y^z \\ 0 & U_z^y & U_z^z \end{bmatrix}, \qquad \begin{bmatrix} M^x \\ T^y \\ T^z \end{bmatrix} = \bar{\boldsymbol{\Gamma}}_2 \begin{bmatrix} \chi_x \\ \gamma_y \\ \gamma_z \end{bmatrix} \quad \text{with } \bar{\boldsymbol{\Gamma}}_2 = \begin{bmatrix} \widetilde{GJ} & 0 & 0 \\ 0 & \widetilde{GA_y} & 0 \\ 0 & 0 & \widetilde{GA_z} \end{bmatrix} \tag{7}$$

where, with $i \in \{x, y, z\}$, (U_y^i, U_z^i) are the in-plane SV-functions due to the Poisson's effect and related to (N, M^y, M^z); (ϕ^i) are the out-of-plane SV-warping functions related to (M^x, T^y, T^z); $(\widetilde{EA}, \widetilde{GA_y}, \widetilde{GA_z}, \widetilde{GJ}, \widetilde{EI_y}, \widetilde{EI_z})$ are the six cross-sectional constants of the composite section. Further, due to the double symmetry of the section, (U_y^i, U_z^i, ϕ^i) have the following properties with respect to y and z:

U_y^x	U_z^x	U_y^y	U_z^y	U_y^z	U_z^z	ϕ^x	ϕ^y	ϕ^z
od/y, ev/z	ev/y, od/z	od/y, od/z	ev/y, ev/z	ev/y, ev/z	od/y, od/z	od/y, od/z	od/y, ev/z	ev/y, od/z

where od/y or ev/y means that the function is odd or even with respect to y.

For an x–y–z-orthotropic material, Hooke's law may be splitted to $\boldsymbol{\sigma} = \mathbf{K}_1\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\sigma}_2 = \mathbf{K}_2\boldsymbol{\varepsilon}_2$ where $(\mathbf{K}_1, \mathbf{K}_2, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2)$ are defined by

$$\mathbf{K}_1 = \begin{bmatrix} K_{11} & K_{12} & K_{13} & 0 \\ K_{12} & K_{22} & K_{23} & 0 \\ K_{13} & K_{23} & K_{33} & 0 \\ 0 & 0 & 0 & G_{yz} \end{bmatrix}, \qquad \mathbf{K}_2 = \begin{bmatrix} G_{xy} & 0 \\ 0 & G_{xz} \end{bmatrix}, \qquad \boldsymbol{\sigma}_1 = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \end{bmatrix}, \qquad \boldsymbol{\sigma}_2 = \begin{bmatrix} \tau_{xy} \\ \tau_{xz} \end{bmatrix}$$
$$\boldsymbol{\varepsilon}_1 = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \end{bmatrix}, \qquad \boldsymbol{\varepsilon}_2 = \begin{bmatrix} 2\varepsilon_{xy} \\ 2\varepsilon_{xz} \end{bmatrix} \tag{8}$$

Let us introduce for convenience the constants $\widetilde{R}_y = \frac{\widetilde{GA_y}}{\widetilde{EI_z}}$, $\widetilde{R}_z = \frac{\widetilde{GA_z}}{\widetilde{EI_y}}$ and the following (SV-like) functions:

$$\left.\begin{array}{lll} (V^x, W^x) = (\widetilde{EA}U_y^x, \widetilde{EA}U_z^x), & (V^y, W^y) = (\widetilde{EI}_yU_y^y, \widetilde{EI}_yU_z^y), & (V^z, W^z) = (\widetilde{EI}_zU_y^z, \widetilde{EI}_zU_z^z) \\ \psi^x = \widetilde{GJ}\phi^x, & \psi^y = \widetilde{GA}_y\phi^y, & \psi^z = \widetilde{GA}_z\phi^z \end{array}\right\} \tag{9}$$

One can show from Eqs. (1) and (7), that 3D SV strains and stresses may be splitted and written:

$$\boldsymbol{\varepsilon}_1 = \overbrace{\begin{bmatrix} 1 & z & -y \\ V_{,y}^x & V_{,y}^y & V_{,y}^z \\ W_{,z}^x & W_{,z}^y & W_{,z}^z \\ (V_{,z}^x + W_{,y}^x) & (V_{,z}^y + W_{,y}^y) & (V_{,z}^z + W_{,y}^z) \end{bmatrix}}^{\mathbf{D}_1} \begin{bmatrix} \gamma_x \\ \chi_y \\ \chi_z \end{bmatrix}, \qquad \boldsymbol{\sigma}_1 = \mathbf{K}_1\mathbf{D}_1(\bar{\boldsymbol{\Gamma}}_1)^{-1} \begin{bmatrix} N \\ M^y \\ M^z \end{bmatrix} \tag{10}$$

$$\boldsymbol{\varepsilon}_2 = \overbrace{\begin{bmatrix} (-z + \psi_{,y}^x) & (1 + \psi_{,y}^y - \widetilde{R}_yV^z) & (\psi_{,y}^z + \widetilde{R}_zV^y) \\ (y + \psi_{,z}^x) & (\psi_{,z}^y - \widetilde{R}_yW^z) & (1 + \psi_{,z}^z + \widetilde{R}_zW^y) \end{bmatrix}}^{\mathbf{D}_2} \begin{bmatrix} \chi_x \\ \gamma_y \\ \gamma_z \end{bmatrix}, \qquad \boldsymbol{\sigma}_2 = \mathbf{K}_2\mathbf{D}_2(\bar{\boldsymbol{\Gamma}}_2)^{-1} \begin{bmatrix} M^x \\ T^y \\ T^z \end{bmatrix} \tag{11}$$

Properties of the SV-functions for *so*-CS. The expansion of Eqs. (4) and (5) lead to the detailed properties given by Eqs. (12) and (13), respectively:

$$\left.\begin{array}{l}
\langle K_{11} + K_{12}V^x_{,y} + K_{13}W^x_{,z}\rangle = \widetilde{EA} \\
\langle z(zK_{11} + K_{12}V^y_{,y} + K_{13}W^y_{,z})\rangle = \widetilde{EI_y} \\
\langle -y(-yK_{11} + K_{12}V^z_{,y} + K_{13}W^z_{,z})\rangle = \widetilde{EI_z} \\
\langle -zG_{xy}\psi^x_{,y} + yG_{xz}\psi^x_{,z}\rangle = \widetilde{GJ} - \widetilde{GI_x} \\
\langle G_{xy}(\psi^y_{,y} - \widetilde{R}_y V^z)\rangle = \widetilde{GA_y} - \langle G_{xy}\rangle \\
\langle G_{xz}(\psi^y_{,z} + \widetilde{R}_z W^y)\rangle = \widetilde{GA_z} - \langle G_{xz}\rangle
\end{array}\right\} \tag{12}$$

$$\left.\begin{array}{l}
\langle (K_{11} + K_{12}V^x_{,y} + K_{13}W^x_{,z})\psi^i\rangle = 0 \\
\langle (zK_{11} + K_{12}V^y_{,y} + K_{13}W^y_{,z})\psi^i\rangle = 0 \\
\langle (-yK_{11} + K_{12}V^z_{,y} + K_{13}W^z_{,z})\psi^i\rangle = 0 \\
\langle (G_{xy}(-z + \psi^x_{,y}))V^i + (G_{xz}(y + \psi^x_{,z}))W^i\rangle = 0 \\
\langle (G_{xy}(1 + \psi^y_{,y} - \widetilde{R}_y V^z))V^i + (G_{xz}(\psi^y_{,z} - \widetilde{R}_y W^z))W^i\rangle = 0 \\
\langle (G_{xy}(\psi^z_{,y} + \widetilde{R}_z V^y))V^i + (G_{xz}(1 + \psi^z_{,z} + \widetilde{R}_z W^y))W^i\rangle = 0
\end{array}\right\}; \quad (\forall i \in \{x, y, z\}) \tag{13}$$

where $\widetilde{GI_x} = \langle z^2 G_{xy} + y^2 G_{xz}\rangle$. Further Eq. (6) (written in terms of strains) leads to

$$\left.\begin{array}{l}
\langle (K_{11} + K_{12}V^x_{,y} + K_{13}W^x_{,z})^2\rangle = \widetilde{EA} \\
\langle (zK_{11} + K_{12}V^y_{,y} + K_{13}W^y_{,z})^2\rangle = \widetilde{EI_y} \\
\langle (-yK_{11} + K_{12}V^z_{,y} + K_{13}W^z_{,z})^2\rangle = \widetilde{EI_z} \\
\langle (G_{xy}(-z + \psi^x_{,y})^2 + G_{xz}(y + \psi^x_{,z})^2)\rangle = \widetilde{GJ} \\
\langle (G_{xy}(1 + \psi^y_{,y} - \widetilde{R}_y V^z)^2 + G_{xz}(\psi^y_{,z} - \widetilde{R}_y W^z)^2)\rangle = \widetilde{GA_y} \\
\langle (G_{xy}(\psi^z_{,y} + \widetilde{R}_z V^y)^2 + G_{xz}(1 + \psi^z_{,z} + \widetilde{R}_z W^y)^2)\rangle = \widetilde{GA_z}
\end{array}\right\} \tag{14}$$

3. Non-uniform warping beam theory (NUW-BT)

For the sake of simplicity the beam reference problem is taken similar to the 3D SV-problem defined in Section 2. The kinematical modeling is the following displacement field (Eq. (1)):

$$\boldsymbol{\xi}(\boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{\eta}, \boldsymbol{\alpha}) = \boldsymbol{u} + \boldsymbol{\omega} \wedge \boldsymbol{X} + \alpha_x \begin{bmatrix} 0 \\ V^x \\ W^x \end{bmatrix} + \alpha_y \begin{bmatrix} 0 \\ V^y \\ W^y \end{bmatrix} + \alpha_z \begin{bmatrix} 0 \\ V^z \\ W^z \end{bmatrix} + \eta_x \begin{bmatrix} \psi^x \\ 0 \\ 0 \end{bmatrix} + \eta_y \begin{bmatrix} \psi^y \\ 0 \\ 0 \end{bmatrix} + \eta_z \begin{bmatrix} \psi^z \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

where, with $i \in \{x, y, z\}$, (η_i, α_i) are the out and in warping parameters and (ψ^i, V^i, W^i) are related to *the* out- and in-plane SV-warping functions, respectively. The beam theory that corresponds to this displacement, parametrized by $(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{\eta}, \boldsymbol{\alpha})$, will be derived, in a classical way, by the principle of virtual work. Let us denote by $\widehat{\boldsymbol{\xi}} = \boldsymbol{\xi}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{\omega}}, \widehat{\boldsymbol{\eta}}, \widehat{\boldsymbol{\alpha}})$ a virtual displacement and $\widehat{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}(\widehat{\boldsymbol{\xi}})$ the corresponding strain tensor. The internal virtual work is $W_i = -\int_L \langle \boldsymbol{\sigma} : \widehat{\boldsymbol{\varepsilon}}\rangle \, \mathrm{d}x$; using Eq. (15), W_i takes the form:

$$\begin{aligned}
W_i &= -\int_L \big(\boldsymbol{R} \cdot \widehat{\boldsymbol{\gamma}} + \boldsymbol{M} \cdot \widehat{\boldsymbol{\omega}} + \boldsymbol{M}_\psi \cdot \widehat{\boldsymbol{\eta}}' + \boldsymbol{M}_s \cdot \widehat{\boldsymbol{\eta}} + \boldsymbol{A}_\nu \cdot \widehat{\boldsymbol{\alpha}}' + \boldsymbol{B}_s \cdot \widehat{\boldsymbol{\alpha}}\big)\, \mathrm{d}x \\
&= \int_L \big[\boldsymbol{R}' \cdot \widehat{\boldsymbol{u}} + (\boldsymbol{M}' + \boldsymbol{x} \wedge \boldsymbol{R}) \cdot \widehat{\boldsymbol{\omega}} + (\boldsymbol{M}'_\psi - \boldsymbol{M}) \cdot \widehat{\boldsymbol{\eta}} + (\boldsymbol{A}'_\nu - \boldsymbol{B}) \cdot \widehat{\boldsymbol{\alpha}}\big]\, \mathrm{d}x \\
&\quad - \big[\boldsymbol{R} \cdot \widehat{\boldsymbol{u}} + \boldsymbol{M} \cdot \widehat{\boldsymbol{\omega}} + \boldsymbol{M}_\psi \cdot \widehat{\boldsymbol{\eta}} + \boldsymbol{A}_\nu \cdot \widehat{\boldsymbol{\alpha}} + \big]_O^L
\end{aligned} \tag{16}$$

$(\boldsymbol{R}, \boldsymbol{M})$ are the classical cross-sectional stresses and $\boldsymbol{M}_\psi = (M^x_\psi, M^y_\psi, M^z_\psi)$, $\boldsymbol{M}_s = (M^x_s, T^y_s, T^z_s)$, $\boldsymbol{A}_\nu = (A^x_\nu, A^y_\nu, A^y_\nu)$, and $\boldsymbol{B}_s = (N_s, M^y_s, M^z_s)$ are the new (or additional) ones defined by

$$\left.\begin{array}{ll}
M^x_\psi = \langle \sigma_{xx}\psi^x\rangle, & M^x_s = \langle \tau_{xy}\psi^x_{,y} + \tau_{xz}\psi^x_{,z}\rangle \\
M^y_\psi = \langle \sigma_{xx}\psi^y\rangle, & T^y_s = \langle \tau_{xy}\psi^y_{,y} + \tau_{xz}\psi^y_{,z}\rangle \\
M^z_\psi = \langle \sigma_{xx}\psi^z\rangle, & T^z_s = \langle \tau_{xy}\psi^z_{,y} + \tau_{xz}\psi^z_{,z}\rangle \\
A^x_\nu = \langle \tau_{xy}V^x + \tau_{xz}W^x\rangle, & N_s = \langle \sigma_{yy}V^x_{,y} + \sigma_{zz}W^x_{,y} + \tau_{yz}(V^x_{,z} + W^x_{,y})\rangle \\
A^y_\nu = \langle \tau_{xy}V^y + \tau_{xz}W^y\rangle, & M^y_s = \langle \sigma_{yy}V^y_{,y} + \sigma_{zz}W^y_{,y} + \tau_{yz}(V^y_{,z} + W^y_{,y})\rangle \\
A^z_\nu = \langle \tau_{xy}V^z + \tau_{xz}W^z\rangle, & M^z_s = \langle \sigma_{yy}V^y_{,y} + \sigma_{zz}W^z_{,y} + \tau_{yz}(V^y_{,z} + W^z_{,y})\rangle
\end{array}\right\} \tag{17}$$

$(\boldsymbol{M}_\psi, \boldsymbol{M}_s)$ introduced in [6] are called the *bimoment* vector and the *secondary* internal force vector, and they are both related to out-of-plane warpings. Similarly, it corresponds to the in-plane warping two internal forces $(\boldsymbol{A}_\nu, \boldsymbol{B}_s)$ related to Poisson's effects. The subscript $(.)_s$ as *secondary* has been chosen to indicate that the components (N_s, M^y_s, M^z_s) of $\boldsymbol{B}_s$ may be seen as secondary axial force and secondary bending moments (see the remarks on p. 709). The external virtual work is $W_e = \langle \boldsymbol{H}_O \cdot \widehat{\boldsymbol{\xi}}\rangle_O + \langle \boldsymbol{H}_L \cdot \widehat{\boldsymbol{\xi}}\rangle_L$; it takes the form

$$W_e = \boldsymbol{P}_O \cdot \widehat{\boldsymbol{u}}_O + \boldsymbol{C}_O \cdot \widehat{\boldsymbol{\omega}}_O + \boldsymbol{Q}_O \cdot \widehat{\boldsymbol{\eta}}_O + \boldsymbol{S}_O \cdot \widehat{\boldsymbol{\alpha}}_O + \boldsymbol{P}_L \cdot \widehat{\boldsymbol{u}}_L + \boldsymbol{C}_L \cdot \widehat{\boldsymbol{\omega}}_L + \boldsymbol{Q}_L \cdot \widehat{\boldsymbol{\eta}}_L + \boldsymbol{S}_L \cdot \widehat{\boldsymbol{\alpha}}_L \tag{18}$$

where the 1D external forces $(\boldsymbol{P}, \boldsymbol{C}, \boldsymbol{Q}, \boldsymbol{S})$ are defined by $(i \in \{x, y, z\};\ \boldsymbol{x}_i \in \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\})$

$$\boldsymbol{P} = \langle \boldsymbol{H}\rangle, \qquad \boldsymbol{C} = \langle \boldsymbol{X} \wedge \boldsymbol{H}\rangle, \qquad \boldsymbol{Q} = \langle H_x\psi^i\rangle\boldsymbol{x}_i, \qquad \boldsymbol{S} = \langle H_y V^i + H_z W^i\rangle\boldsymbol{x}_i \tag{19}$$

Thanks to the principle of virtual work, Eqs. (16) and (18) allow to provide the equilibrium equations

$$\boldsymbol{R}' = \boldsymbol{0}, \qquad \boldsymbol{M}' + \boldsymbol{x} \wedge \boldsymbol{R} = \boldsymbol{0}, \qquad \boldsymbol{M}'_\psi - \boldsymbol{M} = \boldsymbol{0}, \qquad \boldsymbol{A}'_\nu - \boldsymbol{B} = \boldsymbol{0} \tag{20}$$

and the boundary conditions

$$x = 0: (\boldsymbol{R}, \boldsymbol{M}, \boldsymbol{M}_\psi, \boldsymbol{A}_\nu) = -(\boldsymbol{P}_O, \boldsymbol{C}_O, \boldsymbol{Q}_O, \boldsymbol{S}_O) \quad \text{and} \quad x = L: (\boldsymbol{R}, \boldsymbol{M}, \boldsymbol{M}_\psi, \boldsymbol{A}_\nu) = (\boldsymbol{P}_L, \boldsymbol{C}_L, \boldsymbol{Q}_L, \boldsymbol{S}_L) \tag{21}$$

Beam structural behavior. Using *all* the properties of (V^i, W^i, ψ^i) detailed in Section 2, it is easy to show that the structural behavior of the beam that corresponds to this beam theory, is expressed by the following (decoupled) 1D-constitutive relations:

$$\begin{gathered}
\begin{bmatrix} N \\ N_s \\ A^x_\nu \end{bmatrix} = \begin{bmatrix} \langle K_{11}\rangle & -(\langle K_{11}\rangle - \widetilde{EA}) & 0 \\ -(\langle K_{11}\rangle - \widetilde{EA}) & \langle K_{11}\rangle - \widetilde{EA} & 0 \\ 0 & 0 & gx \end{bmatrix} \begin{bmatrix} \gamma_x \\ \alpha_x \\ \alpha'_x \end{bmatrix} \\
\begin{bmatrix} M^x \\ M^x_s \\ M^x_\psi \end{bmatrix} = \begin{bmatrix} \widetilde{GI_x} & -(\widetilde{GI_x} - \widetilde{GJ}) & 0 \\ -(\widetilde{GI_x} - \widetilde{GJ}) & \widetilde{GI_x} - \widetilde{GJ} & 0 \\ 0 & 0 & \widetilde{KI^x_\psi} \end{bmatrix} \begin{bmatrix} \chi_x \\ \eta_x \\ \eta'_x \end{bmatrix}
\end{gathered} \tag{22}$$

$$\begin{bmatrix} M^z \\ M^z_s \\ A^z_\nu \\ M^y_\psi \\ T^y \\ T^y_s \end{bmatrix} = \begin{bmatrix} \langle y^2K_{11}\rangle & -(\langle y^2K_{11}\rangle - \widetilde{EI_z}) & 0 & -a_y & 0 & 0 \\ -(\langle y^2K_{11}\rangle - \widetilde{EI_z}) & (\langle y^2K_{11}\rangle - \widetilde{EI_z}) & 0 & a_y & 0 & 0 \\ 0 & 0 & g_z & 0 & f_y & e_y \\ -a_y & a_y & 0 & \widetilde{KI^y_\psi} & 0 & 0 \\ 0 & 0 & f_y & 0 & \langle G_{xy}\rangle & -(\langle G_{xy}\rangle - \widetilde{GA_y}) + \widetilde{R}_y f_y \\ 0 & 0 & e_y & 0 & -(\langle G_{xy}\rangle - \widetilde{GA_y}) + \widetilde{R}_y f_y & \langle G_{xy}\rangle - \widetilde{GA_y} - \widetilde{R}_y(f_y - e_y) \end{bmatrix} \begin{bmatrix} \chi_z \\ \alpha_z \\ \alpha'_z \\ \eta'_y \\ \gamma_y \\ \eta_y \end{bmatrix} \tag{23}$$

$$\begin{bmatrix} M^y \\ M^y_s \\ A^y_\nu \\ M^z_\psi \\ T^z \\ T^z_s \end{bmatrix} = \begin{bmatrix} \langle z^2K_{11}\rangle & -(\langle z^2K_{11}\rangle - \widetilde{EI_y}) & 0 & -a_z & 0 & 0 \\ -(\langle z^2K_{11}\rangle - \widetilde{EI_y}) & (\langle z^2K_{11}\rangle - \widetilde{EI_y}) & 0 & a_z & 0 & 0 \\ 0 & 0 & g_y & 0 & f_z & e_z \\ -a_z & a_z & 0 & \widetilde{KI^z_\psi} & 0 & 0 \\ 0 & 0 & f_z & 0 & \langle G_{xz}\rangle & -(\langle G_{xz}\rangle - \widetilde{GA_z}) - \widetilde{R}_z f_z \\ 0 & 0 & e_z & 0 & -(\langle G_{xz}\rangle - \widetilde{GA_z}) - \widetilde{R}_z f_z & \langle G_{xz}\rangle - \widetilde{GA_z} + \widetilde{R}_z(f_z - e_z) \end{bmatrix} \begin{bmatrix} \chi_y \\ \alpha_y \\ \alpha'_y \\ \eta'_z \\ \gamma_z \\ \eta_z \end{bmatrix} \tag{24}$$

where the new cross-sectional constants that appear in the operators are given by

$$\left.\begin{array}{llll}
\widetilde{KI}^x_\psi = \langle K_{11}(\psi^x)^2\rangle, & g_x = \langle G_{xy}(V^x)^2 + G_{xz}(W^x)^2\rangle, & e_y = \langle G_{xy}\psi^y_{,y}V^z + G_{xz}\psi^y_{,z}W^z\rangle, & a_y = \langle yK_{11}\psi^y\rangle \\
\widetilde{KI}^y_\psi = \langle K_{11}(\psi^y)^2\rangle, & g_y = \langle G_{xy}(V^y)^2 + G_{xz}(W^y)^2\rangle, & e_z = \langle G_{xy}\psi^z_{,y}V^y + G_{xz}\psi^z_{,z}W^y\rangle, & a_z = \langle -zK_{11}\psi^z\rangle \\
\widetilde{KI}^z_\psi = \langle K_{11}(\psi^z)^2\rangle, & g_z = \langle G_{xy}(V^z)^2 + G_{xz}(W^z)^2\rangle, & f_y = \langle G_{xy}V^z\rangle, & f_z = \langle G_{xz}W^y\rangle
\end{array}\right\} \tag{25}$$

Remarks. Among these constants, $([\widetilde{KI}^x_\psi, \widetilde{KI}^y_\psi, \widetilde{KI}^z_\psi]\ ;\ [g_x, g_y, g_z])$ are the six (out and in) *warping rigidities* and $(a_y, f_y, e_y, a_z, f_z, e_z)$ express the coupling between the in and out warpings in the flexural behavior.

Also, one can deduce from these constitutive relations that

$$\left.\begin{array}{lll}
N_p = N + N_s = \widetilde{EA}\gamma_x, & M^y_p = M^y + M^y_s = \widetilde{EI}_z\chi_z, & M^z_p = M^z + M^z_s = \widetilde{EI}_z\chi_z \\
M^x_p = M^x + M^x_s = \widetilde{GJ}\chi_x, & T^y_p = T^y + T^y_s - \widetilde{R}_yA^z_\nu = \widetilde{GA}_y\gamma_y, & T^z_p = T^z + T^z_s + \widetilde{R}_zA^y_\nu = \widetilde{GA}_z\gamma_z
\end{array}\right\} \tag{26}$$

where $(N_p, M^y_p, M^z_p, M^x_p, T^y_p, T^z_p)$, called the *primary* internal forces, obey to constitutive relations similar to those of SV given by Eq. (7). One can see here a justification of the term *primary* and hence *secondary*.

Stresses. Using Hooke's law, we can establish that the stresses $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ may be written:

$$\begin{aligned}
\boldsymbol{\sigma}_1 = {} & \underbrace{\bar{\boldsymbol{\sigma}}\begin{bmatrix} N \\ M^y \\ M^z \end{bmatrix}}_{sv} + \bar{\boldsymbol{\sigma}}\begin{bmatrix} N_s \\ M^y_s \\ M^z_s \end{bmatrix} + \mathbf{K}_1\begin{bmatrix} \psi^x \\ 0 \\ 0 \\ 0 \end{bmatrix}\frac{M^x_\psi}{\widetilde{KI}^x_\psi} + \mathbf{K}_1\begin{bmatrix} 0 \\ V^x_{,y} \\ W^x_{,z} \\ (V^x_{,z} + W^x_{,y}) \end{bmatrix}\frac{N_s}{\langle K_{11}\rangle - \widetilde{EA}} \\
& + \mathbf{K}_1\begin{bmatrix} \psi^y & 0 \\ 0 & V^z_{,y} \\ 0 & W^z_{,z} \\ 0 & (V^z_{,z} + W^z_{,y}) \end{bmatrix}\begin{bmatrix} \widetilde{KI}^y_\psi & a_y \\ a_y & \langle y^2K_{11}\rangle - \widetilde{EI}_z \end{bmatrix}^{-1}\begin{bmatrix} M^y_\psi \\ M^z_s \end{bmatrix} \\
& + \mathbf{K}_1\begin{bmatrix} \psi^z & 0 \\ 0 & V^y_{,y} \\ 0 & W^y_{,z} \\ 0 & (V^y_{,z} + W^y_{,y}) \end{bmatrix}\begin{bmatrix} \widetilde{KI}^z_\psi & a_z \\ a_z & \langle z^2K_{11}\rangle - \widetilde{EI}_y \end{bmatrix}^{-1}\begin{bmatrix} M^z_\psi \\ M^y_s \end{bmatrix}
\end{aligned} \tag{27}$$

$$\begin{aligned}
\boldsymbol{\sigma}_2 = {} & \underbrace{\bar{\boldsymbol{\tau}}\begin{bmatrix} M^x \\ T^y \\ T^z \end{bmatrix}}_{sv} + \bar{\boldsymbol{\tau}}\begin{bmatrix} M^x_s \\ T^y_s - \widetilde{R}_yA^z_\nu \\ T^z_s + \widetilde{R}_zA^y_\nu \end{bmatrix} + \mathbf{K}_2\begin{bmatrix} \psi^x_{,y} \\ \psi^x_{,z} \end{bmatrix}\frac{M^x_s}{\widetilde{GI}_x - \widetilde{GJ}} + \mathbf{K}_2\begin{bmatrix} V^x \\ W^x \end{bmatrix}\frac{A^x_\nu}{g_x} \\
& + \mathbf{K}_2\begin{bmatrix} \psi^y_{,y} & V^z \\ \psi^y_{,z} & W^z \end{bmatrix}\begin{bmatrix} \langle G_{xy}\rangle - \widetilde{GA}_y - \widetilde{R}_y(f_y - e_y) & e_y \\ e_y & g_z \end{bmatrix}^{-1}\begin{bmatrix} T^y_s \\ A^z_\nu \end{bmatrix} \\
& + \mathbf{K}_2\begin{bmatrix} \psi^z_{,y} & V^y \\ \psi^z_{,z} & W^y \end{bmatrix}\begin{bmatrix} \langle G_{xz}\rangle - \widetilde{GA}_z + \widetilde{R}_z(f_z - e_z) & e_z \\ e_z & g_y \end{bmatrix}^{-1}\begin{bmatrix} T^z_s \\ A^y_\nu \end{bmatrix}
\end{aligned} \tag{28}$$

where $\bar{\boldsymbol{\sigma}} = \mathbf{K}_1\mathbf{D}_1(\bar{\boldsymbol{\Gamma}}_1)^{-1}$ and $\bar{\boldsymbol{\tau}} = \mathbf{K}_2\mathbf{D}_2(\bar{\boldsymbol{\Gamma}}_2)^{-1}$ are related to SV-stresses by Eqs. (10) and (11).

4. Comments and first numerical applications

Based on the knowledge of SV-solution, this (first extension of) NUW-BT is, a priori, valid for any shape of *so*-CS. In this theory, traction, torsion and flexural behaviors appear uncoupled and may be studied separately; however in-and-out warpings are coupled in the flexural behaviors. It is worth noting that the closed form expressions of the 3D stresses, easy to compare with those of SV, make clear the contribution of each new or additional internal forces induced by the non-uniformity of the warpings.

In order to apply this NUW-BT, it is imperative to previously compute, for any given cross-section, all its constants and SV-warping functions. This is achieved by the software called *SECOPE* available within the finite element code *CASTEM*.

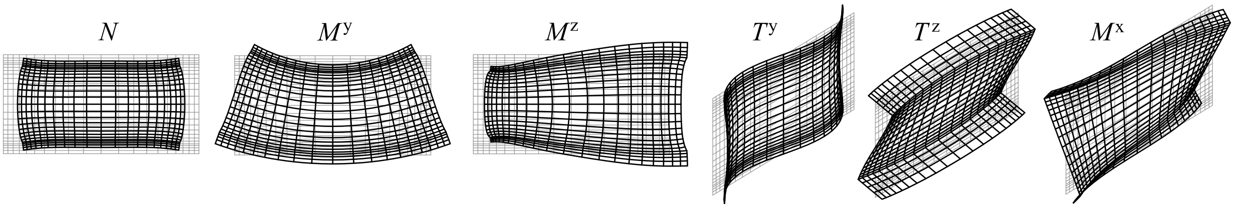

Fig. 2. In-plane warpings (Poisson's effects) due to (N, M^y, M^z) and out-of-plane warpings due to (T^y, T^z, M^x).

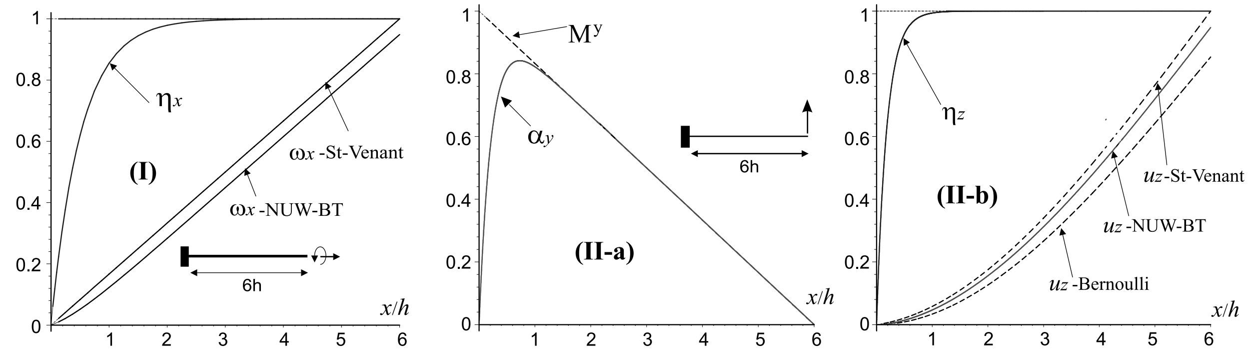

Fig. 3. x-variation of the warping parameters and the displacements for torsion (I) and shear-bending (II-a–b).

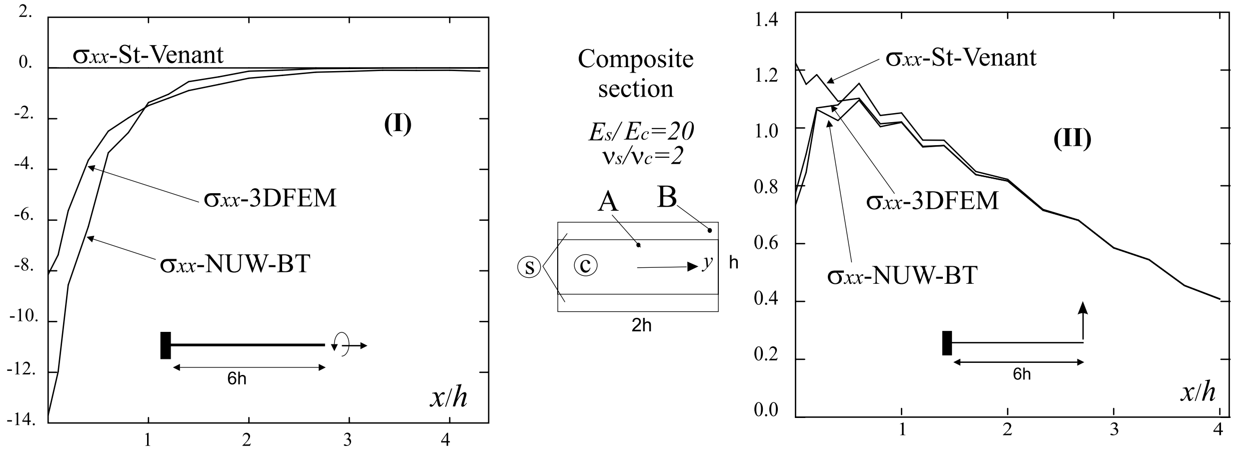

Fig. 4. x-variation of the axial stresses σ_{xx} for torsion (I) and shear-bending (II).

SECOPE has been developed conforming to the numerical method proposed by [2] for the computation of the 3D SV-solution within the framework of the *exact* beam theory [7].

As first applications, we present hereafter some results for a cantilever sandwich beam subjected to a tip torsional moment (C^x) or a tip transversal force (F^z). The cross-section is rectangular $(2h \times h)$ (Fig. 4), the thickness of each skin is $h/10$ and the length of the beam is $6h$. Young's modulus and Poisson's ratio for the skins and the core (which are isotropic) are defined by $E_s/E_c = 20$; $\nu_s/\nu_c = 2$. SV-warping functions are depicted in Fig. 2. For each case of loading, we give some numerical results for the 1D-behavior and the 3D stresses; the stresses are compared with those obtained by full 3D-FEM computations.

Torsion. Fig. 3(I) compares the variations of ω_x and η_x along the span with those of SV-BT. Starting from the built-in section, Fig. 4(I) compares, for the point B (in the skin), the x-variation of the axial stress σ_{xx} with that obtained by a 3D-FEM computation.

Shear-bending. Fig. 3(II) compares the variations of α_y, η_z and u_z along the span with those of SV-BT and Bernoulli-BT. Starting from the built-in section, Fig. 4(II) compares, for the point A (in the core), the x-variation of the axial stress σ_{xx} with that obtained by a 3D-FEM computation.

The results obtained for this section show that for the torsion, warping effect extends slowly from the built-in section to the interior part of the beam to reach SV-results. In contrast, for the shear-bending, warping effects are more localized close to the built-in section. This is confirmed by the variation of the (axial) stresses obtained by both 3D-FEM computations and NUW-BT estimations.

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